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Institute of Mathematics and Computer Science
1st Year Mathematics (Algebra 2)

Series of Tutorial No. 2
A linear mapping

Exercise 1.
Are the following applications from E to F linear? If so, determine a basis of the kernel and a basis of the image.

1. $E = F = \mathbb{R}^2$; *for all* $(x, y) \in \mathbb{R}^2$:
$$f(x, y) = (2x + 3y, x)$$
2. $E = F = \mathbb{R}^2$; *for all* $(x, y) \in \mathbb{R}^2$:
$$f(x, y) = (y, x + y + 1)$$
3. $E = \mathbb{R}^3$, $F = \mathbb{R}$; *for all* $(x, y, z) \in \mathbb{R}^3$:
$$f(x, y, z) = x + 2y + z$$
4. $E = F = \mathbb{R}^2$; *for all* $(x, y) \in \mathbb{R}^2$:
$$f(x, y) = (x + y, xy)$$
5. $E = F = \mathbb{R}$; *for all* $x \in \mathbb{R}$:
$$f(x) = x^2$$

Exercise 2.
In each case, give the dimension of the kernel of f, then the rank of f. Is the application f injective? Surjective? Bijective?

1. $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x, y, z) = (y, z, x)$
2. $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x, y, z) = (x + y, y + z, x - z)$
3. $f : \mathbb{C}^3 \to \mathbb{C}^4$, $f(x, y, z) = (x + y + z)(1, i, -1, i)$
4. $f : \mathbb{C}^2 \to \mathbb{C}^4$, $f(x, y) = (x - y, x + iy, (2 + i)x + y, 3ix + y)$
5. $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x, y, z) = (2x + my - z,\ 2x + 2y,\ x - 2z)$ *depending on the real parameter* m

Exercise 3.
Let f and g be endomorphisms of $\mathbb{R}^2$ defined by:
$$g(x, y) = (y, x) \quad \textit{and} \quad f(x, y) = (x + y, 2x)$$

1. *Show that f and g are isomorphisms of $\mathbb{R}^2$. Determine f^{-1} and g^{-1}.*

2. *Let* $h = f \circ g - g \circ f$. *Justify that* $h \in \mathcal{L}(\mathbb{R}^2)$, *i.e., that* h *is a linear map.*

3. *Do we have* $f \circ g = g \circ f$? *Is* h *injective?*

4. *Is the application* h *surjective?*

Exercise 4.
Let $E = \mathbb{R}^3$, *and let* $B = \{e_1, e_2, e_3\}$ *be the canonical basis of* E. *Let* u *be the endomorphism of* $\mathbb{R}^3$ *defined by the images of the basis vectors:*

$$\begin{aligned} u(e_1) &= -2e_1 + 2e_3 \\ u(e_2) &= 3e_2 \\ u(e_3) &= -4e_1 + 4e_3 \end{aligned}$$

1. *Determine a basis of* $\ker u$. *Is* u *injective? Can it be surjective? Justify your answer.*

2. *Determine a basis of* $\operatorname{Im} u$. *What is the rank of* u?

3. *Show that* $E = \ker u \oplus \operatorname{Im} u$.

Exercise 5.
Let E *and* F *be two finite-dimensional vector spaces, and let* $f : E \to F$ *be a linear map. Show that* f *is an isomorphism if and only if the image under* f *of every basis of* E *is a basis of* F.

Exercise 6.
Let E *be a real vector space of dimension 3, and let* $f \in \mathcal{L}(E)$ *be an endomorphism such that:*

$$f^2 \neq 0 \quad \textit{and} \quad f^3 = 0.$$

Let $x_0 \in E$ *such that* $f^2(x_0) \neq 0$.

1. *Show that the set* $B = \{x_0,\ f(x_0),\ f^2(x_0)\}$ *is a basis of* E.

2. *Let* $g \in \mathcal{L}(E)$ *such that* $g \circ f = f \circ g$.

   (a) *Show that there exist scalars* $\alpha, \beta, \gamma \in \mathbb{R}$ *such that:*

   $$g(x_0) = \alpha x_0 + \beta f(x_0) + \gamma f^2(x_0).$$

   (b) *Show that:*

   $$g = \alpha \operatorname{Id}_E + \beta f + \gamma f^2.$$

   (c) *Deduce the set of endomorphisms of* E *that commute with* f.

Exercise 7.
Let $E = \mathbb{R}^3$ *with its canonical basis* $\{e_1, e_2, e_3\}$, *and let* $f \in \mathcal{L}(E)$ *be defined by:*

$$\begin{aligned} f(e_1) &= (-2,\ -4,\ 5) \\ f(e_2) &= (4,\ 8,\ -10) \\ f(e_3) &= (2,\ 4,\ -5) \end{aligned}$$

1. *Determine the set* $H := \{u \in E \mid f(u) = u\}$, *and show that it is a vector subspace of* E.

2. *Determine the kernel of* f, *i.e.,* $\ker(f)$.

3. *Find a basis of* H *and a basis of* $\ker(f)$.

4. *Show that the union of these two bases forms a basis of* E.

5. *What is the matrix representation (analytical expression) of* f *in this new basis?*