Solution of series $N°1$

Exercise 1:

1. • We suppose that $f(x) = \arcsin x - \dfrac{x}{\sqrt{1-x^2}}$, $\forall x \in]0,1[$. f is defined and differentiable on $]0,1[$, we have:

$$\begin{aligned} f'(x) &= \frac{1}{\sqrt{1-x^2}} - \frac{\sqrt{1-x^2} + \dfrac{2x^2}{2\sqrt{1-x^2}}}{1-x^2} \\ &= \frac{1}{\sqrt{1-x^2}} - \frac{1}{(1-x^2)\sqrt{1-x^2}} \\ &= \frac{1}{\sqrt{1-x^2}}(1 - \frac{1}{1-x^2}) = \frac{-x^2}{\sqrt{1-x^2}(1-x^2)} < 0. \end{aligned}$$

(because $-x^2 < 0,\ \textit{and}\ 1-x^2 > 0,\ \sqrt{1-x^2} \geq 0,\ \forall x \in]0,1[$). So $f'(x) < 0 \Longrightarrow f(x)$ is decreasing $\forall x \in]0,1[$.

We have $x > 0 \Longrightarrow f(x) < f(0) = 0 \Longrightarrow \arcsin x - \dfrac{x}{\sqrt{1-x^2}} < 0$. Therefore $\arcsin x < \dfrac{x}{\sqrt{1-x^2}}$, $\forall x \in]0,1[$.

• We suppose that $f(x) = \arctan x - \dfrac{x}{1-x^2}$, $\forall x > 0$, f is differentiable, and

$$\begin{aligned} f'(x) &= \frac{1}{1+x^2} - \frac{1+x^2-2x^2}{(1+x^2)^2} \\ &= \frac{1}{1+x^2} - \frac{1-x^2}{(1+x^2)^2} = \frac{2x^2}{(1+x^2)^2} > 0 \end{aligned}$$

So f is increasing. We have $x > 0 \Longrightarrow f(x) > f(0) \Longrightarrow \arctan x - \dfrac{x}{1+x^2} > 0$. Then $\arctan x > \dfrac{x}{1+x^2}$, $\forall x > 0$.

• Assume that $f(x) = \arcsin x + \arccos x$, $\forall x \in]-1,1[$, f is defined and differentiable on $]-1,1[$, and $f'(x) = \dfrac{1}{\sqrt{1-x^2}} - \dfrac{1}{\sqrt{1-x^2}} = 0$, so f is a constant $\forall x \in]-1,1[$. We have $f(0) = \arccos 0 + \arcsin 0 = \dfrac{\pi}{2}$. Therefore $\forall x \in]-1,1[$, $f(x) = \dfrac{\pi}{2} \Longrightarrow f(x) = \arccos x + \arcsin x = \dfrac{\pi}{2}$.

- Assume that $f(x)=\arctan x+\arctan\frac{1}{x}$, f is differentiable and $f'(x)=\dfrac{1}{1+x^2}+\dfrac{-(\frac{1}{x^2})}{1+\frac{1}{x^2}}=\dfrac{1}{1+x^2}-\dfrac{x^2}{x^2(1+x^2)}=0$, so f is a constant $\forall x>0$. We have $f(1)=\arctan 1+\arctan 1=\dfrac{\pi}{4}+\dfrac{\pi}{4}=\dfrac{\pi}{2}$. Therefore $f(x)=\arctan x+\arctan\frac{1}{x}=\dfrac{\pi}{2}$, $\forall x>0$.

2.
- $f'(x)=\dfrac{\frac{2x}{2\sqrt{x^2-1}}}{1+(\sqrt{x^2-1})^2}=\dfrac{2x}{2x^2\sqrt{x^2-1}}=\dfrac{1}{x\sqrt{x^2-1}}$.
- $g'(x)=\dfrac{\frac{1}{1+x^2}}{1+\arctan^2 x}=\dfrac{1}{(1+x^2)(1+\arctan^2 x)}$.

Exercise 2:

1. We have

$$\begin{aligned}
ch(x)ch(y) &= \frac{1}{2}(e^x+e^{-x})\times\frac{1}{2}(e^y+e^{-y})\\
&= \frac{1}{4}(e^{x+y}+e^{x-y}+e^{-(x-y)}+e^{-(x+y)})\\
sh(x)sh(y) &= \frac{1}{2}(e^x-e^{-x})\times\frac{1}{2}(e^y-e^{-y})\\
&= \frac{1}{4}(e^{x+y}-e^{x-y}-e^{-(x-y)}+e^{-(x+y)})
\end{aligned}$$

Substracting gives

$$\begin{aligned}
ch(x)ch(y)-sh(x)sh(y) &= 2\times\frac{1}{4}(e^{x-y}+e^{-(x-y)})\\
&= \frac{1}{2}(e^{x-y}+e^{-(x-y)})=ch(x-y).
\end{aligned}$$

2. We have $ch(2x)=\dfrac{1}{2}(e^{2x}+e^{-2x})$, $sh(2x)=\dfrac{1}{2}(e^{2x}-e^{-2x})$. So

$$\begin{aligned}
e^{2x}+e^{-2x}+5e^{2x}-5e^{-2x} &= 5\\
6e^{2x}-5-4e^{-2x} &= 0\\
6e^{4x}-5e^{2x}-4 &= 0\\
(3e^{2x}-4)(2e^{2x}+1) &= 0
\end{aligned}$$

so $e^{2x} = \frac{4}{3}$ *or* $e^{2x} = -\frac{1}{2}$. The only real solution occurs when $e^{2x} > 0$, therefore $2x = \ln\frac{4}{3} \Longrightarrow x = \frac{1}{2}\ln\frac{4}{3}$.

Exercise 3:

1. $\forall x \in \left[0, \frac{\pi}{2}\right],\ x - \frac{x^3}{6} \leq \sin x \leq x - \frac{x^3}{6} - \frac{x^5}{125}$. Using Taylor-Lagrange formula for $f(x) = \sin x,\ x_0 = 0$, and $n = 3;\ n = 5$ we get

   - For $n = 3 \Longrightarrow \sin x = x - \frac{x^3}{6} + \frac{\sin c}{125}x^4$ where $x \in \left[0, \frac{\pi}{2}\right]$, and c is a real number between x and 0.
   - For $n = 5 \Longrightarrow \sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + \frac{\sin k}{720}x^6$ where $x \in \left[0, \frac{\pi}{2}\right]$, and k is a real number between x and 0.

   Since $\forall x, c, k \in \left[0, \frac{\pi}{2}\right]:\ -\frac{\sin k}{720}x^6 \leq 0$, and $\frac{\sin c}{120}x^4 \geq 0$. We get

$$\forall x \in \left[0, \frac{\pi}{2}\right],\ x - \frac{x^3}{6} \leq \sin x \leq x - \frac{x^3}{6} - \frac{x^5}{125}.$$

Exercise 4:

1. Let us compute $LD_3(0)$ for:

   - $x \longrightarrow \sin\left(\frac{1}{1-x} - 1\right)$. Since

$$\begin{aligned} \frac{1}{1-x} - 1 &= -x + x^2 - x^3 + o(x^3) \\ \sin x &= x - \frac{x^3}{6} + o(x^3) \end{aligned}$$

   Then, we compose, only considering terms of order 3 or less:

$$\begin{aligned} \sin\left(\frac{1}{1-x} - 1\right) &= -x + x^2 - x^3 - \frac{1}{6}(-x)^3 + o(x^3) \\ &= -x + x^2 - \frac{5x^3}{6} + o(x^3). \end{aligned}$$

   - $x \longrightarrow \frac{1}{(1-x)^2}$. Since $\frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)'$, and

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + o(x^4).$$

Derive the $LD_4(0)$ of $\dfrac{1}{1-x}$, we obtain $LD_3(0)$ for $\dfrac{1}{(1-x)^2}$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + o(x^3).$$

2. We show that $\forall x > 0,\ x - \dfrac{x^2}{2} < \ln(x+1) < x$. Maclaurin's formula is

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}.$$

- We show that $x - \dfrac{x^2}{2} < \ln(x+1)$. We have

$$\begin{aligned}
f(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^3(\theta x)}{3!}x^3,\ 0 < \theta < 1\\
f(x) &= \ln(1+x) \Longrightarrow f(0) = 0\\
f'(x) &= \frac{1}{1+x} \Longrightarrow f'(0) = 1\\
f''(x) &= -\frac{1}{(1+x)^2} \Longrightarrow f''(0) = -1\\
f^{(3)}(x) &= \frac{2}{(1+x)^3} \Longrightarrow f^{(3)}(\theta x) = \frac{2}{(1+\theta x)^3},\ 0 < \theta < 1.
\end{aligned}$$

So, $f(x) = \ln(1+x) = x - \dfrac{1}{2}x^2 + \dfrac{1}{3!}\dfrac{2}{(1+\theta x)^3}x^3$. We have $\dfrac{1}{3(1+\theta x)^3}x^3 > 0$ (because $x > 0$, and $0 < \theta < 1$), therefore, $x - \dfrac{1}{2}x^2 + \dfrac{1}{3(1+\theta x)^3}x^3 > x - \dfrac{x^2}{2} \Longrightarrow \ln(1+x) > x - \dfrac{x^2}{2}$.

- Now, we show that $\ln(x+1) < x$. Let

$$f(x) = \ln(1+x) = f(0) + f'(0)x + \frac{f''(\theta x)}{2!}x^2,\ 0 < \theta < 1.$$

We have:

$$f(0) = 0,\ f'(0) = 1,\ f''(x) = -\frac{1}{(1+x)^2} \Longrightarrow f''(\theta x) = -\frac{1}{(1+\theta x)^2}.$$

Then, $f(x) = \ln(1+x) = x - \dfrac{1}{2(1+\theta x)^2}x^2,\ 0 < \theta < 1$.

We have $-\dfrac{1}{2(1+\theta x)^2}x^2 < 0$, (because $x > 0$, and $0 < \theta < 1$), so $x - \dfrac{1}{2(1+\theta x)^2}x^2 < x \Longrightarrow \ln(1+x) < x$. Therfore

$$x - \frac{x^2}{2} < \ln(1+x) < x, \ \forall x > 0.$$

Exercise 5:

1. Let $f_a(x) = \arctan(u(x))$ such that $u(x) = \dfrac{x+a}{1-ax}$. $f'_a(x) = \dfrac{u'(x)}{1+u^2(x)}$, thus

$$u'(x) = \frac{1 \times (1-ax) - (x+a) \times (-a)}{(1-ax)^2} = \frac{1+a^2}{(1-ax)^2}$$

$$1 + (u(x))^2 = 1 + \left(\frac{x+a}{1-ax}\right)^2 = \frac{(1-ax)^2 + (x+a)^2}{(1-ax)^2}$$

$$= \frac{1 + a^2x^2 + x^2 + a^2}{(1-ax)^2} = \frac{(1+a^2)(1+x^2)}{(1-ax)^2}.$$

Now,

$$f'_a(x) = \frac{\dfrac{1+a^2}{(1-ax)^2}}{\dfrac{(1+a^2)(1+x^2)}{(1-ax)^2}} = \frac{1}{1+x^2}.$$

$$f'_a(x) = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2(n-1)} + o(x^{2n-1}).$$

2. We have

$$\begin{aligned} f_a(x) &= f_a(0) + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n-1}}{2n-1} + o(x^{2n}) \\ &= \arctan(a) + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n-1}}{2n-1} + o(x^{2n}). \end{aligned}$$

Exercise 6:

1. The Maclaurin formula of order n is

$$f(x) = e^x = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k + \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}, \ 0 < \theta < 1$$

We have $f^{(n)}(x) = e^x \Longrightarrow f^{(n)}(0) = 1$, and $f^{(n+1)}(\theta x) = e^{\theta x}$. Then

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} x^k + \frac{e^{\theta x}}{(n+1)!} x^{n+1}, \ 0 < \theta < 1.$$

2. We show that

$$1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} < e < 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{e}{(n+1)!}$$

We have for $x = 1$ in the previous formula:

$$\begin{aligned} f(1) = e &= 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{e^\theta}{(n+1)!}, \ 0 < \theta < 1 \\ &= \sum_{k=0}^{n} \frac{1}{k!} + \frac{e^\theta}{(n+1)!}, \ 0 < \theta < 1 \end{aligned}$$

We have

$$0 < \theta < 1 \Longrightarrow 1 < e^\theta < e \Longrightarrow \frac{1}{(n+1)!} < \frac{e^\theta}{(n+1)!} < \frac{e}{(n+1)!}$$

therefore

$$\begin{aligned} \frac{1}{(n+1)!} &< e - \sum_{k=0}^{n} \frac{1}{k!} < \frac{e}{(n+1)!} \\ 0 &< e - \sum_{k=0}^{n} \frac{1}{k!} < \frac{e}{(n+1)!} \\ \sum_{k=0}^{n} \frac{1}{k!} &< e < \frac{e}{(n+1)!} + \sum_{k=0}^{n} \frac{1}{k!} \end{aligned}$$

so

$$1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} < e < 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{e}{(n+1)!}.$$

3. Deduce the limit of the sequence $u_n = \sum_{k=0}^{n} \frac{1}{k!}$. We have

$$\begin{aligned} 0 &< e - \sum_{k=0}^{n} \frac{1}{k!} < \frac{e}{(n+1)!} \\ 0 &< \lim_{n \to +\infty} (e - \sum_{k=0}^{n} \frac{1}{k!}) < \lim_{n \to +\infty} \frac{e}{(n+1)!} = 0 \end{aligned}$$

then $\lim\limits_{n\to+\infty}\left(e-\sum\limits_{k=0}^{n}\frac{1}{k!}\right)=0\Longrightarrow\lim\limits_{n\to+\infty}\sum\limits_{k=0}^{n}\frac{1}{k!}=e$. Thus

$$\lim_{n\to+\infty}u_n=\lim_{n\to+\infty}\sum_{k=0}^{n}\frac{1}{k!}=e.$$

Exercise 7:

Let $u_n=\prod\limits_{k=1}^{n}\left(1+\frac{k}{n^2}\right)$, $\forall n\in\mathbb{N}^*$. Assume that

$$v_n=\ln u_n\Longrightarrow v_n=\ln\prod_{k=1}^{n}\left(1+\frac{k}{n^2}\right)=\sum_{k=1}^{n}\ln\left(1+\frac{k}{n^2}\right)$$

according to exercise 3

$$\begin{aligned}&\frac{k}{n^2}-\frac{k^2}{2n^4}<\ln\left(1+\frac{k}{n^2}\right)<\frac{k}{n^2}\\ \Longrightarrow\quad&\sum_{k=1}^{n}\frac{k}{n^2}-\sum_{k=1}^{n}\frac{k^2}{2n^4}<\sum_{k=1}^{n}\ln\left(1+\frac{k}{n^2}\right)<\sum_{k=1}^{n}\frac{k}{n^2}\end{aligned}$$

$$\Longrightarrow\quad\frac{1}{n^2}\sum_{k=1}^{n}k-\frac{1}{2n^4}\sum_{k=1}^{n}k^2<\sum_{k=1}^{n}\ln\left(1+\frac{k}{n^2}\right)<\frac{1}{n^2}\sum_{k=1}^{n}k$$

We have: $\sum\limits_{k=1}^{n}k=\frac{n(n+1)}{2}$, and $k^2\le n^2\Longrightarrow\sum\limits_{k=1}^{n}k^2\le\sum\limits_{k=1}^{n}n^2=n.n^2$, so

$$\frac{1}{2n^4}\sum_{k=1}^{n}k^2\le\frac{1}{2n^4}n.n^2=\frac{1}{2n}\Longrightarrow-\sum_{k=1}^{n}\frac{k^2}{2n^4}\ge-\frac{1}{2n}.$$

Therefore,

$$\begin{aligned}\frac{1}{n^2}\frac{n(n+1)}{2}-\frac{1}{2n}&<\sum_{k=1}^{n}\ln(1+\frac{k}{n^2})<\frac{1}{n^2}.\frac{n(n+1)}{2}\\ \lim_{n\to+\infty}\frac{1}{n^2}\frac{n(n+1)}{2}-\frac{1}{2n}&<\lim_{n\to+\infty}\sum_{k=1}^{n}\ln(1+\frac{k}{n^2})<\lim_{n\to+\infty}\frac{1}{n^2}.\frac{n(n+1)}{2}\end{aligned}$$

Thus,

$$\begin{aligned}\lim_{n\to+\infty}\sum_{k=1}^{n}\ln(1+\frac{k}{n^2})=\frac{1}{2}&\Longrightarrow\lim_{n\to+\infty}v_n=\frac{1}{2}\\ &\Longrightarrow\lim_{n\to+\infty}u_n=\lim_{n\to+\infty}e^{v_n}=e^{\frac{1}{2}}.\end{aligned}$$

Exercise 8:

1. $f(x)=x(ch\,x)^{\frac{1}{x}}=xe^{\ln(ch\,x)}$, $x_0=0$, $n=3$. We have $ch\,x=1+\frac{x^2}{2}+o(x^2)\Longrightarrow$ $\ln(ch\,x)=\ln(1+\frac{x^2}{2}+o(x^2))$. We pose $y=\frac{x^2}{2}+o(x^2)$, if $x\longrightarrow0$, then $y\longrightarrow0$, then

$$\begin{array}{rcl} \ln(1+y) & = & y - \frac{1}{2}y^2 + o(y^2) \\ \Longrightarrow \ln(1+\frac{x^2}{2}+o(x^2)) & = & \ln(ch\ x) = \frac{x^2}{2} - \frac{1}{8}x^4 + o(x^4) \\ \Longrightarrow \frac{1}{x}\ln(ch\ x) & = & \frac{x}{2} - \frac{1}{8}x^3 + o(x^3) \\ \Longrightarrow e^{\frac{1}{x}\ln(ch\ x)} & = & e^{\frac{x}{2}-\frac{1}{8}x^3+o(x^3)}. \end{array}$$

Now, we pose $z = \frac{x}{2} - \frac{1}{8}x^3 + o(x^3),\ x \longrightarrow 0 \Longrightarrow z \longrightarrow 0$, so

$$\begin{array}{rcl} e^z & = & 1 + z + z^2 + \frac{1}{3!}z^3 + o(z^3) \\ \Longrightarrow e^{\frac{1}{x}\ln(ch\ x)} & = & 1 + \frac{x}{2} - \frac{1}{8}x^3 + \frac{1}{2}\left(\frac{x}{2} - \frac{x^3}{8}\right)^2 + \frac{1}{6}\left(\frac{x}{2} - \frac{x^3}{8}\right)^3 + o(x^3) \\ & = & 1 + \frac{x}{2} + \frac{x^2}{8} - \frac{5x^3}{48} + o(x^3) \\ \Longrightarrow f(x) = xe^{\frac{1}{x}\ln(ch\ x)} & = & x\left(1 + \frac{x}{2} + \frac{x^2}{8} - \frac{5x^3}{48} + o(x^3)\right) \\ & = & x + \frac{x^2}{2} + \frac{x^3}{8} + o(x^3). \end{array}$$

2. $f(x) = \ln(1+\sin x),\ x_0 = 0,\ n = 3$. We have $\sin x = x - \frac{1}{3!}x^3 + o(x^3) \Longrightarrow$ $f(x) = (1 + x - \frac{1}{3!}x^3 + o(x^3))$. We pose $y = x - \frac{1}{3!}x^3 + o(x^3)$, then $f(x) = \ln(1+y) = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 + o(y^3)$, therefore

$$\begin{array}{rcl} f(x) & = & \ln(1+\sin x) = x - \frac{1}{6}x^3 - \frac{1}{2}\left(x - \frac{1}{6}x^3\right)^2 + \frac{1}{3}\left(x - \frac{1}{6}x^3\right)^3 + o(x^3) \\ & = & x - \frac{x^3}{6} - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3) \\ & = & x - \frac{x^2}{2} + \frac{x^3}{6} + o(x^3). \end{array}$$

3. $f(x) = \tan x,\ x_0 = 0,\ n = 5$. We have $\tan x = \frac{\sin x}{\cos x}$ with $\cos(0) = 1 \neq 0$, and $\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5),\ \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5)$, so

$$
\begin{array}{r|l}
x-\frac{x^3}{6}+\frac{x^5}{120}+o(x^5) & 1-\frac{x^2}{2}+\frac{x^4}{24}+o(x^5) \\
x-\frac{x^3}{2}+\frac{x^5}{24}+o(x^5) & \\
\hline
\frac{x^3}{3}-\frac{x^5}{30}+o(x^5) & x+\frac{x^3}{3}+\frac{2x^5}{15} \\
\frac{x^3}{3}-\frac{x^5}{6}+o(x^5) & \\
\hline
\frac{2x^5}{15}+o(x^5) & \\
\frac{2x^5}{15}+o(x^5) & \\
\hline
o(x^5) &
\end{array}
$$

Then $\tan x = x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + o(x^5)$.

4. $f(x) = \dfrac{\ln(1+x)}{1+x}$, $x_0 = 0$, $n = 3$. We have

$$
\begin{aligned}
\frac{\ln(1+x)}{1+x} &= \ln(1+x) \times \frac{1}{1+x} = \left(x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)\right)(1 - x + x^2 - x^3 + o(x^3)) \\
&= x + \left(-1 - \frac{1}{2}\right)x^2 + \left(1 + \frac{1}{2} + \frac{1}{3}\right)x^3 + o(x^3) \\
&= x - \frac{3x^2}{2} + \frac{11x^3}{6} + o(x^3).
\end{aligned}
$$

5. $f(x) = e^{3x}\sin(2x)$, $x_0 = 0$, $n = 4$. We have

$$
\begin{aligned}
f(x) &= e^{3x}\sin(2x) = \left(1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!} + o(x^4)\right)\left(2x - \frac{(2x)^3}{3!} + o(x^4)\right) \\
&= \left(1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + \frac{27x^4}{8} + o(x^4)\right)\left(2x - \frac{4x^3}{3} + o(x^4)\right) \\
&= 2x + 6x^2 + \left(\frac{9}{2} \times 2 - \frac{4}{3}\right)x^3 + \left(3 \times (-\frac{4}{3}) + \frac{9}{2} \times 2\right)x^4 + o(x^4) \\
&= 2x + 6x^2 + \frac{23}{3}x^3 + 5x^4 + o(x^4).
\end{aligned}
$$

6. $f(x) = e^{\sqrt{x}}$, $x_0 = 1$, $n = 3$. We pose $y = x - 1 \Rightarrow x = y + 1$, $(If\ x \to 1 \Longrightarrow$ $y \to 0)$, so $e^{\sqrt{x}} = e^{\sqrt{y+1}}$. We have

$$\begin{aligned}
\Rightarrow \sqrt{y+1} &= 1+\frac{y}{2}-\frac{y^2}{8}+\frac{y^3}{16}+o(y^3)\\
\Rightarrow e^{\sqrt{y+1}} &= e^{1+\frac{y}{2}-\frac{y^2}{8}+\frac{y^3}{16}+o(y^2)} = e.e^{\overbrace{\frac{y}{2}-\frac{y^2}{8}+\frac{y^3}{16}+o(y^3)}_{z}}, \ (if \ y\to 0 \Rightarrow z\to 0)\\
\Rightarrow e^{z} &= 1+z+\frac{z^2}{2}+\frac{z^3}{6}+o(z^3)\\
\Rightarrow e^{\sqrt{y+1}} = e.e^{z} &= e.(1+z+\frac{z^2}{2}+\frac{z^3}{6}+o(z^3))\\
&= e.\left[1+\frac{y}{2}-\frac{y^2}{8}+\frac{y^3}{16}+\frac{1}{2}\left(1+\frac{y}{2}-\frac{y^2}{8}+\frac{y^3}{16}\right)^2+\frac{1}{6}\left(1+\frac{y}{2}-\frac{y^2}{8}+\frac{y^3}{16}\right.\right.\\
&= e.\left[1+\frac{y}{2}-\frac{y^2}{8}+\frac{y^3}{16}+\frac{y^2}{8}-\frac{y^3}{32}-\frac{y^3}{32}+\frac{y^3}{48}+o(y^3)\right]\\
&= e.\left[1+\frac{y}{2}+\frac{y^3}{48}+o(y^3)\right]
\end{aligned}$$

so $f(x) = e^{\sqrt{x}} = e.\left[1+\frac{x-1}{2}+\frac{(x-1)^3}{48}+o((x-1)^3)\right] = e+\frac{e(x-1)}{2}+\frac{e(x-1)^3}{48}+o((x-1)^3)$.

7. $f(x)=\frac{x^3+2}{x-1}, \ x_0=+\infty, \ n=3$. We pose $y=\frac{1}{x}\Rightarrow x=\frac{1}{y}$, if $x\to\infty, \ y\to 0$.

$$\begin{aligned}
f(x) &= \frac{x^3+2}{x-1} = \frac{(\frac{1}{y})^3+2}{\frac{1}{y}-1} = \frac{\frac{1+2y^3}{y^3}}{\frac{1-y}{y}} = \frac{1}{y^2}\left(\frac{1+2y^3}{1-y}\right)\\
&= \frac{1}{y^2}\left(1+2y^3\right)\left(\frac{1}{1-y}\right)\\
&= \frac{1}{y^2}\left(1+2y^3\right)\left(1+y+y^2+y^3+y^4+y^5+o(y^5)\right)\\
&= \frac{1}{y^2}\left(1+y+y^2+y^3+y^4+y^5+2y^3+2y^4+2y^5+o(y^5)\right)\\
&= \frac{1}{y^2}+\frac{1}{y}+1+3y+3y^2+3y^3+o(y^3)\\
&= x^2+x+1+\frac{3}{x}+\frac{3}{x^2}+\frac{3}{x^3}+o\left(\left(\frac{1}{x}\right)^3\right).
\end{aligned}$$

Exercise 9:

1.
$$\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{\ln(1+x)}-\frac{1}{x}\right) &= \lim_{x\to 0}\left(\frac{1}{x-\frac{x^2}{2}+o(x^2)}-\frac{1}{x}\right)\\
&= \lim_{x\to 0}\frac{1}{x}\left(\frac{1}{1-\frac{x}{2}}-1\right)=\lim_{x\to 0}\frac{1}{x}\left(1+\frac{x}{2}+o(x^2)-1\right)\\
&= \lim_{x\to 0}\left(\frac{1}{2}-\underbrace{\frac{o(x)}{x}}_{0}\right)=\frac{1}{2}.
\end{aligned}$$

2.
$$\begin{aligned}
\lim_{x\to 0}\frac{e^{3x}\sin 3x}{sh(-2x)} &= \lim_{x\to 0}\frac{(1+3x+o(x))(3x+o(x))}{-2x+o(x)}\\
&= \lim_{x\to 0}\frac{3x+o(x)}{-2x+o(x)}=\lim_{x\to 0}\left(\frac{-3}{2}+\frac{\frac{o(x)}{x}}{\frac{o(x)}{x}}\right)\\
&= -\frac{3}{2}.
\end{aligned}$$

3. $\lim\limits_{x\to+\infty}\left(1+\frac{1}{x}\right)^{x}=\lim\limits_{y\to 0}(1+y)^{\frac{1}{y}}$, we pose $y=\frac{1}{x}$, so

$$\begin{aligned}
\lim_{y\to 0}(1+y)^{\frac{1}{y}} &= \lim_{y\to 0}e^{\frac{1}{y}(\ln(1+y))}=\lim_{y\to 0}e^{\frac{1}{y}(y-\frac{y^2}{2}+o(y^2))}\\
&= \lim_{y\to 0}e^{1-\frac{y}{2}+o(y)}=\lim_{y\to 0}e.e^{-\frac{y}{2}+o(y)}\\
&= e.
\end{aligned}$$

4.
$$\begin{aligned}
\lim_{x\to 0}\frac{x-\sin x}{x^3} &= \lim_{x\to 0}\frac{x\left(x-\frac{x^3}{6}+\frac{x^5}{120}+o(x^5)\right)}{x^3}\\
&= \lim_{x\to 0}-\frac{1}{6}-\frac{x^2}{120}+o(x^2)\\
&= -\frac{1}{6}.
\end{aligned}$$

5. $\lim\limits_{x\to+\infty}x^2\left(e^{\frac{1}{x}}-e^{\frac{1}{1+x}}\right)$, First, we put $t=\frac{1}{x}$, and then $t\to 0$ when $x\to+\infty$ and

$$x^2\left(e^{\frac{1}{x}}-e^{\frac{1}{1+x}}\right)=\frac{1}{t^2}\left(e^t-e^{\frac{t}{1+t}}\right).$$

Since, $\frac{t}{1+t}=t-t^2+o(t^2)$, and $e^t=1+t+\frac{t^2}{2}+o(t^2)$ we obtained,

$$e^{\frac{t}{1+t}}=1+t-t^2+\frac{(t-t^2)^2}{2}+o(t^2)=1+t-\frac{t^2}{2}+o(t^2)$$

Hence,

$$\begin{aligned}\frac{1}{t^2}\left(e^t - e^{\frac{t}{1+t}}\right) &= \frac{1}{t^2}\left(\left(1+t+\frac{t^2}{2}\right) - \left(1+t-\frac{t^2}{2}\right) + o(t^2)\right) \\ &= \frac{1}{t^2}\left(t^2 + o(t^2)\right) \\ &= 1 + o(1)\end{aligned}$$

Consequently,

$$\begin{aligned}\lim_{x\to+\infty} x^2\left(e^{\frac{1}{x}} - e^{\frac{1}{1+x}}\right) &= \lim_{t\to 0}\frac{1}{t^2}\left(e^t - e^{\frac{t}{1+t}}\right) \\ &= \lim_{t\to 0}(1+o(1)) \\ &= 1.\end{aligned}$$