Chapter 3

Usual Probability Lows

A probability distribution is a mathematical function that theoretically describes a random experiment. Probability distributions are essential in biology for quantifying and predicting variability in various biological processes. They allow biologists to analyze data, formulate hypotheses, and make decisions. These mathematical laws thus contribute to a better understanding of random phenomena in the living world.

Contents

3.1 Random Variable

Definition 3.1.1. *A random variable X is an application from the sample space* Ω *to* $\mathbb{R}$*:*

$$\begin{aligned} X: \ \Omega \rightarrow & \quad \mathbb{R} \\ w \quad \mapsto & \quad X(w) \end{aligned}$$

There are two types of random variables:

3.1.1 Discrete Random Variable

Definition 3.1.2. *A random variable X is discrete if it can take on a finite number of distinct values.*

Probability low

Definition 3.1.3. *Let X be a random variable on* Ω*, and* $X(\Omega) = \{x_1, x_2, ..., x_n\}$*. The probability low of X is given by:*

x_i	x_1	x_2	$\dots$	x_n
$p(X = x_i)$	$p_1 = p(X = x_1)$	$p_2 = p(X = x_2)$	...	$p_n = p(X = x_n)$

Distribution Function

:

Definition 3.1.4. *The distribution function of D.R X is:*

$$F_X(x) = P(X \leq x) = \sum_{x_i \leq x} p(X = x_i)$$

$$F_X(x) = \begin{cases} 0 & \text{if } x < x_1 \\ p_1 & \text{if } x_1 \leq x < x_2 \\ p_1 + p_2 & \text{if } x_2 \leq x < x_3 \\ \vdots & \\ 1 & \text{if } x \geq x_n \end{cases}$$

Expected Value

Definition 3.1.5. *The expected value of the random variable X is:*

$$E(X) = \sum_{i=1}^{n} x_i p(X = x_i)$$

Variance and Standard Deviation:

The variance of the random variable X is:

Definition 3.1.6.

$$V(X) = E\left((X - E(X))^2\right) = \sum_{i=1}^{n} (x_i - E(X))^2 p(X = x_i)$$

Alternatively,

$$V(X) = E(X^2) - (E(X))^2 = \sum_{i=1}^{n} x_i^2 p(X = x_i) - \left(\sum_{i=1}^{n} x_i p(X = x_i)\right)^2$$

The standard deviation δ_X is:

Definition 3.1.7.

$$\delta_X = \sqrt{V(X)}$$

Example 3.1.1. *Consider the random experiment "rolling a six-sided die and observing the result." The game is as follows:*

- *If the result is even, you win 2 DA.*
- *If the result is 1, you win 3 DA.*
- *If the result is 3 or 5, you lose 4 DA.*

Define a random variable X which gives the gain in this game.

1. *The sample space is* $\Omega = \{1, 2, 3, 4, 5, 6\}$
2. $X(\Omega) = \{-4, 2, 3\}$

3. *The probability distribution of X is:*

x_i	-4	2	3
$p(X = x_i)$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{1}{6}$

4. *The DF of X is:*

$$F_X(x) = \begin{cases} 0 & \text{if } x < -4 \\ \frac{2}{6} & \text{if } -4 \leq x < 2 \\ \frac{5}{6} & \text{if } 2 \leq x < 3 \\ 1 & \text{if } x \geq 3 \end{cases}$$

5. *The expected value $E(X)$ is:*

$$E(X) = -4 \times \frac{2}{6} + 2 \times \frac{3}{6} + 3 \times \frac{1}{6} = \frac{1}{6}$$

6. *The variance $V(X)$ is:*

$$V(X) = 8.8 \quad \text{and} \quad \delta_X = \sqrt{V(X)} = 2.97$$

3.1.2 Continuous Random Variable

Definition 3: A random variable X is said to be continuous if it can take any value within an interval.

Probability Density Function (PDF)

Let X be a continuous random variable. We say that $f : \mathbb{R} \to \mathbb{R}^+$ is the probability density function (PDF) of X if:

- $f(x) \geq 0$ for all $x \in \mathbb{R}$,
- f is continuous on $\mathbb{R}$,
- $\int_{-\infty}^{+\infty} f(x)\, dx = 1$.

Distribution Function

The distribution function of a continuous random variable X is defined by:

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\, dt$$

Proposition 3.1.1. • *$F_X(x)$ is positive,*

- $\lim_{x\to-\infty} F_X(x) = 0$ *and* $\lim_{x\to+\infty} F_X(x) = 1.$

Remark 3.1.1. • *$P(X = x) = 0,$*

- *$P(X \leq x) = P(X < x),$*
- *The probability that $X \in [a, b]$ is given by:*

$$\begin{aligned} P(a \leq X \leq b) &= P(a < X \leq b) \\ &= P(a \leq X < b) \\ &= P(a < X < b) \\ &= \int_a^b f(t)\, dt \\ &= F_X(b) - F_X(a) \end{aligned}$$

Expected Value (Mathematical Expectation)

The expected value of the continuous random variable X is given by:

$$E(X) = \int_{-\infty}^{+\infty} x f(x)\, dx$$

Variance and Standard Deviation

The variance of the continuous random variable X is given by:

$$V(X) = \int_{-\infty}^{+\infty} (X - E(X))^2 f(x)\, dx$$

or,

$$V(X) = E(X^2) - (E(X))^2 = \int_{-\infty}^{+\infty} x^2 f(x)\, dx - \left(\int_{-\infty}^{+\infty} x f(x)\, dx\right)^2$$

3.2 Usual Probability lows

3.2.1 Discrete Lows

Bernoulli Low

- Any random experiment with two possible outcomes: success and failure, is called a **Bernoulli experiment**.
- Probability low: The random variable $X = 1$ in case of success with probability p, and $X = 0$ in case of failure with probability $q = 1 - p$.

The probability distribution is given by:

$$p(X = x) = \begin{cases} p & \text{if } x = 1 \\ q & \text{if } x = 0 \end{cases}$$

Notation: $X \sim B(p)$

$$E(X) = p$$

$$V(X) = pq$$

$$\delta_X = \sqrt{V(X)} = \sqrt{pq}$$

Examples 3.2.1. *We flip a fair coin. If we get heads, it's a success.* $\{\Omega\}$ *={heads, tails}=**{A}**= {heads} ,* $\bar{A}$ *= {tails}*

Probability distribution:

The probability of success $p = p(A) = \frac{1}{2}$

The probability of failure $q = p(\bar{A}) = \frac{1}{2}$

$$p(X = x) = \begin{cases} \frac{1}{2} & \textit{if } x = 1 \\ \frac{1}{2} & \textit{if } x = 0 \end{cases}$$

Calculating the values:

$$E(X) = p = \frac{1}{2}$$

$$V(X) = pq = \frac{1}{4}$$

$$\delta_X = \sqrt{V(X)} = \frac{1}{2}$$