Chapitre 01

Electrostatic

Electrostatic charges and fields

Electrostatics is a branch of physics that studies the phenomena created by electrical charges in a state of rest.

In this course we will see the explanation of some basic notions which is summarized in the notion of electric charge and electric force and the field and electric potential.

A manifestation of static electricity consists of the attraction of small light bodies (e.g. pieces of paper) with rubbed bodies (rulers, etc.). It was in the 19th century that the unified theory of electrical and magnetic phenomena, called electromagnetism, was developed. At this time the word “static” appeared to designate electrostatic phenomena.

1.1- Electrification

The phenomenon of electrification is the production of electricity during an interaction between two electrical charges (particles) located in space. By experience we can prove the presence of two types of charge: positive charge and negative charge; Two charges of the same type repel each other, but two charges of different types attract each other. On the other hand, a non-electrified (charged) body is said to be neutral.

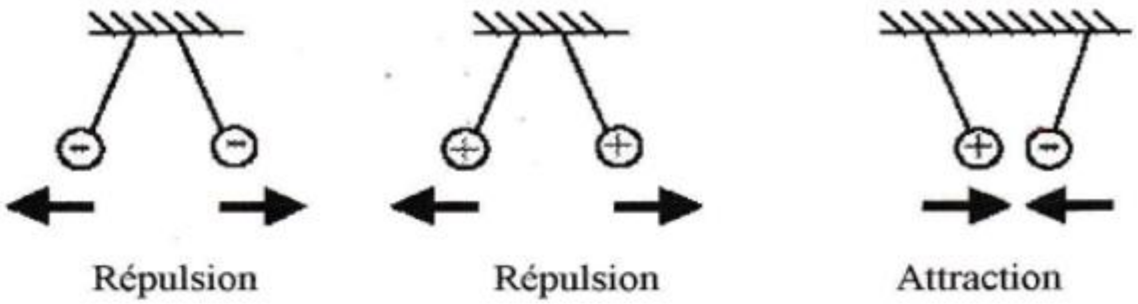


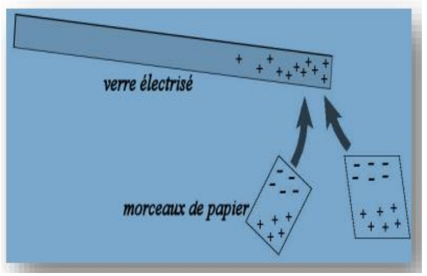
Figure 1.1: Interaction between two electric charges

But this experiment also shows us that electricity is capable not only of acting at a distance (repulsion or attraction) but also of moving from one body to another.

There are three types of electrification:

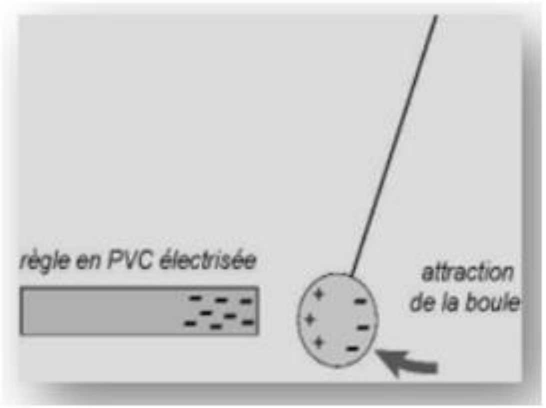
- **Electrification by friction.**
- **Electrification by contact.**
- **Electrification by induction.**

- **Électrisation par frottement** : Ce phénomène se produit lorsqu'une substance acquiert ou perd des électrons lorsqu'elle est frottée contre une autre. Par exemple, frotter une règle en plastique avec un chiffon peut transférer des électrons d'un matériau à l'autre, les chargeant électriquement.

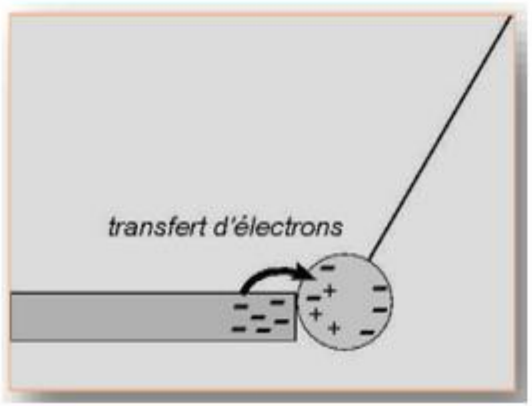


Exempe 1 A glass rod rubbed with a wool cloth is brought close to small pieces of paper, which are then attracted to the rod.

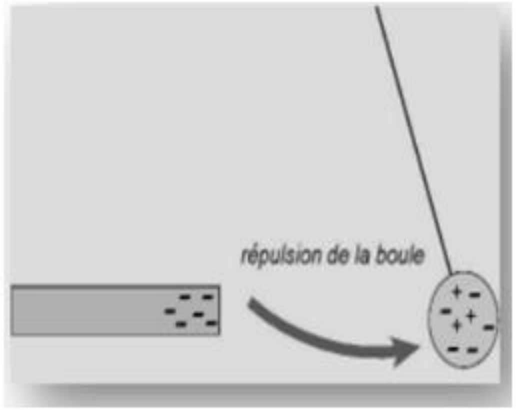
- **Électrisation par contact** : Elle se produit lorsqu'un matériau électrisé est mis en contact avec un autre matériau. Les charges électriques se répartissent entre les deux objets, les rendant tous deux électrisés.
- **Électrisation par influence** : Ce processus consiste à approcher un matériau électrisé d'un autre sans les mettre en contact. Les charges à l'intérieur du matériau approché se réorganisent en réponse au champ électrique du premier matériau, entraînant une séparation des charges au sein du matériau neutre.



A) Attraction of the ball



B) Electron transfer during electrification by contact



C) Repulsion of the ball

1.2- Electric Charge

Electric charge is a fundamental property of matter that causes it to experience a force when placed in an electric field. It exists in two types: **positive charge** (carried by protons) and **negative charge** (carried by electrons).

The atom is composed of electrons, protons, and neutrons. The charge of electrons is negative, while the charge of protons is positive. Neutrons, on the other hand, are neutral. The charge of an electron (or a proton) is called the **elementary charge**, with a value of **e=1.6×10−19e = 1.6 \times 10^{-19}e=1.6×10−19 C**.

Toute autre charge Q est un multiple entier de cette quantité : Q= n.e

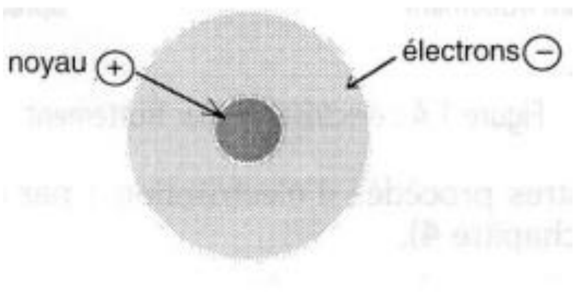


Figure 1.3: Components of an Atom

	Charge (c : coulomb)	La masse (Kg)
Le proton (p)	$1.6*10^{-19}$	$1.672*10^{-27}$
Le Neutron (n)	0	$1.675*10^{-37}$
L'électron (e)	$-1.6*10^{-19}$	$9.11*10^{-31}$

micro C (1µC = 10^{-6}C) ; nano C (1nC = 10^{-9}C) ; pico C (1pC = 10^{-12}C)

An atom consists of three main subatomic particles:

- **Electrons (-)**: Negatively charged particles that orbit around the nucleus.
- **Protons (+)**: Positively charged particles located in the nucleus.
- **Neutrons (0)**: Neutral particles also found in the nucleus.

The nucleus (composed of protons and neutrons) forms the dense core of the atom, while electrons move in energy levels or shells around it. The balance between protons and electrons determines the atom's overall charge.

2- Electrostatic Force

2. Coulomb's Law

In order to determine the properties of the electrostatic force exerted between two point charges, consider two identical point charges $\mathbf{q_A}$ and $\mathbf{q_B}$ separated by a distance **"r"**.

Each charge exerts an electrostatic interaction force on the other, which is:

•**Proportional** to the product of the two charges.

•**Inversely proportional** to the square of the distance between them.

•**Directed along** the straight line joining the two charges.

This follows **Coulomb's Law**, which mathematically expresses the force as:

$$\vec{F}_{AB} = K\frac{q_A q_B}{r^2}\vec{u} = -\vec{F}_{BA} \quad and \quad \vec{u} = \frac{\overrightarrow{AB}}{\left\|\overrightarrow{AB}\right\|}. \quad r = \left\|\overrightarrow{AB}\right\| \;\Rightarrow\; \vec{F}_{AB} = K\frac{|q_A q_B|}{\left\|\overrightarrow{AB}\right\|^3}\overrightarrow{AB}$$

This relationship represents the force exerted by charge q_A located at point A, on charge q_B located at point B.

Where:

F is the electrostatic force,

k is Coulomb's constant ($8.99 \times 10^9\ N \cdot m^2/C^2$),

q_A and q_B are the magnitudes of the charges,

r is the distance between the charges.

The force exerted by charge $\boldsymbol{q_B}$ on charge $\boldsymbol{q_A}$ is given by: $\vec{F}_{BA} = -K\frac{|q_A q_B|}{r^2}\vec{u} = -\vec{F}_{AB}$

meaning they are equal in magnitude but opposite in direction.

Coulomb's Constant K:

The constant K is given by:

$$K = \frac{1}{4\pi\varepsilon_0}$$

where:

- ε_0 is the **permittivity of free space**, with a value of

$$\varepsilon_0 = 8.85 \times 10^{-12}\, C^2 N^{-1} m^{-2}$$

- Substituting this value, we get:

$$K = \frac{1}{4\pi(8.85 \times 10^{-12})} \approx 8.99 \times 10^9\, N \cdot m^2/C^2$$

This constant is used to calculate electrostatic forces in **vacuum or air**.

- If q_A and q_B have the **same sign** (both positive or both negative), the force is **repulsive**: the charges push each other away.
- If q_A and q_B have **opposite signs** (one positive, one negative), the force is **attractive**: the charges pull each other together.

This behavior follows **Coulomb's Law**, which describes how electric charges interact based on their magnitude and sign.

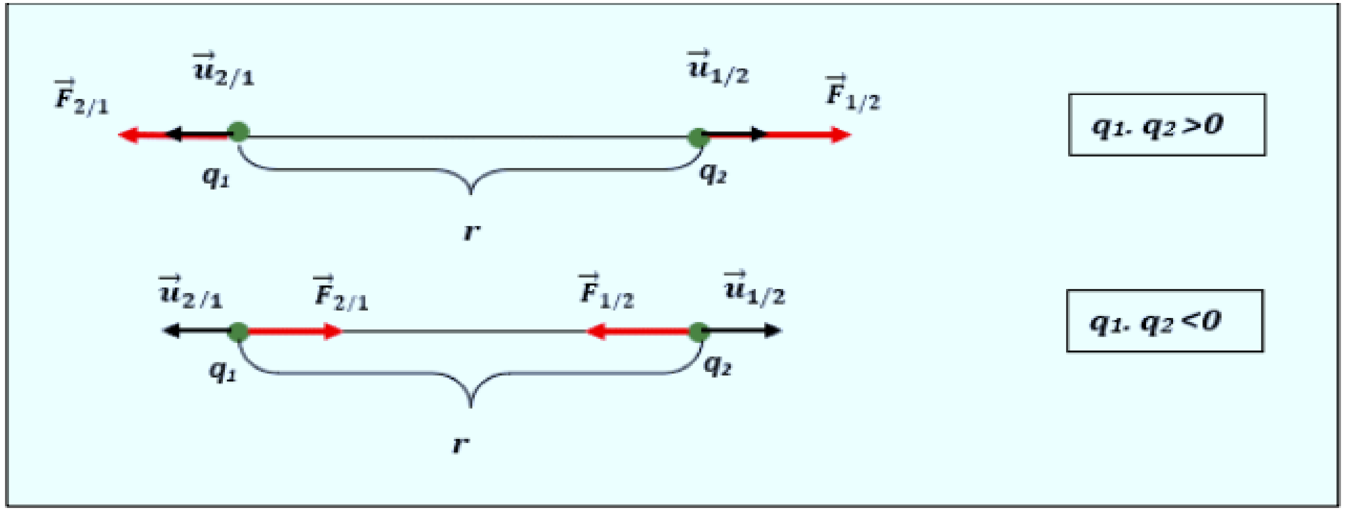


Figure 1.4: Coulomb Forces Between Two Point Charges

Characteristics of Electrostatic Force

- **Direction**: The force acts along the **straight line** connecting the two charges.
- **Direction of the Force**: Depends on the sign of the charges:
  - $+q_1 \cdot +q_2 > 0$ → **Repulsion**
  - $-q_1 \cdot +q_2 < 0$ → **Attraction**
  - $-q_1 \cdot -q_2 > 0$ → **Repulsion**
- **Point of Application**: The force is applied to each charge q_i.
- **Action and Reaction Principle**:

$$\vec{F}_{12} = -\vec{F}_{21}$$

Each charge exerts a force on the other that is equal in magnitude but opposite in direction.

Charge of the Electron

- The elementary charge of an electron is:

$$q_e = -1.6 \times 10^{-19}\ \mathrm{C} \quad (\mathrm{Coulomb})$$

- Similarly, the charge of a proton is **positive** with the same absolute value:

$$q_p = +1.6 \times 10^{-19}\ \mathrm{C}$$

These principles govern electrostatic interactions between charged particles.

Superposition Principle

The **superposition principle** states that when multiple charges exert forces on a given charge, the **resultant force** is the **vector sum** of all individual forces acting on it.

$$\vec{F}_{\text{total}} = \vec{F}_1 + \vec{F}_2 + \ldots + \vec{F}_n$$

where each force $\vec{F}_i$ is calculated using **Coulomb's Law.**

$$\overrightarrow{F_3} = \overrightarrow{F_{13}} + \overrightarrow{F_{23}} = k\frac{q_1 q_3}{r_{13}^2}\overrightarrow{u_{23}}$$

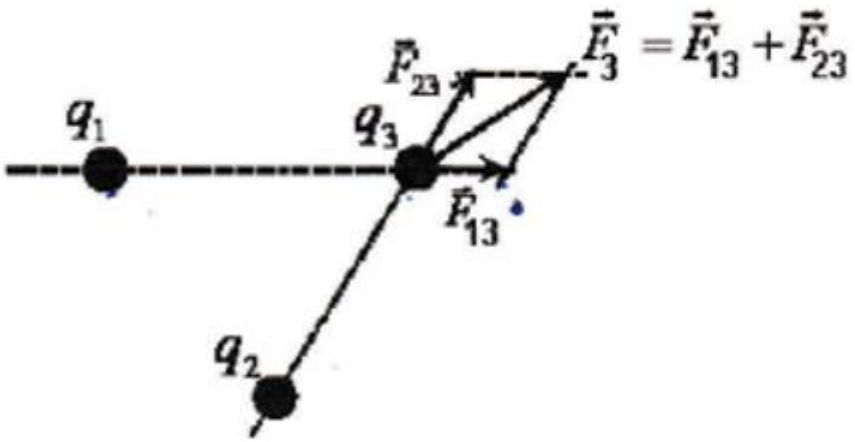
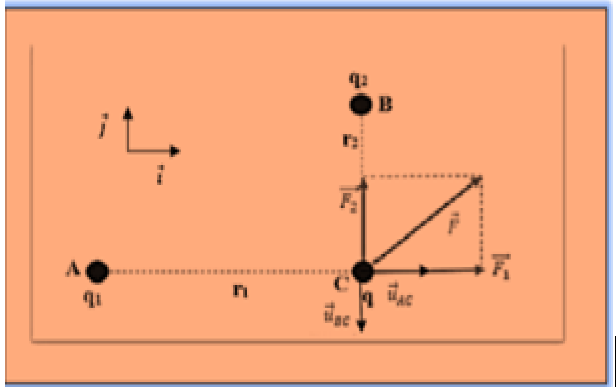


Figure 1.5 Representation of Electrostatic Forces

If **n** charges are present, the total electrostatic force **exerted on charge mmm** is given by the **superposition principle**:

$$\vec{F}_m = \sum_{i=1}^{n} \vec{F}_i \qquad \overrightarrow{F_m} = \overrightarrow{F_{1m}} + \overrightarrow{F_{2m}} + \cdots + \overrightarrow{F_{mm}} = \sum_{i=1}^{n} \overrightarrow{F_{1m}}$$

Example: knowing that $q_1 = -1{,}5.\ 10^{-3} C\ ;\ q_2 = 0{,}5.\ 10^{-3} C\ ;\ q = -0{,}2.\ 10^{-3} C\ ;\ r_1 = 1{,}2\ m$ et $r_2 = 0{,}5$ m



q_1 and q have the same sign, in this case $\vec{F_1}$ is repulsive : $\vec{F_1} = \frac{k\ q_1 q}{r_1^2}\overrightarrow{u_{AC}}$

q_2 and q have an opposite sign, in this case $\vec{F_2}$ is attractive $\vec{F_2} = \frac{k\ q_2 q}{r_2^2}\overrightarrow{u_{BC}}$

The total force exerted on q: $\vec{F} = \vec{F_1} + \vec{F_2} = \frac{k\ q_1 q}{r_1^2}\overrightarrow{u_{AC}} + \frac{k\ q_2 q}{r_2^2}\overrightarrow{u_{BC}}$

We have: $\overrightarrow{u_{AC}} = \vec{i}\ ;\ \overrightarrow{u_{BC}} = -\vec{j}$ *then* $\vec{F} = \frac{k\ q_1 q}{r_1^2}\vec{i} + \frac{k\ q_2 q}{r_2^2}\vec{j}$

the modulus of force $\vec{F}$

$$|\vec{F}| = \sqrt{\left(\frac{k\ q_1 q}{r_1^2}\right)^2 + \left(\frac{k\ q_2 q}{r_2^2}\right)^2}\ ;\ \underline{\text{A.N :}}\ \ |\vec{F}| = 4.1 * 10^3 N$$

Champ et potentiel électriques (cas d'une distribution ponctuelle de charge)

Notion de Champ électrique

The influence between two electric charges in the absence of a material medium is explained by **the concept of the electrostatic field** and **the action-at-a-distance interaction** described by Coulomb's law.

The electric force occurs between two charges. How does this influence take place between the two charges in the absence of a material medium?

The answer is that this medium is the **electric field**. The electric field is defined as the region of space where any particle is subjected to the action of an electric force due to the presence of other charges.
This concept was introduced to explain the phenomenon of interaction between charges and to characterize each charge with its own specific field rather than describing the interaction solely through the electric force.

Let **E** be the electric field created at a point M by a charge q_0 . If a charge q is placed at M, the Coulomb force exerted by q_0 on q can be determined by the equation: $\vec{F} = \boldsymbol{q}\vec{E}$……..

This equation expresses that the force experienced by a charge q at a given point in an electric field E is directly proportional to the magnitude of the charge and follows the direction of the electric field.

$\vec{F} = q\vec{E}$ $\qquad$ $\vec{E} = k\frac{q_0}{r^2}\vec{u}$ $\qquad$ N/C or V/m $\qquad$ and $\vec{F} = q\,k\frac{q_0}{r^2}\vec{u} = k\frac{qq_0}{r^2}\vec{u}$

The field created by the charge q_0 at a distance r follows the direction of $\vec{u}$

The electric field created by a charge exists at every point in its space, whereas the electric force can only exist if there are at least two charges.

According to the equation: $\vec{F} = q\vec{E}$

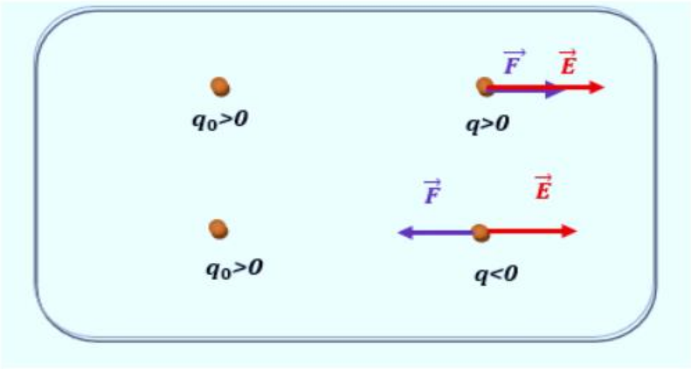
If $q >$: have the same meaning.

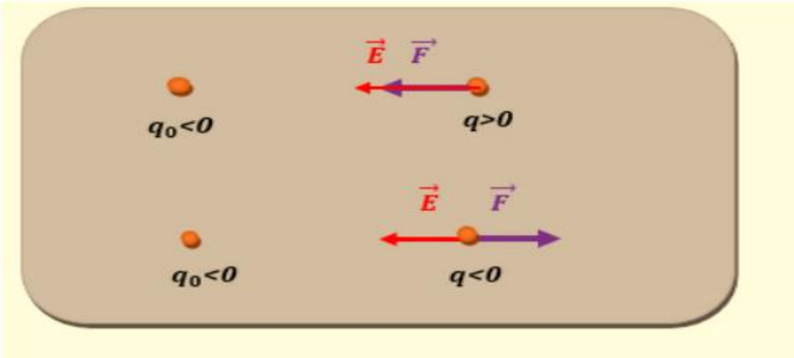
If $q <$: have opposite signs

Explanation

In the event that

$q_0 > 0$





$q_0 < 0$

- The vector $\vec{E}$ created by a charge **q>0** at **M** is directed from **q** to **M.**

- The vector $\vec{E}$ created by a charge **q<0** at **M is** directed from **M** to **q.**

Champ électrostatique créé par une charge ponctuelle unique

Given an isolated and immobile point charge $q1$, in its region, any charge $q2$ experiences an electric force

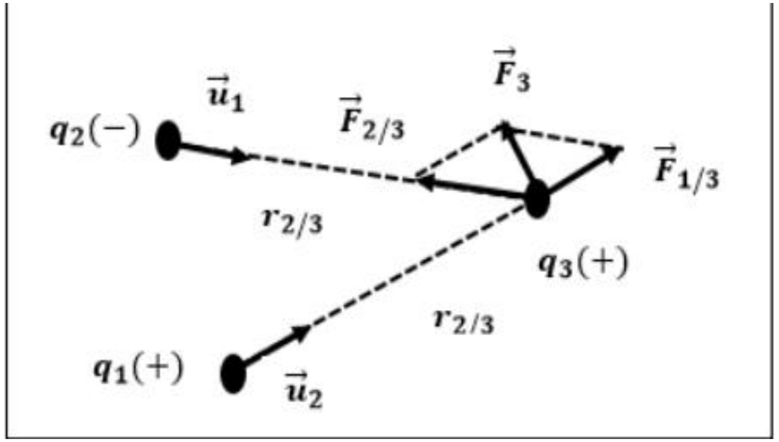
$$\overrightarrow{F_{1/2}} = q_2 . \overrightarrow{E_{1/2}}$$

Hence the electric field is $\vec{E} = \frac{\vec{F}}{q}$ Then, the expression of the field is written $\vec{E}_1 = \frac{k\, q_1}{r^2} \overrightarrow{u_{1/2}}$.

Field created by a set of point charges (principle of superposition)

Given three point charges $\boldsymbol{q_1}$, $\boldsymbol{q_2}$ separated by $r_{i.j}$ as indicated in ***(figure 1.6)*** the electric field $\vec{\boldsymbol{E}}$, created at a point **M** by the two point charges, is the sum of the two fields $\overrightarrow{E_1}$ and $\overrightarrow{E_2}$ created by each of the charges $\boldsymbol{q_1}$ and $\boldsymbol{q_2}$.

Case of **n** point loads $\quad q_1, q_2, q_3 \dots\dots\dots q_n$: $\quad \vec{E} = \sum_{i=1}^{n} \vec{E} = \sum_{i=1}^{n} k \frac{q_i}{r_i^2} \overrightarrow{u_i}$



Principle of superposition

If a charge $\boldsymbol{q_3}$ is located at **M** in the region of space where the field $\vec{E}$ reigns,

it will experience an electric force

The force $\overrightarrow{\boldsymbol{F_3}}$ acting on $\boldsymbol{q_3}$ due to $\boldsymbol{q1}$ and $\boldsymbol{q2}$ will be the resultant of the forces

$$\overrightarrow{\mathbf{F_{1/3}}} \ \textbf{\textit{and}} \ \overrightarrow{\boldsymbol{F_{2/3}}}$$

His expression is written $\quad \overrightarrow{F_3} = \overrightarrow{F_{1/3}} + \overrightarrow{F_{2/3}}$

$$\overrightarrow{F_3} = k\frac{q_1 q_3}{r_{1/3}^2}\overrightarrow{u_{1/3}} + k\frac{q_2 q_3}{r_{2/3}^2}\overrightarrow{u_{2/3}} = = q_3\left(\overrightarrow{E_1} + \overrightarrow{E_2}\right) = q_3\vec{E}$$

If the charge moves from point **A** to point **B** following curve

C, otherwise the work of the electric force **F** is:

- Elementary work $dw = \vec{F}.\overrightarrow{dl}$
- The total work of the force: $W_{A\to B} = \int_A^B \vec{F}.\overrightarrow{dl} = \int_A^B \overrightarrow{q\vec{E}}.\overrightarrow{dl}$

Circulation of an electrostatic field $\vec{E}$ created by a charge q

The electrostatic field at a point M created by a fixed and point charge q>0 (see figure 1.7) is $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\vec{u}$

"r" represents the distance between q and point M

For an elementary displacement $\overrightarrow{dl}$ around **M**, the elementary circulation is defined by $dc(\vec{E}) = \vec{E}.\overrightarrow{dl}$

Based on (figure 1.7), let's calculate the dot product $\vec{E}.\overrightarrow{dl}$

$$\vec{E}.\overrightarrow{dl} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\vec{u}.\overrightarrow{dl} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}dr$$

$dr = \vec{u}.\overrightarrow{dl}$ (dr is the projection of $\overrightarrow{dl}$ on to $\vec{u}$.) While $\vec{E}.\overrightarrow{dl} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}dr$

The elementary circulation is then written $dC(\vec{E}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}dr$

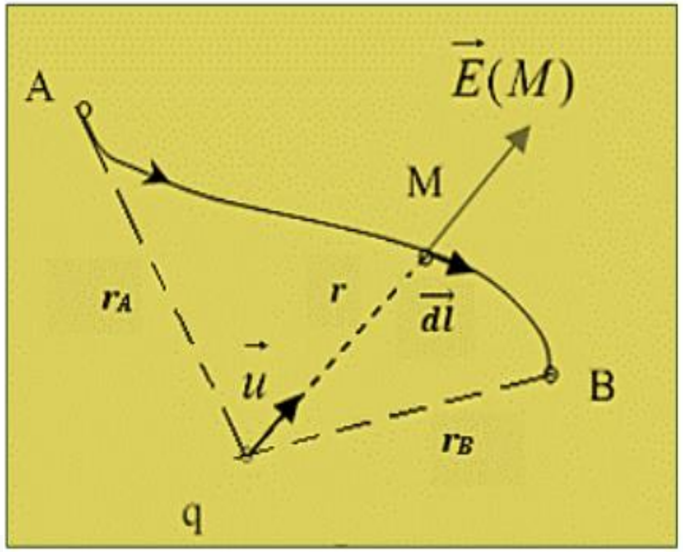


Figure1.7 Circulation of an electrostatic field $\vec{E}$

The total circulation of $\vec{E}$ from point A to point B is

$$C_A^B(\vec{E}) = \int_A^B dC = \frac{q}{4\pi\varepsilon_0}\int_A^B \frac{1}{r^2}dr = \frac{q}{4\pi\varepsilon_0}\left[\frac{-1}{r}\right]_A^B = \frac{-q}{4\pi\varepsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$$

$$C_A^B(\vec{E}) = \frac{-q}{4\pi\varepsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$$

r_A and r_B represent the positions of the two points **A** and **B** with respect to the charge q which creates the field $\vec{E}$

> CONCLUSION
>
> *La circulation de $\vec{E}$ entre deux points ne dépend que de leurs positions initiales et finales et non pas du trajet suivi. On dit que $\vec{E}$ est à circulation conservative.*

- Un champ électrostatique $\vec{E}$ part du point **A** et revient au point **A** selon la trajectoire fermée **(figure 1.8)**.

  La circulation du champ $\vec{E}$ dans ce cas est donnée par :

$$C_A^A(\vec{E}) = \int_A^A \vec{E}.\vec{dl} = \oint_A^A \vec{E}.\overrightarrow{dl} = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_A}\right) = 0$$

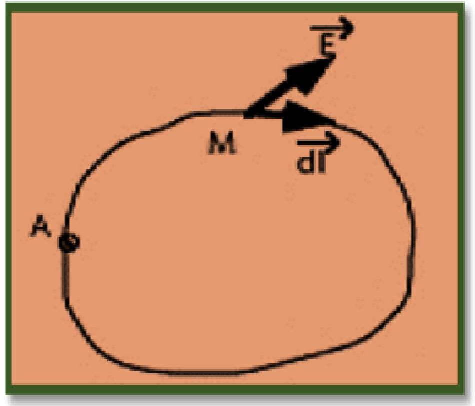


Figure 1.8 Circulation du champ $\vec{E}$ sur un parcours fermé

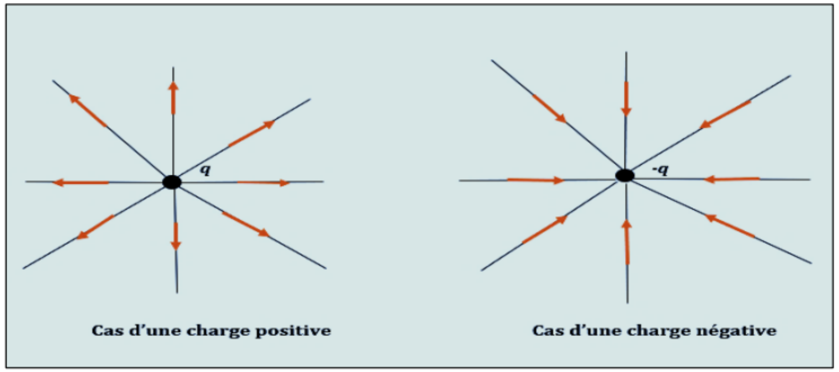
NOTE

La circulation d'un champ électrostatique $\vec{E}$ à travers un contour fermé est nulle.

Electrostatic field lines

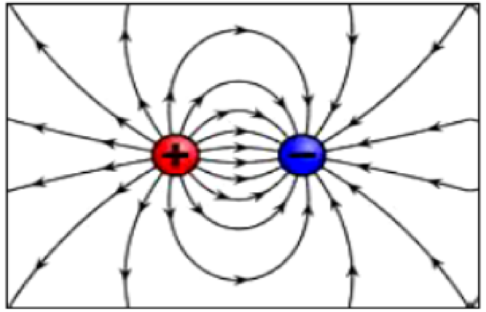
A field line is a curve such that at each of its points, the electric field is carried by the tangent to the curve.

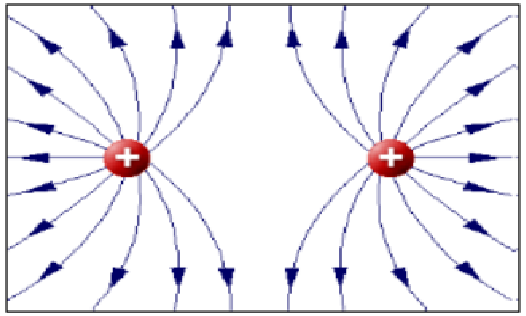
Case of **a single charge** In this case the field lines are straight lines which intersect at the point where the charge is placed.



Case of two point charges

The field lines in the case of two point charges of the same sign and those of two charges of opposite sign are shown in the diagram below





Q: Determine the electrostatic field created by three identical point charges **(q>0)** placed at the vertices of an equilateral triangle, at its geometric center **G**.

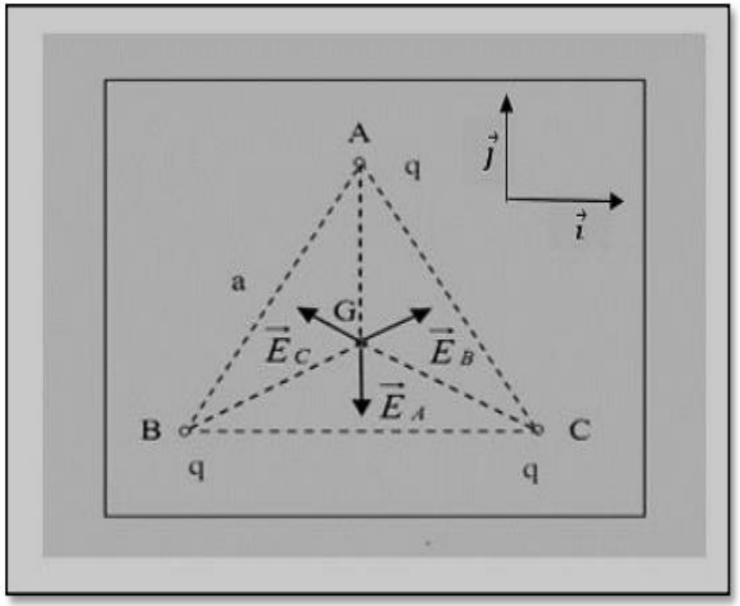
R: Applying the principle of superposition, the total field at point **G** is (see diagram below

$$\vec{E}_G = \vec{E}_A + \vec{E}_B + \vec{E}_C$$

$$\vec{E}_G = \frac{Kq}{(AG)^2}\vec{u}_{AG} + \frac{Kq}{(BG)^2}\vec{u}_{BG} + \frac{Kq}{(CG)^2}\vec{u}_{CG}$$

$\Delta(ABC)$ is an equilateral triangle then the distance (AG)= (BG)= (CG), the total field is

$$\vec{E}_G = \frac{Kq}{(AG)^2}(\vec{u}_{AG} + \vec{u}_{BG} + \vec{u}_{CG})$$



La projection des trois vecteurs unitaires $\vec{u}_{AG}, \vec{u}_{BG}$ et $\vec{u}_{CG}$ sur la base $(\vec{\imath}, \vec{\jmath})$ donne :

$$\vec{u}_{BG} = \cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath}$$

$$\vec{u}_{AG} = -\vec{\jmath}$$

$$\vec{u}_{CG} = -\cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath}$$

$\boldsymbol{\theta}$ est l'angle entre la droite (BC) et la droite (BG) ; $\theta = \frac{\pi}{6}$

Le champ total au point $\boldsymbol{G}$ est :

$$\vec{E}_G = \frac{Kq}{(AG)^2}\left(-\vec{\jmath} + 2\sin\left(\frac{\pi}{6}\right)\vec{\jmath}\right)$$

Alors : $\vec{E}_G = \vec{0}$

Field produced by a continuous distribution of charges

Let there be a set of charges qi distributed over an element. Each charge dq located in this element creates an elementary field

Distribution linéique on a wire of length dl with a linear density ? (figure 1.7-a): dq = λ dl.

$$\overrightarrow{dE} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dl}{r^2}\vec{u}$$

The field created by q is written:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int\frac{\lambda dl}{r^2}\vec{u}$$

Surface distribution: on a charged surface with a surface density ? (figure 1.7-b): dq= σ dS.

$$\vec{E} = k\iint\sigma\frac{ds}{r^2}\overrightarrow{u_i} = \frac{1}{4\pi\varepsilon_0}\iint\sigma\frac{ds}{r^2}\overrightarrow{u_i}$$

Volume distribution on a volume V charged with a volume density ?(figure 1.7c): $dq = \rho dV$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\iiint\rho\frac{dV}{r^2}\overrightarrow{u_i}$$

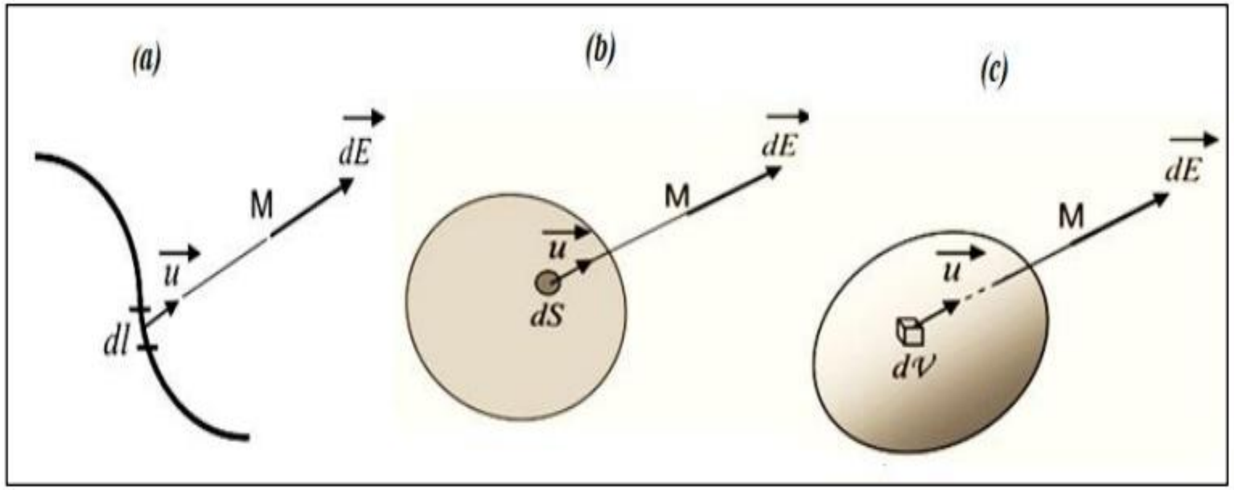


Figure 1.7: Continuous load distribution

dq = λ dl.

$$\overrightarrow{dE} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dl}{r^2}\vec{u}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda dl}{r^2}\ \vec{u}$$

dq= σ dS.

$$\vec{E} = k\iint \sigma\frac{ds}{r^2}\overrightarrow{u_i} = \frac{1}{4\pi\varepsilon_0}\iint \sigma\frac{ds}{r^2}\overrightarrow{u_i}$$

$$dq = \rho dV$$

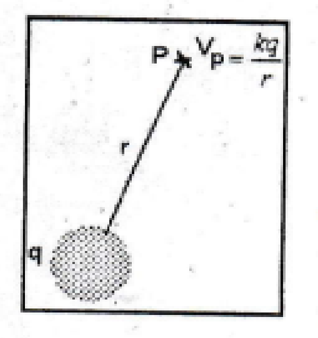
$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\iiint \rho\frac{dV}{r^2}\overrightarrow{u_i}$$

Potentiel électrostatique

A charge q creating an electric field vector $\vec{E}$; from its elementary displacement **dl** has a potential energy $\mathbf{U_p = qV}$

V represents the potential induced by the charge **q** in a point **p** in the space surrounding and distant from **r**

$$dC(\vec{E}) = \vec{E}.\vec{dl} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}dr = -dV$$



The electric potential is written: $V = -\int \vec{E}.\vec{dl} = -\int \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}dr$

$$\boldsymbol{V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} + Cte.}$$

- **The unit of electrostatic potential V in the SI system is the SI Volt $[V]$**

In the case of a point charge the origin of the potential measurement is: $\boldsymbol{V}(\boldsymbol{r} \to \infty) = 0$

$$V(r \to \infty) = 0 \Rightarrow \quad Cte = 0$$

$$\Rightarrow \boldsymbol{V = \frac{k\,q}{r}} \quad ; \quad \left(k = \frac{1}{4\pi\varepsilon_0}\right).$$

Exemple :

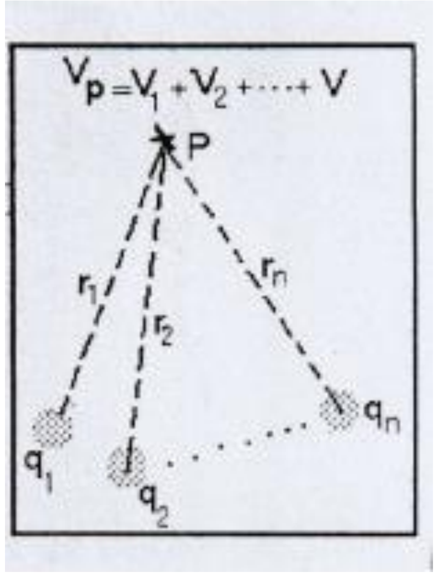
in the case of Cartesian coordinate systems if the position of the charge q is the point $A(x_A, y_A, z_A)$ the electric potential at the point $P(x_A, y_A, z_A)$ is:

$$V = \frac{k\,q}{r} = \frac{k\,q}{\|\overrightarrow{AP}\|} = \frac{kq}{\sqrt{(x_A - x_p)^2 + (y_A - y_p)^2 + (z_A - z_p)^2}}$$

Summation principale

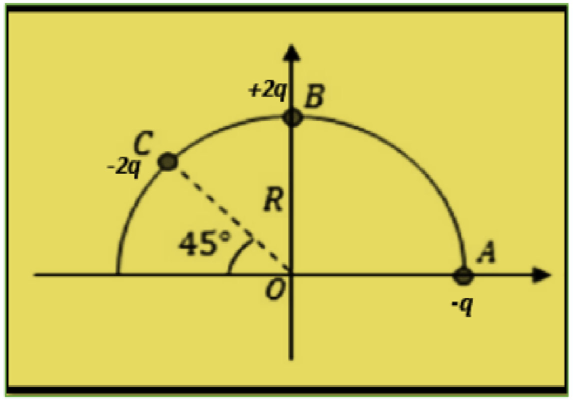
If we have a set of charges, the potential at a point of the potentials induced by each charge.

$$V = V_1 + V_2 + \cdots .. V_n = \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} \frac{kq_i}{r_i} = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{q_i}{r_i}$$



Exemple

Three point charges q_A, q_B and q_C are placed respectively at points **A, B** and **C** belonging to a circle with center **O** and radius **R** according to the figure below.
- Calculate the electrostatic potential at point **O**



Applying the principle of superposition, the electrostatic potential at point O is:

$$V_O = V_A + V_B + V_C$$

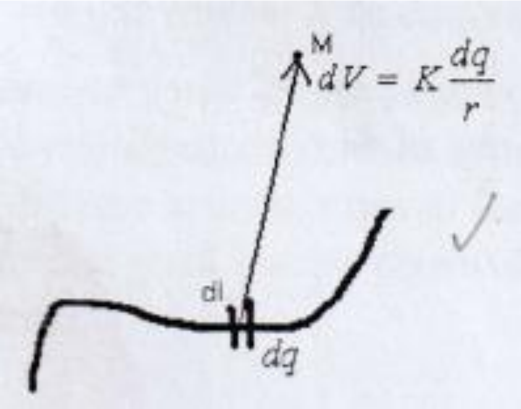
$$V_A = \frac{kq_A}{r_A} = k\frac{(-q)}{R} \quad ; \quad V_B = \frac{kq_B}{r_B} = k\frac{(2q)}{R} \quad ; \quad V_c = \frac{kq_c}{r_c} = k\frac{(-2q)}{R}$$

$$V_O = k\frac{(-q)}{R} + k\frac{(2q)}{R} + k\frac{(-2q)}{R}$$

$$V_O = -k\frac{q}{R}$$

THE ELECTRIC POTENTIAL IN THE CASE OF A CONTINOUS DISTRIBUTION OF CHARGES:

a) Linear distribution



$$dV = \frac{k\,dq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r}$$; Linear charge density $\lambda = \frac{dq}{dl} \Rightarrow dq = \lambda . dl$

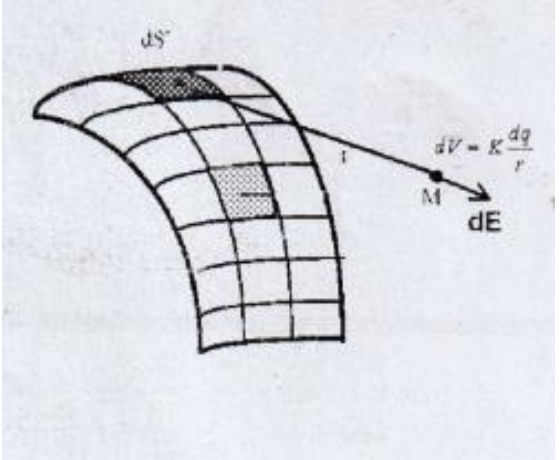
$$V(M') = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda dl}{r}$$

b) Surfac distribution

The potential induced by an elementary charge **dq** of a surface **ds** in a point **M'** is:

$$dV(M') = \frac{kdq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma ds}{r}$$

Knowing that: σ=dq/ds represents the surface density of the charge.



The total potential induced by the surface is:

$$V(M') = \frac{1}{4\pi\varepsilon_0}\iint \frac{\sigma ds}{r}$$

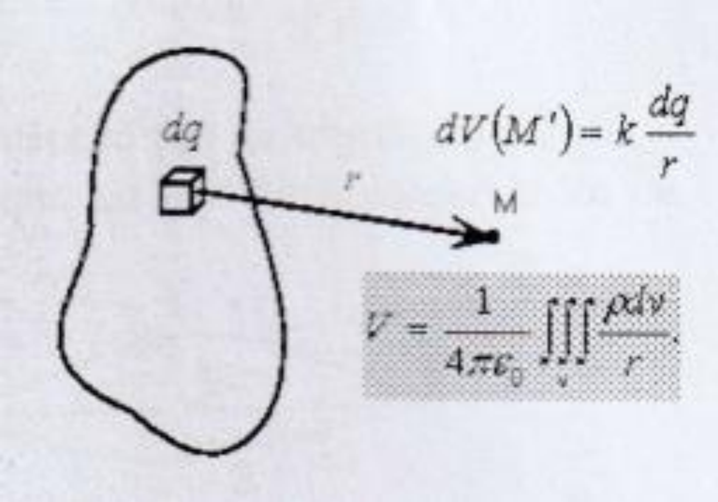
c) Volume distribution

The potential due to an elementary charge dq in a point M': $$dV(M') = \frac{kdq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\rho dv}{r}$$

Knowing that: ρ=dq/dv represents the volume density of the charge

The total volume potential V:

$$V(M') = \frac{1}{4\pi\varepsilon_0}\iiint \frac{\rho dv}{r}$$



The relationship between the field $\vec{E}$ and the potential V

The electrostatic potential was defined from the elementary circulation of $\vec{E}$

$$dC(\vec{E}) = \vec{E}.\overrightarrow{dl} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}dr = -dV$$

In Cartesian coordinates :

Let V be a scalar function: $dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$

$$\vec{E} = E_x\vec{\imath} + E_y\vec{\jmath} + E_z\vec{k} \quad ; \quad \overrightarrow{dl} = dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k} \qquad dV = -\vec{E}.\overrightarrow{dl}$$

So: $$\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = -(E_x\vec{\imath} + E_y\vec{\jmath} + E_z\vec{k}).(dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k})$$

$$\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = -E_x dx - E_y dy - E_z dz$$

By identification: $$\begin{cases} E_x = -\frac{\partial V}{\partial x} \\ E_y = -\frac{\partial V}{\partial y} \\ E_z = -\frac{\partial V}{\partial z} \end{cases}$$

The electrostatic field can be written: $\vec{E} = -\left(\frac{\partial V}{\partial x}\vec{\imath} + \frac{\partial V}{\partial v}\vec{\jmath} + \frac{\partial V}{\partial z}\vec{k}\right) \Rightarrow \vec{\boldsymbol{E}} = -\overrightarrow{\boldsymbol{grad}}\,\boldsymbol{V}$

- *Le champ électrostatique dérive du potentiel électrostatique V.*
- *Le champ électrostatique est orienté vers les potentiels décroissants*

don't forget

Quand on te demande de ***déduire*** $\vec{E}$ *ou V*, utiliser l'équation :

$$\vec{E} = -\overrightarrow{grad}\, V$$

- From the following relation for a scalar function f: $\overrightarrow{Rot}\left(\overrightarrow{grad}\, f\right) = \vec{0}$

So: $\overrightarrow{Rot}\left(\overrightarrow{grad}\, V\right) = \vec{0}$ But: $\overrightarrow{grad}\, V = -\vec{E}$ $\Rightarrow$ $\overrightarrow{Rot}\left(\overrightarrow{grad}\, V\right) = -\overrightarrow{Rot}\, \vec{E} = \vec{0}$

The rotational electric field is zero $\overrightarrow{Rot}\, \vec{E} = \vec{0}$

- We say that the field is derived from a potential. **Therefore the electric field is conservative**.

- $\vec{E} = -\overrightarrow{grad}\, V$, $q\vec{E} = -q\ \overrightarrow{grad}\, V = -\overrightarrow{grad}(q\, V)$ $\Rightarrow \vec{F} = -\overrightarrow{grad}\, U_p$ $\left(U_p = q\, V\right)$

$\overrightarrow{Rot}\, \vec{E} = \vec{0}$ $\Rightarrow q\overrightarrow{Rot}\, \vec{E} = \overrightarrow{Rot}\left(q\vec{E}\right) = \vec{0}$ $\Rightarrow \overrightarrow{Rot}\left(\vec{F}\right) = \vec{0}$ $\vec{F} = -\overrightarrow{grad}\, U_p$

The electric force $\vec{F}$ is derived from a potential energy U_p as well as $\overrightarrow{Rot}\left(\vec{F}\right) = \vec{0}$ The electric force $\vec{F}$ is a conservative force).

Equipotential surface It is the set of points that have the same potential $V(M) = V_0 = C^{te}$

It is the set of points that have the same potential

Electrostatic Potential Energy

The electrostatic potential energy of a charged particle placed in an electrostatic field is equal to the work required to bring this particle quasi-statically from infinity to its current position.

$$W = \int_A^B \vec{F}\,.\,\overrightarrow{dl} = \int_A^B q\vec{E}\,.\,\overrightarrow{dl} \qquad et \qquad \vec{E} = \sum_{i=1}^{n} \overrightarrow{E_i}$$

Hence the working formula is :

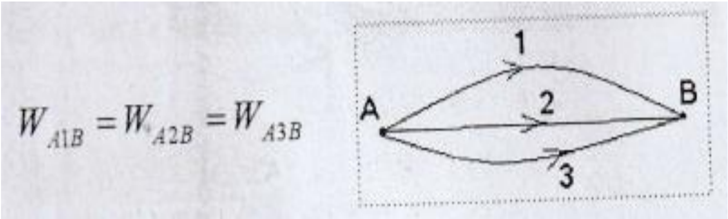
$$W = \int_A^B \vec{F}\,.\,\overrightarrow{dl} = q\int_A^B \sum_{i=1}^{n} \overrightarrow{E_i}\,.\,\overrightarrow{dl} = q\sum_{i=1}^{n} \int_A^B \overrightarrow{E_i}\,.\,\overrightarrow{dl}$$

$$W = q\sum_{i=1}^{n} \int_A^B \frac{kq_i}{r_i}\overrightarrow{u_i}.\overrightarrow{dl_i}$$

$$W = q\sum_{i=1}^{n} \int_A^B \frac{kq_i}{r_i}\,\overrightarrow{dr_i} = -q\sum_{i=1}^{n} \left(\frac{kq_i}{r_i}\right)_A^B = -q\sum_{i=1}^{n}\left(k\frac{q_i}{r_{iB}} - k\frac{q_i}{r_{iA}}\right)$$

This relationship clearly shows that the work of the electric force does not depend on the trajectory travelled, but rather on the initial and final points. We say that the electric force is a conservative force, or one derived from a potential.

$$W = -q\sum_{i=1}^{n}\left(\frac{kq_i}{r_i}\right)_A^B = -q\sum_{i=1}^{n}(V_i)_A^B = -\,q\left[\sum_{i=1}^{n}V_i(B) - \sum_{i=1}^{n}V_i(A)\right] = -q$$



$$\boldsymbol{W}_{(\vec{F}_{ext})} = -\boldsymbol{q}[\Delta \boldsymbol{V}]$$

So the work done by the electric force on the charge is equal to the value of the charge q multiplied by the potential difference between points B and A.

Knowing that: q represents the value of the electric charge; **V(B)** the potential at point **B** and **V(A)** the potential at point **A**.

q V(A)= U(A) and **q V(B)= U(B)**

The quantity $U_p = qV$ is called **potential energy**.

U(A)= q V(A) is the potential energy of the charge q at point A

U(B)= q V(B) is the potential energy of the charge q at point B

According to the definition of the potential energy of charge q : $U_p = qV = q\frac{q_0}{4\pi\varepsilon_0 r} = \frac{qq_0}{4\pi\varepsilon_0 r}$

The work done by the electric force on the charge q is equal to minus the variation in the potential energy of the charge between the two points **B** and **A**. $W_{A\to B} = -\left(U(B) - U(A)\right)$

- Assuming we have a charge **Q** at point **M**, how much work is required to bring the charge q from point A (point A is at infinity: $\boldsymbol{A \to \infty}$ **et** $\boldsymbol{V(\infty) = 0}$ to point **B** with potential $\boldsymbol{V_B}$

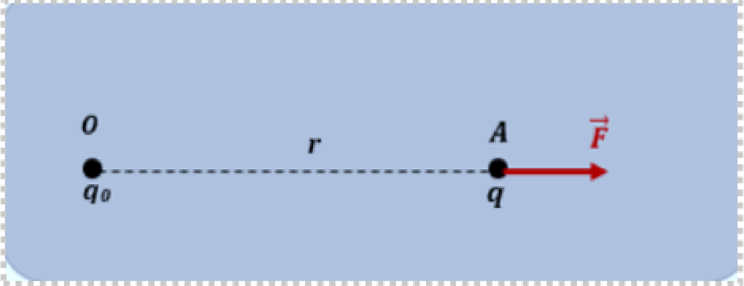
A force equal and inverse to the electric force due to the electric charge **Q** must be supplied. The work performed is equal to and inverses the work of the electric force:

$$\boldsymbol{W_{A\to B} = -W_{ext} = \big(U(B) - U(A)\big) = U_p(B) = qV_B \qquad \{U(A) = 0\}}$$

And equal to the potential energy of the charge q at point B , this is the definition of potential energy.

Example

Electrostatic energy: Let $\mathbf{q_0}$ and $\mathbf{q}$ be two positive point charges, separated by a distance r and $\vec{F}$ is the force exerted by $\mathbf{q_0}$ on $\mathbf{q}$ as shown in the diagram below.



To transport the load q from infinity to position A, a force **(-$\vec{F}$)** must be applied, where :

$$dW_{\infty\to A} = -\vec{F}.\vec{dl} \Rightarrow W_{\infty\to A} = \int_{\infty}^{A} -\vec{F}.\vec{dl}$$

$$W_{\infty\to A} = \int_{\infty}^{A} -q\vec{E}.\vec{dl} = -q\int_{\infty}^{A} \vec{E}.\vec{dl}$$

Where : $\int_{\infty}^{A} \vec{E}.\vec{dl} = -\int_{\infty}^{A} dV \qquad W_{\infty\to A} = q\left(V_A - \underbrace{V_\infty}_{0}\right)$

$$\boldsymbol{W_{\infty\to A} = qV_A}$$

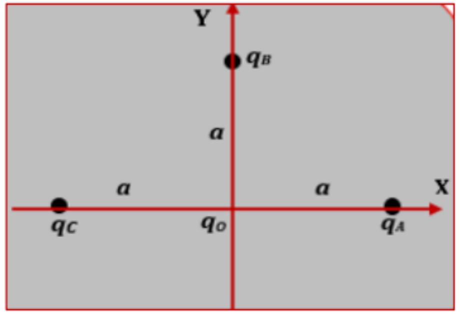
don't forget

electrostatic potential at infinity far from charges is zero

Exercice :

Four electric point charges q_0, q_A, q_B and q_C are placed at points (0,0); A(a,0); B (0, a) and C(-a,0) respectively as shown in the figure below.

Calculate the electrostatic potential energy of the charge q_O in the case where: $q_A = q = 2.10^{-9}$ C ; $q_B = -2q$; $q_C = 2q$; $q_O = -q$ and a = 5 cm.



The electrostatic energy of the charge q_O :

$$U_p = \frac{kq_Aq_0}{a} + \frac{kq_Bq_0}{a} + \frac{kq_Cq_0}{a}$$

$$U_p = \frac{kq_0}{a}(q_A + q_B + q_C)$$

$$U_p = \frac{kq_0}{a}(q - 2q + 2q) = -\frac{kq^2}{a}$$

A.N : $\boldsymbol{U_p = -7.2\ 10^{-7} joule}$

Note

- The unit of electrical potential is the **Volt (V)**
- The unit of potential energy is the **joule (J)** and is relatively large for small bodies, so a new unit is introduced: the **electron volt (eV)**, which is the kinetic energy of an electron accelerated by a potential of **1V**.

$$\mathbf{U = q\,V} \qquad\qquad \boldsymbol{1eV = 1.6 * 10^{-19} J}$$

- Definition of potential: from the relation **U=q V** we have $\boldsymbol{V = \frac{U}{q}}$ if **q=1 ⇒ V=U** we say that potential is potential energy per unit charge. (It agrees with the definition of electric field $\boldsymbol{\vec{E} = \frac{\vec{F}}{q}}$ the field is the force per unit charge).

Electric field flux: Gauss's theorem

- Gauss's theorem is based on a hypothetical closed surface called the Gaussian surface. This surface can take any shape, but the most useful is one that reproduces the symmetry of the problem to be solved. Therefore, the Gaussian surface is often a sphere, cylinder, or other symmetrical shape. It must always be a closed surface, so that a clear distinction can be made between points that lie inside the surface, on the surface, and outside it.
- Gauss's theorem establishes the relationship between the electric fields at points on a (closed) Gaussian surface and the net charge inside it.

Solid angle

Representation of a surface A surface S is decomposed into small elements dS; represented as vectors and directed in an arbitrary direction which will be preserved for all the elements of the surface S

(figure I.12) So : $\boldsymbol{S} = \iint |\boldsymbol{dS}|$

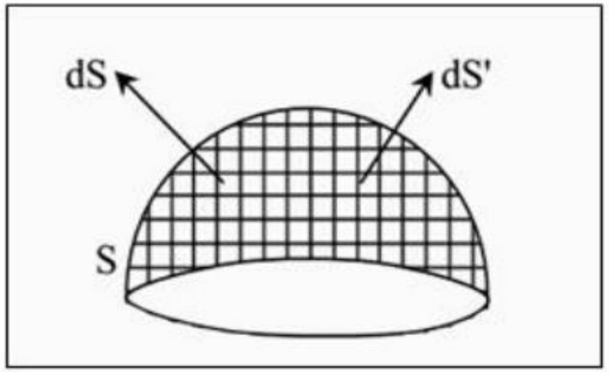
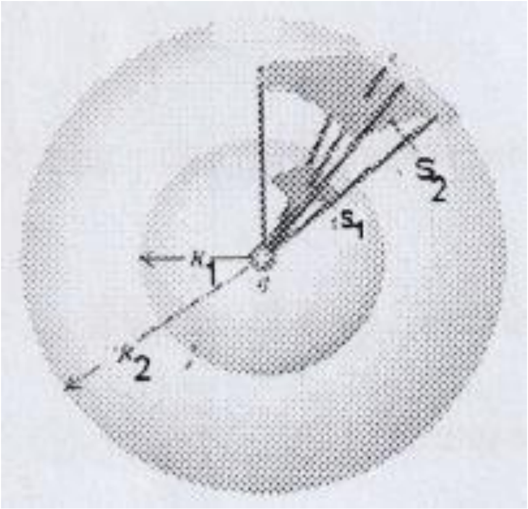
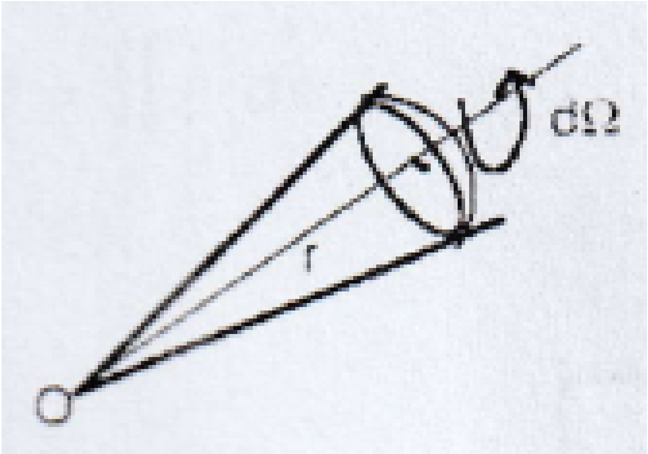


Figure I. 12. Representation of a surface

The solid angle Ω is defined as having the ratio between the interpreted surface S and the radius of the sphere





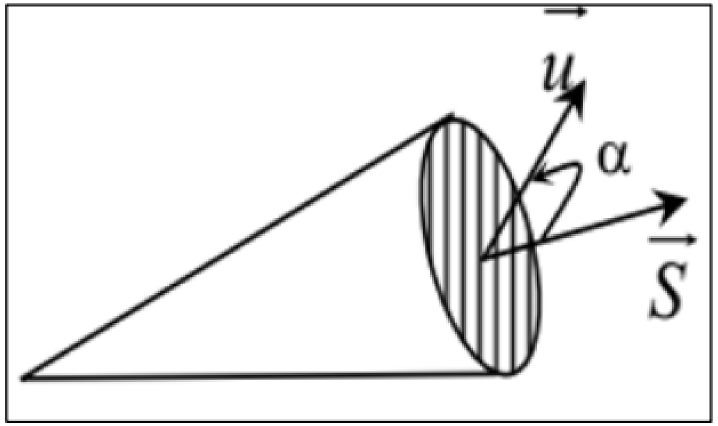


Figure I. 12. Solid angle

- The unit of solid angle is steradians (st)

For all space

$$\boldsymbol{\Omega = \frac{S}{R^2} = \frac{4\pi R^2}{R^2} = 4\pi\ st}$$

$$\boldsymbol{\Omega = \frac{S}{R^2} = \frac{S_1}{R_1^2} = \frac{S_2}{R_2^2}}$$

✓ *The elementary solid angle*: let **ds** be an infinitesimal surface. What is the solid angle from which we see the surface **ds** from a point **O** for example.

Let **M** be a mean point of the surface **ds**: $\boldsymbol{\overrightarrow{oM} = r\vec{u}}$

Let $\mathbf{ds_0}$ be the perpendicular surface on $\boldsymbol{\overrightarrow{oM}}$ and which contains **M**

$$\mathbf{d\Omega} = \boldsymbol{\frac{dS_0}{r^2}}$$

$dS_0 = dS\ cos\theta$ ($\boldsymbol{\theta}$ is the angle between the surface $\mathbf{ds}$ and $\mathbf{ds_0}$)

$$\mathrm{d}\Omega = \frac{dS_0}{r^2} = \frac{dS\ cos\theta}{r^2} = \frac{\overrightarrow{ds}.\vec{u}}{r^2}$$

The solid angle from which we see a surface **S** is: $\boldsymbol{\Omega} = \int \mathbf{d\Omega} = \iint \boldsymbol{\frac{\overrightarrow{ds}.\vec{u}}{r^2}}$

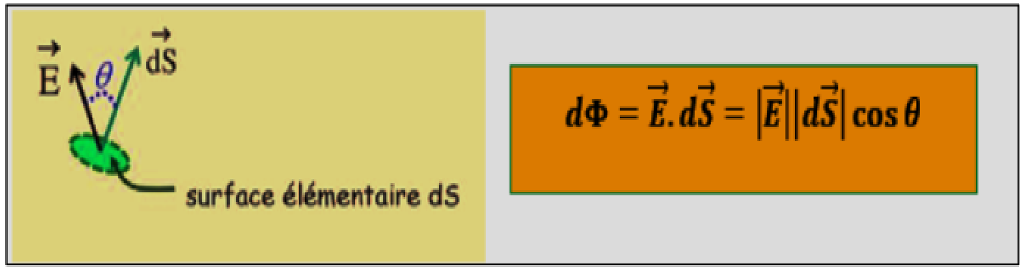
Concept of Flow

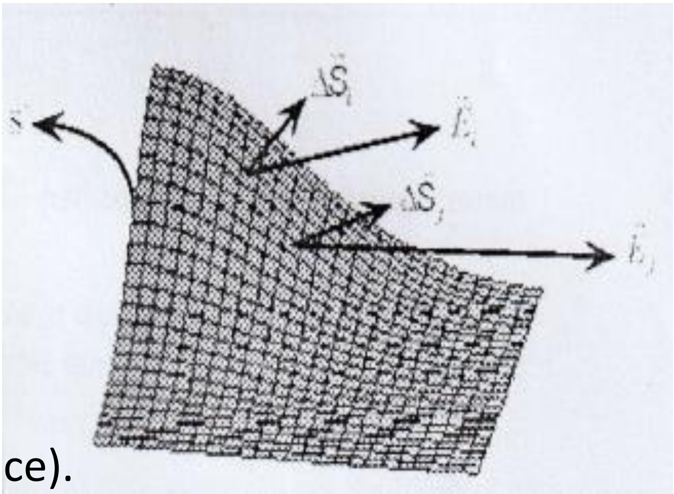
Let the vector field $\vec{b}$ be the field flux $\vec{b}$ through an infinitesimal surface **ds** and the quantity: $d\phi = \vec{b}.\overrightarrow{ds}$

The total flux through the surface S: $\boldsymbol{\phi} = \iint d\boldsymbol{\phi} = \iint \vec{\boldsymbol{b}}.\overrightarrow{\boldsymbol{ds}}$

The elementary flux of a field $\vec{E}$ through a surface dS

The elementary flux of a field $\vec{E}$ through a surface **dS** is the scalar quantity



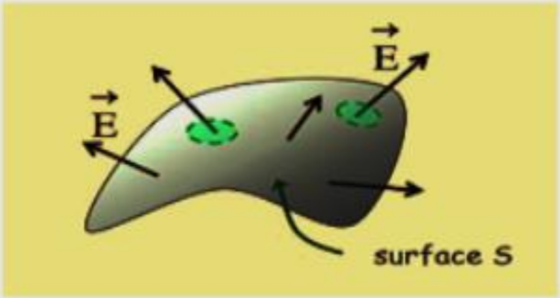


$\overrightarrow{dS}$ is called the elementary vector of the surface **S**

$\overrightarrow{dS}$ is always normal to S (it is perpendicular to the tangent to the surface).

$\overrightarrow{dS}$ is directed towards the outside of the volume limited by S.

The flux of a field $\vec{E}$ through a finite surface **S** is the scalar quantity:



$$\Phi = \iint_S \vec{E}.d\vec{S}$$

The flux of a vector field through a surface is a measure of the number of lines of this field crossing this surface

Application

Let **S** be a closed spherical surface, containing a positive charge **q** at its center **(see Figure 1.13).**

1. Calculate the flux of the field created by this charge at a point **M** at a distance **r** from **q**.

2. Deduce the value of the flux created by a set of charges $\mathbf{q_i}$ located inside the sphere.

Solution

Let us first draw the diagram to designate the vector $\vec{dS}$ and the field $\vec{E}$:

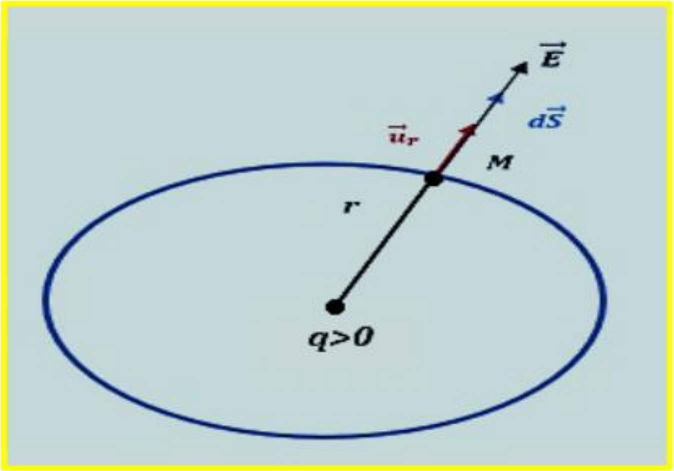


Figure 1.13

By definition, the field flux through the surface element **dS** is: $d\Phi = \vec{E}.d\vec{S} \Rightarrow \Phi = \oiint \vec{E}.d\vec{S}$

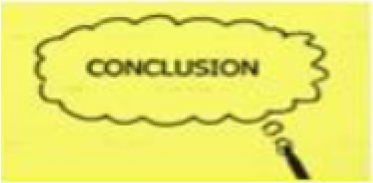
$\oiint$ *Represents an integral over a closed surface*

The electric field created by q in M is: $\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2}\vec{u}_r$

Then: $$\Phi = \oiint E.dS.\cos 0 = \oiint E.dS$$

$$\Phi = E\oiint dS = E.S \Longrightarrow \Phi = \frac{q}{4\pi\varepsilon_0 r^2}.4\pi r^2$$

$$\boldsymbol{\Phi} = \frac{\boldsymbol{q}}{\boldsymbol{\varepsilon_0}}$$



The flow of the electric field through a closed sphere depends only on the charge that creates this field.

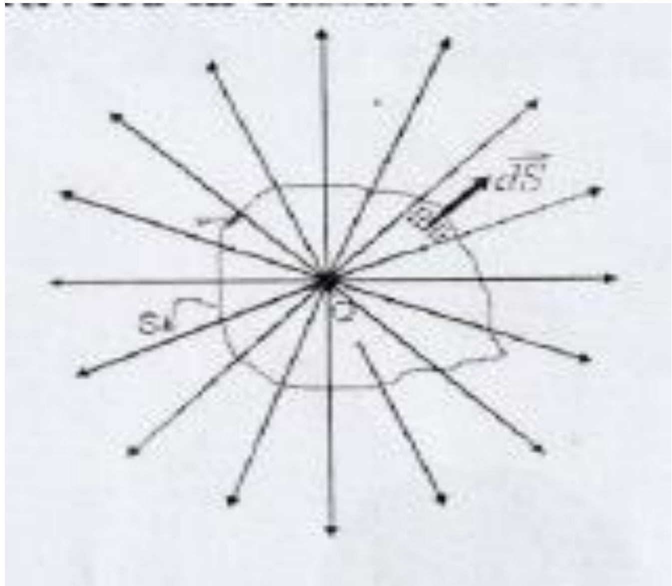
- By virtue of the superposition principle, this result is easily generalized to any set of charges $\mathbf{q_i}$.

$$\phi = \frac{\sum_i q_i}{\varepsilon_0}$$

Gauss's theorem

Given a charge **Q** inside a surface **S**, the flux of the field due to the charge **Q** through the surface **S** is: $\boldsymbol{\phi} = \frac{\boldsymbol{Q}}{\boldsymbol{\varepsilon_0}}$

$$\phi = \iint \vec{E}.\overrightarrow{ds} = \iint \frac{kQ}{r^2}\,\vec{u}.\overrightarrow{ds} = kQ\iint \frac{\vec{u}.\overrightarrow{ds}}{r^2} = kQ\iint d\Omega = kQ(\Omega)$$

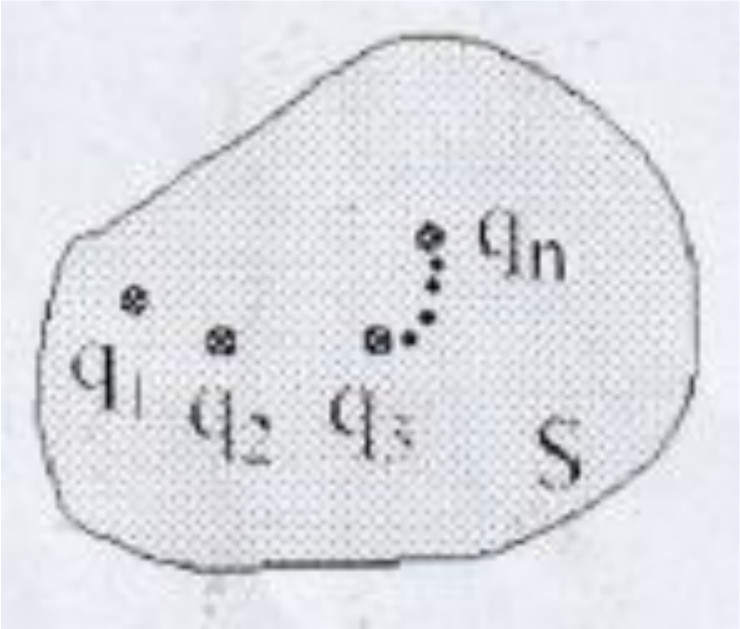


The surface **S** is closed, so the solid angle through which we see all of space is **Ω=4π**.
Hence the flux of the electric field through the closed surface **S** is:

$$\phi = kQ(\Omega) = kQ4\pi = \frac{1}{4\pi\varepsilon_0}Q4\pi = \frac{Q}{\varepsilon_0}$$

If several charges $\boldsymbol{Q_1}, \boldsymbol{Q_2}, \dots \dots, \boldsymbol{Q_n}$ are inside the closed surface the total flux of the electric field through the surface **S** is:

$$\boldsymbol{\phi} = \phi_1 + \phi_2 + \cdots + \phi_n = \frac{Q_1}{\varepsilon_0} + \frac{Q_2}{\varepsilon_0} + \cdots + \frac{Q_n}{\varepsilon_0} = \frac{1}{\varepsilon_0}\sum_{i=1}^{n} Q_i = \frac{\boldsymbol{Q_{int}}}{\boldsymbol{\varepsilon_0}}$$



$\boldsymbol{Q_{int}}$: represents the set of charges inside the surface

<u>Gauss's theorem</u> establishes the relationship between the net flux ϕ of an electric field through a closed surface and the net charge Q_{int} found inside this surface:

$$\boldsymbol{\phi} = \iint \vec{\boldsymbol{E}} . \overrightarrow{\boldsymbol{ds}} = \frac{\boldsymbol{Q_{int}}}{\boldsymbol{\varepsilon_0}}$$

Note:

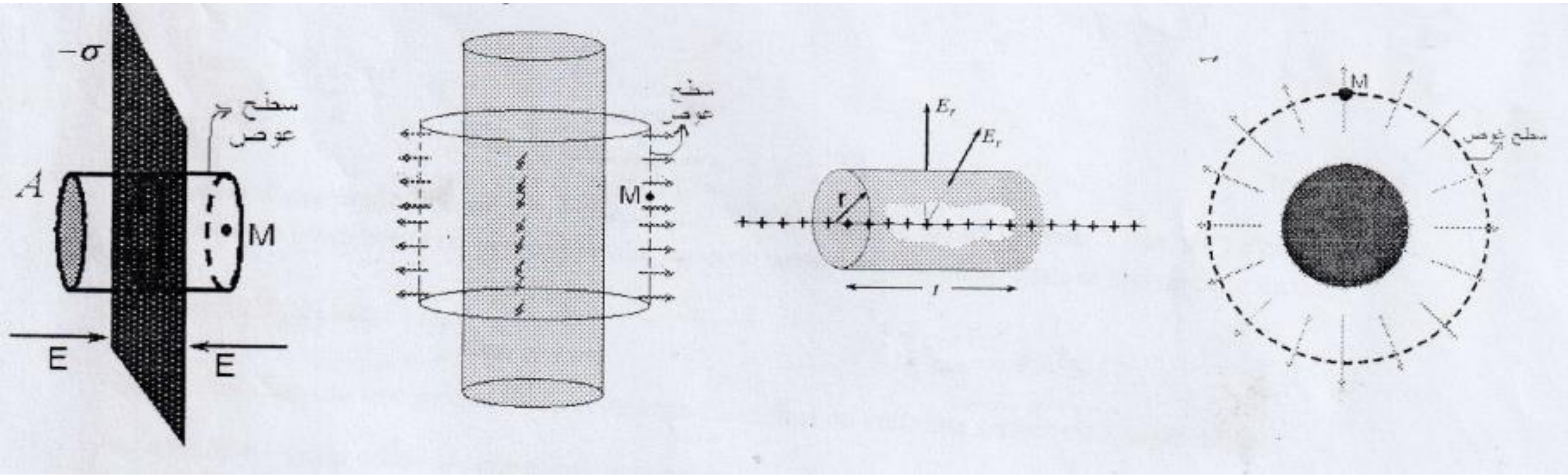
- If the charge $\mathbf{Q}$ is outside ***the closed surface***: $\boldsymbol{\phi} = \mathbf{0}$ because the field line entering through the surface will certainly exit.
- ***Gauss's theorem*** creates a link between the flow of the electrostatic field and the charges called source charges that create this field
- ***Gauss's theorem*** is valid for any closed surface when the source charges have high symmetries.

The steps to follow for the application of Gauss's theorem (calculation of a field $\vec{E}$ at a point M by applying Gauss's theorem):

1. We choose a closed surface (it is an imaginary surface) which contains the point at which we want to calculate the electric field. The Gaussian surface is chosen so that the field at the points of the surface is zero or constant or perpendicular or parallel to the surface, this is to facilitate the calculations.

   The most common cases are:

   - If the charge distribution is spherically symmetric, the Gaussian surface is chosen to be spherical in shape.
   - If the charge distribution is cylindrically symmetric, the Gaussian surface is chosen to be cylindrical in shape.
   - If the charge distribution is along an infinite plane, the Gaussian surface is chosen to be cylindrical in shape



2- Write the definition of the flow and calculate the integral $\boldsymbol{\phi} = \iint \vec{E}.\overrightarrow{ds}$

3- Calculate the total charge that exists inside the surface **S**. Then write that $\boldsymbol{\phi} = \iint \vec{E}.\overrightarrow{ds} = \frac{Q_{int}}{\varepsilon_0}$ and deduce the field $\vec{E}$.