Contents

Chapter 1

Integral Calculus

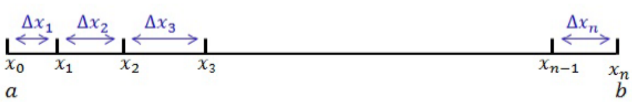
Calculus is built on two operations ”differentiation” and ”integration”. Integration, at its most basic level, allows us to analyze the area under a curve. Of course, its application and importance extend far beyond areas and it plays a central role in solving differential equations.

1.1 The Definite Integrals

1.1.1 Partition

Definition 1.1.1. *Let f be a function defined and bounded on $[a, b]$. A set $P = (x_0, x_1, x_2, \cdots, x_n)$ is called a partition of a closed interval $[a, b]$ if for any positive integer n,*

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$



Notes:

- The division of the interval $[a, b]$ by the partition P generates n subintervals:

$$[x_0, x_1], [x_1, x_2], [x_2, x_3], \cdots, [x_{n-1}, x_n]$$

.

- The length of each subinterval $[x_{k-1}, x_k]$ is $\Delta x_k = x_k - x_{k-1}$.
- The union of subintervals gives the whole interval $[a, b]$.
- The strictly positive real $\delta(P) = \max(x_k - x_{k-1})$ is the maximum length of a subintervals.

1.1.2 Definition of Darboux Sums

Definition 1.1.2. *Let f be a bounded function defined on a closed bounded interval $[a, b]$, and P is a partition of $[a, b]$. Let m_k and M_k be the infimum and the supremum of f over $[x_{k-1}, x_k]$*

$$m_k = \inf_{x \in [x_{k-1}, x_k]} f(x)$$

$$M_k = \sup_{x \in [x_{k-1}, x_k]} f(x)$$

Consider the lower and upper Darboux sums of f corresponding to a partition P:

$$L(f, P) = \sum_{k=1}^{n} m_k (x_k - x_{k-1})$$

$$U(f, P) = \sum_{k=1}^{n} M_k (x_k - x_{k-1})$$

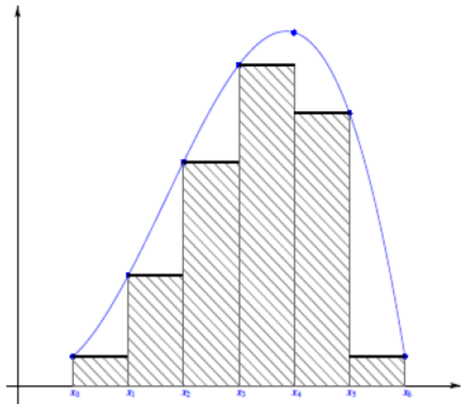
Definition 1.1.3. ***(Upper and Lower Integrals)***

Suppose $f : [a, b] \longrightarrow \mathbb{R}$ is bounded. Then we define the lower integral of f on $[a, b]$ as:

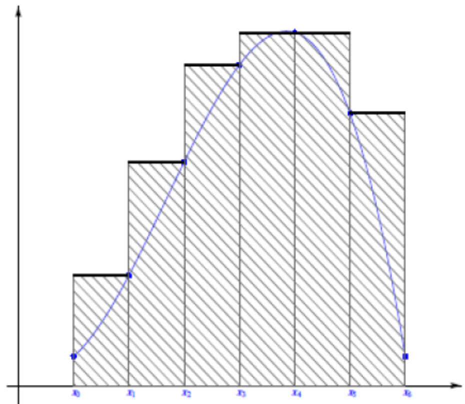
$$\int_{*a}^{b} f(x)dx = \sup(L), \text{ where } L = \{L(f, P) | P \text{ is a partition of } [a, b]\},$$

and the upper integral of f on $[a, b]$ as:

$$\int_{a}^{*b} f(x)dx = \inf(U), \text{ where } U = \{U(f, P) | P \text{ is a partition of } [a, b]\}.$$



Lower Darboux sum



Upper Darboux sum

1.1.3 Properties of Darboux Sums

- For all partition $P \subset [a, b]$, we have $U(f, P) \geq L(f, P)$.
- If $P \subset P'$, then $L(f, P) \leq L(f, P')$,and $U(f, P) \geq U(f, P')$.
- $\forall P,\ P' \subset [a, b],\ L(f, P) \leq \int_{*a}^{b} f(x)\ dx \leq \int_{a}^{b} f(x)\ dx \leq \int_{a}^{*b} f(x)\ dx \leq U(f, P').$

1.1.4 Integrable functions, Riemann integrals

Definition 1.1.4. *Suppose $f : [a, b] \longrightarrow \mathbb{R}$ is bounded. We say that f is integrable in the sense of Riemann if:*

$$\int_{*a}^{b} f(x)\ dx = \int_{a}^{*b} f(x)\ dx.$$

The value of the lower integral and the upper integral is called the Riemann integral of f over $[a, b]$ and is denoted by $\int_{a}^{b} f(x)\ dx$.

Theorem 1.1.5. *A function f is Riemann integrable on $[a, b]$ if and only if for every $\varepsilon > 0$ there is a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$.*

Theorem 1.1.6.
- *Any bounded and monotonic function on $[a, b]$ is integrable on $[a, b]$.*
- *Any continuous function on $[a, b]$ is integrable on $[a, b]$.*

1.1.5 Riemann Sums

Let $f : [a, b] \longrightarrow \mathbb{R}$ be a bounded function, and $P = x_0, x_1, \cdots, x_n$ a partition of $[a, b]$.

Definition 1.1.7. *The sum* $\sigma(f, P) = \sum_{k=1}^{n} f(\zeta_k)(x_k - x_{k-1})$ *where* $\zeta_k \in [x_{k-1}, x_k],\ \ k = \overline{1, n}$ *is said to be the Riemann sum of* f *corresponding to* P *and the system of points* $\zeta = (\zeta_1, \zeta_2, \cdots, \zeta_n)$.

Definition 1.1.8. *We say that the number* A *is the limit of* $\sigma(f, P, \zeta)$ *when* $\delta(P)$ *tends to 0 and we write* $A = \lim\limits_{\delta(P)\to 0} \sigma(f, P, \zeta)$ *if:*

$$\forall \varepsilon > 0,\ \exists\ \tau > 0,\ \forall \sigma(f, P, \zeta),\ \delta(P) < \tau \Longrightarrow |\sigma(f, P, \zeta) - A| < \varepsilon.$$

Theorem 1.1.9. *If* $f : [a, b] \longrightarrow \mathbb{R}$ *is integrable then:*

$$A = \lim_{\delta(P)\to 0} \sigma(f, P, \zeta) = \int_a^b f(x)\ dx.$$

Particular case:

$$\int_a^b f(x)\ dx = \lim_{n\to+\infty} \frac{b-a}{n} \textstyle\sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right).$$

Example 1.1.10. *Calculate:* $I = \lim\limits_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} (\ln(n+k) - \ln(n))$.

$$\begin{aligned} I &= \lim_{n\to+\infty} \frac{1}{n} \textstyle\sum_{k=1}^{n} (\ln(n+k) - \ln(n)) \\ &= \lim_{n\to+\infty} \frac{1}{n} \textstyle\sum_{k=1}^{n} \ln\left(\frac{n+k}{n}\right) \\ &= \lim_{n\to+\infty} \frac{1}{n} \textstyle\sum_{k=1}^{n} \ln\left(1 + \frac{k}{n}\right) \end{aligned}$$

Therefore, $f(x) = \ln(x),\ a = 1,$ *and* $b = 2$.

$$\begin{pmatrix} b - a = 1 \\ \textit{and} \\ 1 + \frac{k}{n} = a + k\frac{b-a}{n} \end{pmatrix} \Longrightarrow \begin{pmatrix} b = a + 1 \\ 1 + \frac{k}{n} = a + k\frac{(a+1-a)}{n} \end{pmatrix} \Longrightarrow a = 1, \ \ \textit{and}\ \ b = 2$$

So

$$\lim_{n\to+\infty} \frac{1}{n} \textstyle\sum_{k=1}^{n} (\ln(n+k) - \ln(k)) = \int_1^2 \ln(x)\ dx.$$

1.1.6 Properties of the Riemann integral

Let f and g be two bounded and integrable functions on $[a,b]$, $c \in [a,b]$ and $\alpha \in \mathbb{R}$

1. $\int_{\alpha}^{\alpha} f(x)\ dx = 0.$
2. $\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx.$
3. $\int_a^b (f(x)+g(x))\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx.$
4. $\int_a^b (\alpha f(x))\ dx = \alpha \int_a^b f(x)\ dx.$
5. $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx.$
6. $\left| \int_a^b f(x)\ dx \right| \le \int_a^b |f(x)|\ dx.$
7. $f(x) \le g(x),\ \forall x \in [a,b] \Longrightarrow \int_a^b f(x)\ dx\ \le \int_a^b g(x)\ dx.$
8. $f(x) \ge 0 \Longrightarrow \int_a^b f(x)\ dx \ge 0.$

   but if $f \ge 0$ and continuous on $[a,b]$ and if $\int_a^b f(x)\ dx = 0 \Longrightarrow f(x) = 0$ on $[a,b]$.

1.1.7 Mean Theorem

If f and g are two bounded and integrable functions on $[a,b]$. If $g \ge 0$ and if $m \le f(x) \le M$, $\forall\ m,\ M \in \mathbb{R}$ then:

$$m\int_a^b g(x)\ dx \le \int_a^b f(x).g(x)\ dx \le M\int_a^b g(x)\ dx.$$

Particular case:

- If $g = 1$, we obtain:

$$m(b-a) \le \int_a^b f(x)\ dx \le M(b-a).$$

- First formula for the average: If f is a continuous function on $[a, b]$ and if g is a bounded integrable function, of constant sign on $[a, b]$, then there exists an element $c \in [a, b]$ such that:

$$\int_a^b f(x).g(x)\ dx = f(c) \int_a^b g(x)\ dx.$$

If $g = 1$, we obtain the

- Second average formula: If a function f is continuous on $[a, b]$, there exists an element $c \in [a, b]$ such that:

$$\int_a^b f(x)\ dx = (b - a)f(c).$$

The number $f(c)$ is called the average value of f over $[a, b]$.

1.1.8 Cauchy-Schwarz Inequality

If f and g are two bounded and integrable functions on $[a, b]$, we have:

$$\left(\int_a^b f(x).g(x)\ dx \right)^2 \leq \int_a^b f^2(x)\ dx\ . \int_a^b g^2(x)\ dx.$$

1.1.9 Antiderivative of a continuous function

Definition 1.1.11. *Let $f : [a, b] \longrightarrow \mathbb{R}$ a function. We say that a differentiable function $F : [a, b] \longrightarrow \mathbb{R}$ is an antiderivative of f if: $\forall x \in [a, b],\ F'(x) = f(x)$*

Example 1.1.12. *1. $F(x) = \frac{1}{3}x^3 + 5x + 2$ is an antiderivative of $f(x) = x^2 + 5$, since $F'(x) = (\frac{1}{3}x^3 + 5x + 2)' = x^2 + 5$.*

2. e^x is an antiderivative of e^x, since $(e^x)' = e^x$.

Theorem 1.1.13. *Any continuous function $f : [a, b] \longrightarrow \mathbb{R}$ admits an antiderivative. The application $x \longrightarrow F(x) = \int_a^b f(t)\ dt$ is an antiderivative of f.*

1.1.10 General method of calculating the integral

1. **Integration by parts method:**

   Proposition 1.1.14. *Let f and g be two functions in class C^1 on an open interval $I = [a, b]$, we have:*

$$\int_a^b f'(x)g(x)\ dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)\ dx.$$

   Example 1.1.15. *Calculate:* $I = \int_0^1 \arctan(x)\ dx$

$$\begin{cases} f(x) & = & \arctan(x) \\ & and & \\ g'(x) & = & dx \end{cases} \Longrightarrow \begin{cases} f'(x) & = & \dfrac{1}{1+x^2} \\ & and & \\ g(x) & = & x \end{cases}$$

   we have:

$$\begin{aligned} I &= \int_0^1 \arctan(x)\ dx \\ &= x.\arctan(x)|_0^1 - \int_0^1 \frac{x}{1+x^2} dx \\ &= 1.\arctan(1) - 0 - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\ dx \\ &= \frac{\pi}{4} - \frac{1}{2}\ln\left(1+x^2\right)|_0^1 \\ &= \frac{\pi}{4} - \frac{1}{2}\ln\left(2\right). \end{aligned}$$

2. **Integration By Substitution (Change of Variables):**

   Proposition 1.1.16. *Let f be a continuous function on $[a, b]$ and $g \in C^1([a, b])$, then:*

$$\int_{g(a)}^{g(b)} f(x)\ dx = \int_a^b f(g(t))g'(t)\ dt.$$

   Example 1.1.17. *Calculate the integral* $I = \int_0^\pi \cos^4(x)\sin(x)\ dx$.

   We set $t = \cos(x) \Longrightarrow dt = -\sin(x)\ dx$, therefore

$$\begin{cases} if \quad x = 0 \Longrightarrow t = 1 \\ if \quad x = \pi \Longrightarrow t = -1 \end{cases}$$

so

$$\begin{aligned} I &= \int_0^{\pi} \cos^4(x)\sin(x)\ dx \\ &= \int_1^{-1} t^4(-dt) = \int_{-1}^{1} t^4\ dt \\ &= \frac{1}{5}t^5|_{-1}^{1} = \frac{2}{5} \end{aligned}$$

1.2 Indefinite integral

Definition 1.2.1. *The indefinite integral of $f(x) : I \longrightarrow \mathbb{R}$ is the collection of all antiderivatives of $f(x)$, denoted by $\int f(x)\ dx$.*

If we have: F is an antiderivative of f on I, then we write:

$$\int f(x)\ dx = F + c, \quad \forall c \in \mathbb{R}.$$

Example 1.2.2. *We have:*

- *The indefinite integral of the function $\frac{1}{x}$ on $]-\infty, 0[$ is defined by:*

$$\int \frac{dx}{x} = \ln(-x) + c_1, \quad c_1 \in \mathbb{R}$$

- *The indefinite integral of the function $\frac{1}{x}$ on $]0, +\infty[$ is defined by:*

$$\int \frac{dx}{x} = \ln(x) + c_2, \quad c_2 \in \mathbb{R}$$

$\Longrightarrow$ *The indefinite integral of the function $\frac{1}{x}$ on $\mathbb{R}^*$ is defined by:*

$$\int \frac{dx}{x} = \ln|x| + c, \quad c \in \mathbb{R}$$

1.2.1 Existence of the Indefinite Integral

Theorem 1.2.3. *Let $f : I \longrightarrow \mathbb{R}$ be a function defined on I, then we have the following implication:*

$$f \text{ is continuous on } I \Longrightarrow f \text{ admits an antiderivative on } I.$$

1.2.2 Change of variable and integration by parts in indefinite integrals

1. **Change of Variables:**

   Let I and J be two intervals and $h : J \longrightarrow I$ of class $C^1(J)$.

   Theorem 1.2.4. *Let $f \in C(I)$, then*

   $$F = \int f(x)\ dx = F \circ h = \int (f \circ h)h'(x).$$

   In other words, if F is a primitive of f then $F \circ h$ is a primitive of $(f \circ h)h'$.
   $\int f(x)\ dx = \int f(h(t))h'(t)\ dt, \quad x = h(t) \Longrightarrow dx = h'(t)dt$. *Indeed the change of variable* $x = h(t)$.

   Example 1.2.5. *Calculate:* $I = \int \cos^3(t)\ dt$.
   $\int \cos^3(t)\ dt = f(h(t))h'(t)dt$, *so*

   $$\cos^3(t)\ dt = \cos^2(t)\cos(t)\ dt = \cos^2(t)(\sin(t))'dt = (1-\sin^2(t))(\sin(t))'dt.$$

   We set $x = \sin(t)$, we obtain

   $$\begin{aligned} \cos^2(t)\cos(t)dt &= (1-x^2)\ dx \\ \int \cos^3(t)\ dt &= \int (1-x^2)\ dx = x - \frac{x^3}{3} + c = \sin(t) - \frac{1}{3}\sin^3(t) + c. \end{aligned}$$

2. **Integration by parts:**

   Theorem 1.2.6. *Let $u,\ v : I \longrightarrow \mathbb{R}$, $u,\ v \in C^1(I)$ then:*

   $$\int u'(x)v(x)\ dx = u(x)v(x) - \int u(x)v'(x)\ dx$$

   Example 1.2.7. *(a) Compute* $\int xe^x\ dx$. *We set $u(x) = x$, and $v'(x) = e^x$, then we get:*

   $$\begin{aligned} \int xe^x\ dx &= xe^x - \int e^x\ dx \\ &= xe^x - e^x + c \\ &= e^x(x-1) + c. \end{aligned}$$

(b) Compute $\int x\ln(x)\,dx$. *We set* $u(x)=\ln x$, *and* $v'(x)=x$, *then we get:*

$$\begin{aligned}\int x\ln x\,dx &= \frac{1}{2}x^2\ln x-\frac{1}{2}\int x^2.\frac{1}{x}\,dx\\ &= \frac{1}{2}x^2\ln x-\frac{1}{4}x^2+c.\end{aligned}$$

1.2.3 Antiderivative of rational fractions

Any rational fraction can be uniquely decomposed into the sum of a polynomial and a finite number of simple rational fractions (simple elements) of the form $\frac{A}{(x-a)^k}$, $\frac{Mx+N}{((x-\alpha)^2+\beta^2)^k}, \cdots$. We are therefore brought back to looking for antiderivatives of the polynomial and each of the simple elements there is no difficulty with integral polynomial as well as the simple elements of the first kind $\frac{A}{(x-a)^k}$.

$$\int\frac{1}{(x-a)^k}dx = -\frac{1}{(k-1)(x-a)^{k-1}}+c,\quad k>1,\quad a\in\mathbb{R}$$

$$\int\frac{1}{(x-a)}dx = \ln|x-a|+c,\qquad a\in\mathbb{R}$$

The integration of simple elements of the second kind $\frac{Mx+N}{((x-\alpha)^2+\beta^2)^k}$

By applying the substitution $x=\alpha+\beta t$, we calculate the integrals:

$$I_k=\int\frac{t}{(1+t^2)^k}dt,\qquad J_k=\int\frac{1}{(1+t^2)^k}dt.$$

- The calculation of I_k is immediate in fact the change of variable $u=1+t^2$ then $I_k=\frac{1}{2}\int\frac{du}{u^k}$ hence
  - For $k=1$: $I_1=\frac{1}{2}\ln(1+t^2)+c.$
  - For $k>1$: $I_k=\frac{-1}{2(k-1)(1+t^2)^{k-1}}+c$
- For the calculation of J_k an integration by parts gives us:

$$\begin{cases} u' = dt \\ v = \dfrac{1}{(1+t^2)^k} \end{cases} \Longrightarrow \begin{cases} u = 1 \\ v' = -k\dfrac{2t(1+t^2)^{k-1}}{(1+t^2)^{2k}} = -k\dfrac{2t}{(1+t^2)^{k+1}} \end{cases}$$

so

$$\begin{aligned} J_k &= \int \frac{1}{(1+t^2)^k}\ dt \\ &= \frac{t}{(1+t^2)^k} + 2k\int \frac{t^2}{(1+t^2)^{k+1}}dt \\ &= \frac{t}{(1+t^2)^k} + 2k\int \frac{1}{(1+t^2)^k}dt - 2k\int \frac{1}{(1+t^2)^{k+1}}dt \end{aligned}$$

we have $J_{k+1} = \int \frac{1}{(1+t^2)^{k+1}}dt$, then

$$J_k = \frac{t}{(1+t^2)^k} + 2kJ_k - 2kJ_{k+1}$$

hence the recurrence relation:

$$2kJ_{k+1} = \frac{t}{(1+t^2)^k} + (2k-1)J_k.$$

The calculation comes down to that of:

$$J_1 = \int \frac{1}{1+t^2}dt = \arctan(t) + c.$$

Example 1.2.8. *Calculate* $I = \int \frac{x+3}{x^2-3x+2}\ dx$.

We have

$$\begin{aligned} \frac{x+3}{x^2-3x+2} &= \frac{x+3}{(x-2)(x-1)} = \frac{a}{(x-2)} + \frac{b}{(x-1)} \\ &= \frac{5}{(x-2)} - \frac{4}{(x-1)} \end{aligned}$$

so

$$\begin{aligned} I &= \int \frac{x+3}{x^2-3x+2}\ dx \\ &= \int \left(\frac{5}{x-2} - \frac{4}{x-1}\right) dx \\ &= \int \frac{5}{x-2}dx - \int \frac{4}{x-1}dx \\ &= 5\ln|x-2| - 4\ln|x-1| + c \end{aligned}$$

Example 1.2.9. *Calculate* $J = \int \frac{5x-1}{(x+2)^2(x^2-1)}dx$. *The decomposition into simple elements gives*

$$\begin{aligned}\frac{5x-1}{(x+2)^2(x^2+1)} &= \frac{a}{x+2} + \frac{b}{(x+2)^2} + \frac{c}{x-1} + \frac{d}{x+1} \\ &= \frac{1}{9}\left(\frac{-29}{x+2} - \frac{33}{(x+2)^2} + \frac{2}{x-1} + \frac{27}{x+1}\right)\end{aligned}$$

Now, we integrat the simple elements obtained after decomposition. From the property of indefinite integrals we have:

$$\begin{aligned}J &= \frac{1}{9}\int\left(\frac{-29}{x+2} - \frac{33}{(x+2)^2} + \frac{2}{x-1} + \frac{27}{x+1}\right)dx \\ &= \frac{1}{9}\int\frac{-29}{x+2}dx - \frac{1}{9}\int\frac{33}{(x+2)^2}dx + \frac{1}{9}\int\frac{2}{x-1}dx + \frac{1}{9}\int\frac{27}{x+1}dx \\ &= \frac{-29}{9}\ln|x+2| + \frac{11}{3}\frac{1}{x+2} + \frac{2}{9}\ln|x-1| + 3\ln|x+1| + c.\end{aligned}$$

Example 1.2.10. *Calculate* $L = \int \frac{x+1}{(x^2-2x+3)^2}dx$.

$$\begin{aligned}L &= \int\frac{x+1}{(x^2-2x+3)^2}dx \\ &= \frac{1}{2}\int\frac{2(x+1)}{(x^2-2x+3)^2}dx \\ &= \frac{1}{2}\int\frac{2x-2+4}{(x^2-2x+3)^2}dx \\ &= \frac{1}{2}\int\frac{2x-2}{(x^2-2x+3)^2}dx + 2\int\frac{1}{(x^2-2x+3)^2}dx\end{aligned}$$

we set $L_1 = \int \frac{2x-2}{(x^2-2x+3)^2}dx$, *and* $L_2 = \int \frac{1}{(x^2-2x+3)^2}dx$

$$L_1 = \int\frac{2x-2}{(x^2-2x+3)^2}dx = -\frac{1}{(x^2-2x+3)}.$$

Now calculate $L_2 = \displaystyle\int \frac{1}{(x^2-2x+3)^2}dx.$

We have $x^2-2x+3 = (x-1)^2+2 = 2\left(\frac{(x-1)^2}{2}+1\right) = 2\left(\left(\frac{x-1}{\sqrt{2}}\right)^2+1\right).$

we set $t = \frac{x-1}{\sqrt{2}} \Longrightarrow dt = \frac{1}{\sqrt{2}}dx \Longrightarrow dx = \sqrt{2}dt$ *hence:*

$$\begin{aligned} L_2 &= \int \frac{1}{(x^2-2x+3)^2}dx = \int \frac{1}{\left(2\left(\left(\frac{x-1}{\sqrt{2}}\right)^2+1\right)\right)^2}dx \\ &= \frac{\sqrt{2}}{4}\underbrace{\int \frac{1}{(t^2+1)^2}dt}_{J_2} \end{aligned}$$

From (I), *for* $k=1$

$$2kJ_{k+1} = \frac{t}{(1+t^2)^k} + (2k-1)J_k \Longrightarrow 2J_2 = \frac{t}{(1+t^2)} + J_1 \Longrightarrow J_2 = \frac{1}{2}\frac{t}{(1+t^2)} + \frac{1}{2}J_1$$

where $J_1 = \displaystyle\int \frac{1}{1+t^2}dt = \arctan t,$ *then*

$$\begin{aligned} L_2 &= \frac{\sqrt{2}}{4}\left(\frac{1}{2}\frac{t}{(1+t^2)} + \frac{1}{2}J_1\right) = \frac{\sqrt{2}}{8}\frac{t}{(1+t^2)} + \frac{\sqrt{2}}{8}J_1 \\ &= \frac{\sqrt{2}}{8}\frac{\frac{x-1}{\sqrt{2}}}{\left(1+\left(\frac{x-1}{\sqrt{2}}\right)^2\right)} + \frac{\sqrt{2}}{8}\arctan\left(\frac{x-1}{\sqrt{2}}\right). \end{aligned}$$

So

$$\begin{aligned} L &= -\frac{1}{2(x^2-2x+3)} + 2\left(\frac{x-1}{4\,(x^2-2x+3)} + \frac{\sqrt{2}}{8}\arctan\left(\frac{x-1}{\sqrt{2}}\right)\right) \\ &= -\frac{1}{2(x^2-2x+3)} + \frac{x-1}{2\,(x^2-2x+3)} + \frac{\sqrt{2}}{4}\arctan\left(\frac{x-1}{\sqrt{2}}\right) + c \\ &= \frac{x-2}{2(x^2-2x+3)} + \frac{\sqrt{2}}{4}\arctan\left(\frac{x-1}{\sqrt{2}}\right) + c. \end{aligned}$$

1.2.4 Integration of Trigonometric Functions

In the calculation of trigonometric function integrals, we distinguish the following cases:

- **Integrals of type** $\int R(\cos(x))\sin(x)dx$ **or** $\int R(\sin(x))\cos(x)dx$ **with** $R(x)$ **is a rational fraction:**

  - If we have: $I = \int R(\cos(x))\sin(x)dx$, we make a change of variable of the form

  $$t = \cos(x), \qquad \text{and} \qquad dt = -\sin(x)\ dx$$

  - If we have: $I = \int R(\sin(x))\cos(x)dx$, we make a change of variable of the form

  $$t = \sin(x), \qquad \text{and} \qquad dt = \cos(x)\ dx$$

  - If we have: $I = \int R(\tan(x))\frac{1}{\cos^2(x)}dx$, we make a change of variable of the form

  $$t = \tan(x), \qquad \text{and} \qquad dt = \frac{1}{\cos^2(x)}\ dx$$

  Example 1.2.11. *Compute* $I = \int \frac{\cos^3(x)}{\sin^2(x)}$. *We have* $\frac{\cos^3(x)}{\sin^2(x)} = \frac{1-\sin^2(x)}{\sin^2(x)}\cos(x) \Longrightarrow I$ *of type* $\int R(\sin(x))\cos(x)\ dx$. *So we perform the following change of variable:*

  $$\begin{aligned} t &= \sin(x) \Longrightarrow dt = \cos(x)dx \\ &\Longrightarrow I = \int \frac{1-t^2}{t^2}dt = \int \frac{1}{t^2}dt - \int dt \\ &\Longrightarrow I = -\frac{1}{t} - t + c, \quad c \in \mathbb{R} \end{aligned}$$

  Finally, if we replace t *by* $\sin(x)$ *we obtain:*

  $$I = -\frac{1}{\sin(x)} - \sin(x) + c, \quad c \in \mathbb{R}.$$

- **Integrals of type** $\int R(cos(x), sin(x))dx$**:**

  The following change of variable can be used for integrals of this kind. We put

  $$t = \tan\left(\frac{x}{2}\right), \quad \cos(x) = \frac{1-t^2}{1+t^2}, \quad \sin(x) = \frac{2t}{1+t^2}, \quad dx = \frac{2dt}{1+t^2}.$$

By replacing the expressions dx, $\cos(x)$ and $\sin(x)$ in the integral, we obtain an integral of type: $\int R(t)dt$ where $R(t)$ is a rational fraction. To return to the variable x in the result, replace t by $\tan\left(\frac{x}{2}\right)$.

Example 1.2.12. *Compute* $I = \int \frac{1}{1-\cos(x)}dx$. *We put:*

$$t = \tan\left(\frac{x}{2}\right), \qquad \cos(x) = \frac{1-t^2}{1+t^2}, \qquad dx = \frac{2dt}{1+t^2}$$

Substituting these values into I*, we get:*

$$I = \int \frac{1}{t^2}dt = -\frac{1}{t} + c, \quad c \in \mathbb{R}$$

Finally, we replace t *by* $\tan\left(\frac{x}{2}\right)$ *we find:*

$$I = -\frac{1}{\tan\left(\frac{x}{2}\right)} + c, \quad c \in \mathbb{R}$$

1.2.5 Integration of rational functions in e^x

This type of integration can always be reduced to a rational fraction by setting $t = e^x$.

Example 1.2.13. *Compute* $I = \int \frac{1}{1+e^x}dx$.

Let

$$t = e^x \Longrightarrow dt = e^x\ dx \Longrightarrow dx = \frac{dt}{e^x} = \frac{dt}{t}.$$

so

$$\begin{aligned}
I &= \int \frac{1}{1+e^x}dx = \int \frac{1}{1+t}\frac{dt}{t} \\
&= \int \left(\frac{a}{1+t} + \frac{b}{t}\right)dt, \qquad (a=-1, \quad b=1) \\
&= \int \frac{-1}{1+t}dt + \int \frac{1}{t}dt \\
&= -\ln|1_t| + \ln|t| + c, \quad c \in \mathbb{R} \\
&= \ln\left|\frac{t}{1+t}\right| + c, \quad c \in \mathbb{R} \\
&= \ln\left|\frac{e^x}{1+e^x}\right| + c, \quad c \in \mathbb{R}
\end{aligned}$$