Introduction to Physics 2: electrostatics and electrokinetics and magnetism

This course is intended for first-year students in Mathematical Sciences

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1. Introduction: Mathematics Revision

This lesson provides a mathematical revision covering fundamental concepts essential for physics and engineering applications. We will review the elements of length, surface area, and volume in different coordinate systems, as well as important mathematical operators and calculus techniques.

2. Elements of Length, Surface Area, and Volume in Different Coordinate Systems

We will analyze the fundamental differential elements in the three primary coordinate systems: Cartesian, cylindrical, and spherical. Understanding these elements is crucial for evaluating integrals in physics and engineering.

2.1 Cartesian Coordinate System

A Cartesian coordinate system is defined by an origin point O and three mutually perpendicular axes (Ox, Oy, Oz). The unit vectors along these axes are $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Any point M in space is represented by the position vector:

$$\vec{R} = \overrightarrow{OM} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

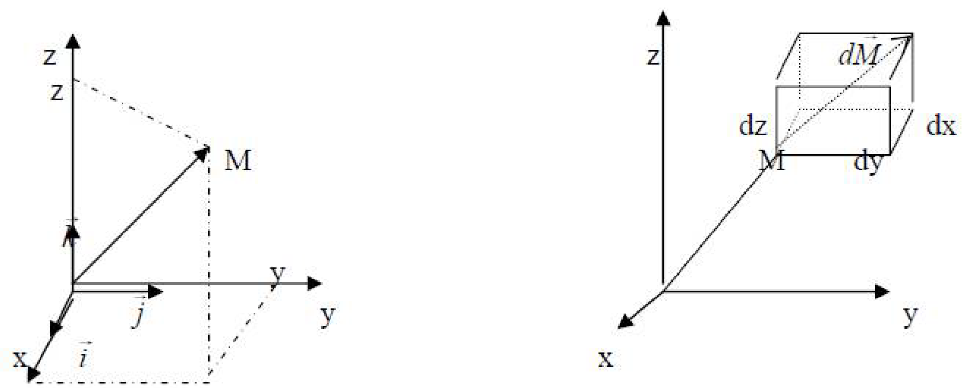

Figure 1: Cartesian basis (a) Position vector and (b) elementary displacement and volume

Example: Consider a straight-line motion along the x-axis where $x = 2t$, $y = 3$, and $z = 0$. The velocity vector is given by:

$$\mathbf{v} = \frac{d\vec{R}}{dt} = \frac{d}{dt}(2t\mathbf{i} + 3\mathbf{j} + 0\mathbf{k}) = 2\mathbf{i}$$

Differential Length Element: The differential displacement is given by:

$$d\overrightarrow{OM} = d\,\vec{l} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$$

Differential Surface Element: The surface element depends on the plane of integration:

$$dS_x = dydz, \quad dS_y = dxdz, \quad dS_z = dxdy$$

Differential Volume Element: The elementary volume is given by:

$$dV = dx\,dy\,dz$$

2.1.1 Cylindrical Coordinate System

In the cylindrical coordinate system (r, θ, z), a point is represented as:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

It should also be noted that we can write:

$$\mathbf{u}_p = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

and derive this vector with respect to θ:

We obtain:

$$d\mathbf{u}_p = d\theta\,(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}),$$

knowing that:

$$\cos\left(\theta + \frac{\pi}{2}\right) = -\sin\theta \quad \text{and} \quad \sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta.$$

Thus:

$\frac{d\mathbf{u}_p}{d\theta}$ can be obtained by rotating $\mathbf{u}_p$ by an angle of $\frac{\pi}{2}$, and we can write:

$$\frac{d\mathbf{u}_p}{d\theta} = \mathbf{u}_\theta.$$

The position vector **DM** is written as:

$$\mathbf{DM} = \rho\mathbf{u}_p + z\mathbf{k} = (x\mathbf{i} + y\mathbf{j}) + z\mathbf{k},$$

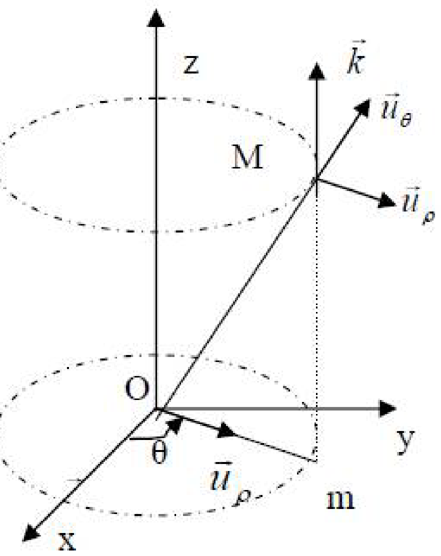

Figure 2: Cylindrical base

where x and y are the Cartesian coordinates of the point M in the Oxy plane, given by:

$$x = \rho\cos\theta, \quad y = \rho\sin\theta, \quad \text{and} \quad z = z.$$

The expression for the elementary displacement is:

$$d\mathbf{DM} = d\rho\,\mathbf{u}_p + \rho\,d\theta\,\mathbf{u}_\theta + dz\,\mathbf{k}.$$

The expression for the elementary surface is:

$$ds = \rho\,d\rho\,d\theta.$$

Example: Find the velocity vector for a particle moving in a circular path where $r = 2$, $\theta = t^2$, and $z = 4t$. The velocity components are:

$$v_r = \frac{dr}{dt} = 0, \quad v_\theta = r\frac{d\theta}{dt} = 2(2t), \quad v_z = \frac{dz}{dt} = 4$$

Thus, the velocity vector is:

$$\mathbf{v} = 0\mathbf{e_r} + 4t\mathbf{e}_\theta + 4\mathbf{e_z}$$

Differential Length Element:

$$d\vec{l} = dr\mathbf{e_r} + rd\theta\mathbf{e}_\theta + dz\mathbf{e_z}$$

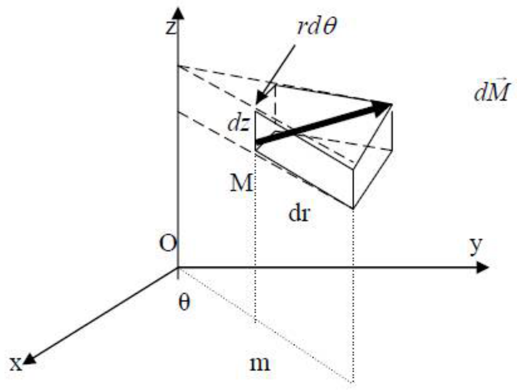

Figure 3: Cylindrical coordinates

Differential Surface Element:

$$dS_r = r d\theta dz, \quad dS_\theta = dr dz, \quad dS_z = r dr d\theta$$

Differential Volume Element:

$$dV = r dr d\theta dz$$

2.2 Spherical Coordinate System

In the spherical coordinate system (r, θ, ϕ), a point is represented as:

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r \cos\theta$$

The position vector of point M in spherical coordinates, meaning in the spherical basis, is written as:

$$\overrightarrow{OM} = r\overrightarrow{u_r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

From the figure, we can express x, y, z in terms of r, θ, ϕ:

$$X = OM \cos\varphi = r \sin\theta \cos\varphi,$$

$$Y = OM \sin\varphi = r \sin\theta \sin\varphi,$$

$$Z = OM \cos\theta = r \cos\theta.$$

Thus, we deduce:

$$\overrightarrow{u_r} = \sin\theta \cos\varphi\,\mathbf{i} + \sin\theta \sin\varphi\,\mathbf{j} + \cos\theta\,\mathbf{k}.$$

The unit vector $\overrightarrow{u_\varphi}$ at OM is written as:

$$\overrightarrow{u_\varphi} = \cos\varphi\,\mathbf{i} + \sin\varphi\,\mathbf{j}.$$

This vector $\overrightarrow{u_\varphi}$ can be obtained either by replacing φ with $\varphi + 2\pi$ or by differentiating $\overrightarrow{u_r}$ with respect to φ:

$$\overrightarrow{u_\varphi} = -\sin\varphi\,\mathbf{i} + \cos\varphi\,\mathbf{j}.$$

This basis vector can also be expressed as the derivative of $\overrightarrow{u_r}$ with respect to φ:

$$\overrightarrow{u_\varphi} = \frac{1}{\sin\theta}\frac{\partial\overrightarrow{u_r}}{\partial\varphi}.$$

The third basis vector in the spherical coordinate system is given by:

$$\overrightarrow{u_\theta} = \frac{\partial\overrightarrow{u_r}}{\partial\theta}.$$

2.2.1 Elementary Displacement:

$$d\overrightarrow{M} = d(r\overrightarrow{u_r}) = dr\overrightarrow{u_r} + rd\overrightarrow{u_r} + r\frac{\partial\overrightarrow{u_r}}{\partial\theta}d\theta + r\frac{\partial\overrightarrow{u_r}}{\partial\varphi}d\varphi.$$

$$= dr\overrightarrow{u_r} + rd\theta\overrightarrow{u_\theta} + r(\sin\theta d\varphi)\overrightarrow{u_\varphi}.$$

2.2.2 Elementary Surface and Volume:

$$dS = r^2\sin\theta\,d\theta\,d\varphi.$$

$$dV = r^2\sin\theta\,dr\,d\theta\,d\varphi.$$

Example: Find the length of an infinitesimal arc in spherical coordinates for a small change in θ while keeping r and ϕ constant.

$$dl = rd\theta$$

Differential Length Element:

$$d\,\overrightarrow{l} = dr\mathbf{e_r} + rd\theta\mathbf{e}_\theta + r\sin\theta d\phi\mathbf{e}_\phi$$

Differential Surface Element:

$$dS_r = r^2\sin\theta d\theta d\phi, \quad dS_\theta = r\sin\theta dr d\phi, \quad dS_\phi = r dr d\theta$$

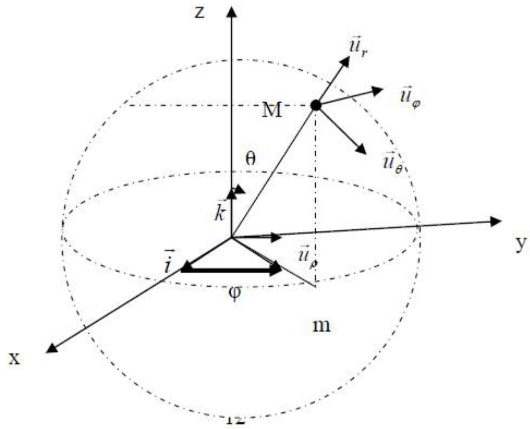

Figure 4: Spherical base

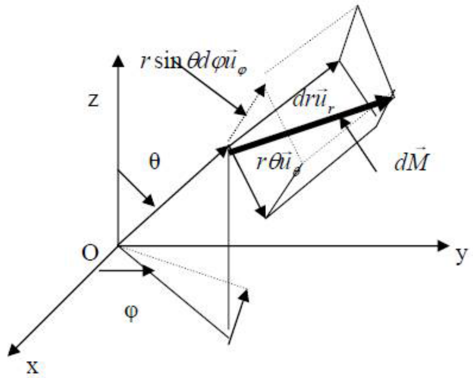

Figure 5: elementary volumes in spherical coordinates

Differential Volume Element:

$$dV = r^2 \sin\theta dr d\theta d\phi$$

2.2.3 Transformational relationships between different coordinates

From	To	Transformation Equations
Cartesian(x,y,z)	Spherical(r,θ,ϕ)	$r=\sqrt{x^2+y^2+z^2}$ $\theta=\arccos\left(\frac{z}{r}\right)$ $\phi=\arctan 2(y,x)$
Spherical(r,θ,ϕ)	Cartesian(x,y,z)	$x=r\sin\theta\cos\phi$ $y=r\sin\theta\sin\phi$ $z=r\cos\theta$
Cartesian(x,y,z)	Cylindrical(ρ,φ,z)	$\rho=\sqrt{x^2+y^2}$ $\varphi=\arctan 2(y,x)$ $z=z$
Cylindrical(ρ,φ,z)	Cartesian(x,y,z)	$x=\rho\cos\varphi$ $y=\rho\sin\varphi$ $z=z$
Spherical(r,θ,ϕ)	Cylindrical(ρ,φ,z)	$\rho=r\sin\theta$ $\varphi=\phi$ $z=r\cos\theta$
Cylindrical(ρ,φ,z)	Spherical(r,θ,ϕ)	$r=\sqrt{\rho^2+z^2}$ $\theta=\arctan\left(\frac{\rho}{z}\right)$ $\phi=\varphi$

2.2.4 Solid Angles

A solid angle $d\Omega$ in spherical coordinates is given by:

$$d\Omega=\sin\theta d\theta d\phi$$

The total solid angle in three-dimensional space is:

$$\Omega=\int_0^{2\pi}\int_0^{\pi}\sin\theta d\theta d\phi=4\pi$$

3. Operators in Vector Calculus

Example: Compute the gradient of the scalar function $f(x,y,z)=x^2+y^2+z^2$:

$$\nabla f=(2x,2y,2z)$$

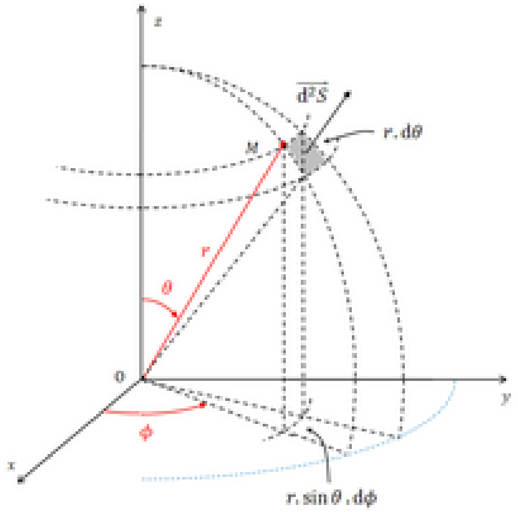

Figure 6: Solid Angles

Example: Compute the divergence of the vector field $\mathbf{A} = (x^2, y^2, z^2)$:

$$\nabla \cdot \mathbf{A} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2) = 2x + 2y + 2z$$

These examples reinforce the mathematical concepts necessary for physics applications.

3.1 Applications:

3.1.1 Calculate the perimeter of a circle C with radius R (simple integral).

Solution:

We have $dl = R\,d\theta$, hence:

$$C = \int_0^{2\pi} R\,d\theta = 2\pi R.$$

3.1.2 Calculate the area of a disk D with radius R (double surface integral).

We use the differential surface element $dS = dp\,p\,d\theta$, hence:

Solution:

$$D = \iint_S dp\,d\theta = \int_0^R \int_0^{2\pi} \rho\,d\rho\,d\theta.$$

Evaluating the integral:

$$D = \int_0^{2\pi} d\theta \int_0^R \rho\,d\rho = 2\pi \times \frac{R^2}{2} = \pi R^2.$$

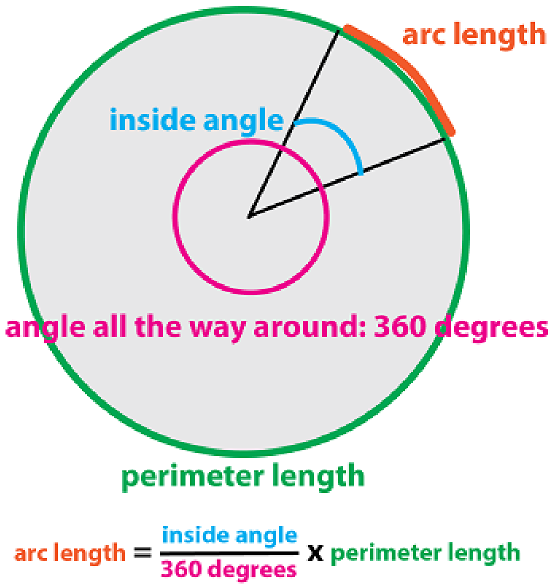

Figure 7: Perimeter of a circle

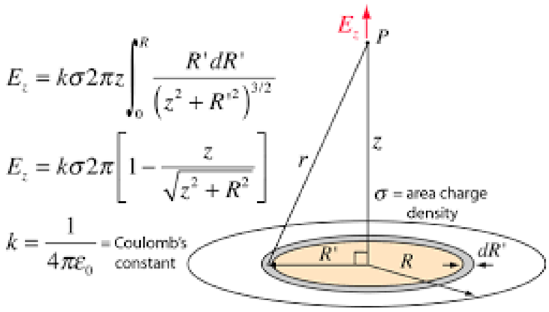

Figure 8: Area of a disk

3.1.3 Calculate the volume of a cylinder V with radius R and height H (triple volume integral).

We use the differential volume element $dV = dp\, p\, d\theta\, dz$, hence:

Solution:

$$V = \iiint_V dp\, d\theta\, dz = \int_0^R \rho\, d\rho \int_0^{2\pi} d\theta \int_0^H dz.$$

Evaluating the integral:

$$V = \int_0^H dz \int_0^{2\pi} d\theta \int_0^R \rho\, d\rho.$$

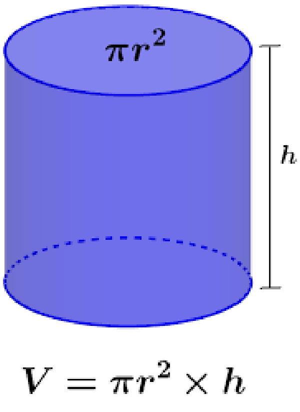

Figure 9: Volume of a cylinder

$$V = H \times 2\pi \times \frac{R^2}{2} = \pi R^2 H.$$

3.1.4 Calculate the surface area of a hemisphere D with radius R (excluding the horizontal disk) (double surface integral).

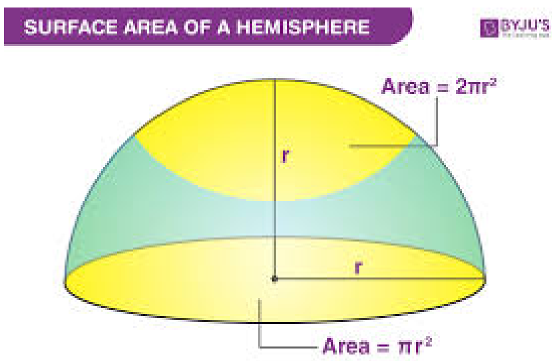

Figure 10: Surface area of a hemisphere

We use the differential surface element $dS = R^2 \sin\theta \, d\theta \, d\phi$, hence:
Solution:

$$D = \iint_S R^2 \sin\theta \, d\theta \, d\phi.$$

Evaluating the integral:

$$D = R^2 \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi.$$

$$D = R^2(-\cos\theta\Big|_0^{\pi}) \times (2\pi) = R^2(1+1) \times 2\pi = 2\pi R^2.$$

3.1.5 Calculate the volume of a sphere V with radius R (triple volume integral).

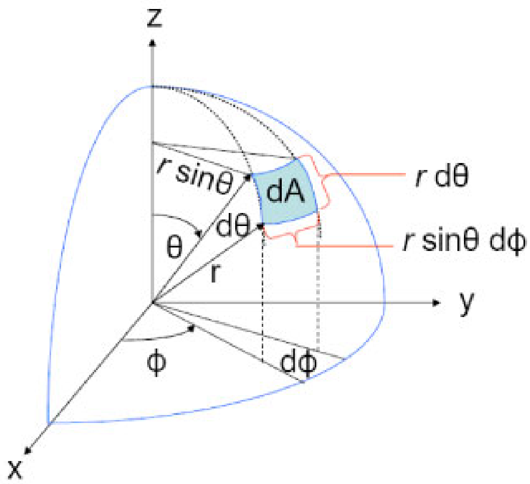

Figure 11: Volume of a sphere

We use the differential volume element $dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$, hence:
Solution:

$$V = \iiint_V r^2 \sin\theta\, dr\, d\theta\, d\phi.$$

Evaluating the integral:

$$V = \int_0^R r^2\, dr \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi.$$

$$V = \left(\frac{R^3}{3}\right) \times (-\cos\theta\Big|_0^{\pi}) \times 2\pi.$$

$$V = \frac{R^3}{3} \times (1+1) \times 2\pi = \frac{R^3}{3} \times 2 \times 2\pi = \frac{4}{3}\pi R^3.$$

4. Elementary Electric Charges

The electrical properties of matter originate at the atomic level. Matter is composed of atoms, each consisting of a nucleus around which a cloud of electrons orbits. These electrons repel each other but remain positioned around the nucleus. The nucleus consists of protons, which carry positive charges, and neutrons, which are neutral. The set of particles forming the nucleus is called nucleons.

Electrons and protons carry the same electric charge in absolute value, denoted by e. This electric charge, known as the elementary charge, has a value of:

$$e = 1.602 \times 10^{-19}\ \mathrm{C} \tag{4.1}$$

The electric force acting between positively charged protons and negatively charged electrons is responsible for the cohesion of atoms and molecules. The total charge of non-ionized atoms (i.e., those that have neither lost nor gained electrons) is zero.

An electric charge cannot take arbitrary values; it is always an integer multiple of the elementary charge:

$$Q = \pm ne \quad (\mathrm{C}) \tag{4.2}$$

This expresses the fundamental principle of charge quantization.

5. Electrification Experiment

When a glass rod is rubbed with a piece of silk and brought close to small pieces of paper, the paper pieces are attracted to the rod, indicating that electrons have been removed from the rod.

5.0.1 First Experiment

A small ball made of elderberry wood or polystyrene is suspended by a thread. A glass or amber rod, previously rubbed, is brought near the ball. Each rod first attracts and then repels the ball after contact (Figure 2.1a). However, if both rods are brought close to the ball simultaneously, nothing happens (Figure 2.1b).

5.0.2 Second Experiment

If two balls are electrified by contact with a rubbed glass rod, they repel each other. However, if each ball has touched different rubbed rods made of different materials, they attract each other.

These experiments demonstrate the existence of two states of electrification, corresponding to two types of electric charges: positive and negative. We recall the fundamental rule:

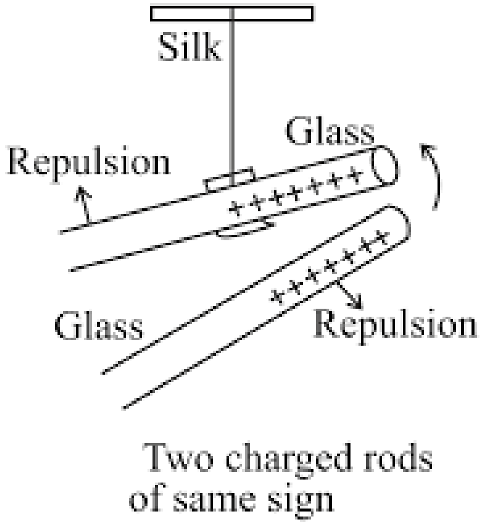

Figure 12: Electrification experiment

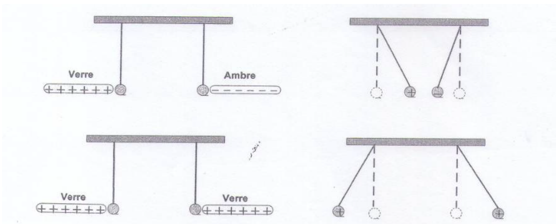

Figure 13: Electrification experience

Two bodies with the same type of charge repel each other, while bodies with opposite charges attract each other.

6. Coulomb's Law

Consider two point charges q_1 and q_2 placed in a vacuum. The first exerts a force proportional to q_1 on the second, and vice versa. The force between the two charges, known as electrostatic force, is proportional to the product of their charges:

$$\mathbf{F_e} = K\frac{q_1 q_2}{r^2}\mathbf{U_{12}} \tag{6.3}$$

where r is the distance between the two charges, and K is given by:

$$K = \frac{1}{4\pi\epsilon_0}, \quad \text{with} \quad \epsilon_0 = 8.854 \times 10^{-12} \text{ F/m} \tag{6.4}$$

Application: Calculate the force exerted by charge $q_1 = 3 \times 10^{-3}$ C on charge $q_2 = -5 \times 10^{-4}$ C separated by a distance of 20 mm.
Solution:

$$F = K\frac{q_1 q_2}{r^2} = 9 \times 10^9 \times \frac{(3 \times 10^{-3})(-5 \times 10^{-4})}{(20 \times 10^{-3})^2} \tag{6.5}$$

$$F = 33.75 \times 10^6 \text{ N} \tag{6.6}$$

7. Superposition Principle

Consider a charge q at point M in the presence of other charges q_i located at points M_i. The force $\mathbf{F}$ acting on charge q is:

$$\mathbf{F} = \sum_i K\frac{q q_i}{r_i^2}\mathbf{U_{iM}} \tag{7.7}$$

Application: Compute the resultant force acting on q_3 due to q_1 and q_2.

8. Electrostatic Field

An electric field exists at a point in space if a test charge q_0 at that point experiences an electrostatic force $\mathbf{F_e}$ such that:

$$\mathbf{E} = \frac{\mathbf{F_e}}{q_0} \tag{8.8}$$

8.1 Electric Field of a Point Charge

A charge Q at point O creates an electric field at any point M given by:

$$\mathbf{E}(M) = \frac{KQ}{r^2}\mathbf{U_{OM}} \tag{8.9}$$

9. Electrostatic Potential

9.1 Electric Potential

The work required to move a charge q_0 from point A to point B in an electric field is:

$$W_{AB} = q_0 \int_A^B \mathbf{E} \cdot d\mathbf{l} \tag{9.10}$$

The electric potential difference is defined as:

$$U_{AB} = V_B - V_A = -\int_A^B \mathbf{E} \cdot d\mathbf{l} \tag{9.11}$$

9.2 Potential of a Point Charge

For a charge Q at point O, the electric potential at a distance r is:

$$V = K\frac{Q}{r} \tag{9.12}$$

assuming $V = 0$ at infinity.

10. Electrostatic Potential of continuous charge distribution

The potential at a distance r from a charge q is given by:

$$V(r) = \frac{Kq}{r} \tag{10.13}$$

The potential remains constant on spheres of radius r centered around the charge q, which are called equipotential surfaces.

10.1 Potential Created by Multiple Distinct Point Charges

We start from the relationship between the electric field $\mathbf{E}$ and the potential V, more precisely from the differential relation:

$$dV = \mathbf{E}(M) \cdot \overrightarrow{dl}$$

For a set of charges q_i, concentrated at point M, and using the superposition theorem:

$$dV = -\mathbf{E}(M) \cdot \overrightarrow{dl} = -\sum_{i=1}^{N} [\mathbf{E}_i(M)] \cdot \overrightarrow{dl} = \sum_{i=1}^{N} [-\mathbf{E}_i(M)] \cdot \overrightarrow{dl} = \sum_{i=1}^{N} dV_i$$

The sum of a set of differentials being the differential of the sum:

$$dV = \sum_{i=1}^{N} dV_i = d\left(\sum_{i=1}^{N} V_i\right)$$

$$V(M) = \sum_{i=1}^{N} V_i = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{q_i}{r_i} \tag{10.14}$$

Where r_i is the distance between q_i and point M. The charge q_i can be positive or negative, which is why it must be taken with its sign.

Proof. Using the relationship between the electric field $\mathbf{E}$ and potential V, we obtain:

$$V(M) = \sum_i \frac{Kq_i}{r_i} \quad (10.15)$$

where r_i is the distance between charge q_i and point M. The charges q_i can be positive or negative.

□

10.2 Potential Due to a Continuous Charge Distribution

For a continuous charge distribution, integration is used:

$$V(M) = K\int \frac{dq}{r} \quad (10.16)$$

10.2.1 Volume Distribution

$$V(M) = K\iiint \frac{\rho dV}{r} \quad (10.17)$$

where ρ is the volume charge density.

10.2.2 Surface Distribution

$$V(M) = K\iint \frac{\sigma dS}{r} \quad (10.18)$$

where σ is the surface charge density.

10.2.3 Linear Distribution

$$V(M) = K\int \frac{\lambda dl}{r} \quad (10.19)$$

where λ is the linear charge density.

Here is the combined and corrected translation of the text from the images into English, rewritten in LaTeX:

—

c) If the distribution is linear:

$$V(M) = \int_C \frac{\lambda dl}{4\pi\epsilon_0 r} \quad (II-26)$$

where λ is the linear charge density.

10.3 Applications:

10.3.1 Field and Potential Created by a Ring:

A ring with center O and radius R carries a charge q uniformly distributed with a linear charge density $\lambda > 0$.

1. Calculate the potential created at point M on the axis Oy located at a distance y from O. 2. Deduce the electric field vector at point M.

10.3.2 Solution:

For the given point M, the quantities r, y, and R are constant. Starting from Figure I.8 and setting $K = \frac{1}{4\pi\epsilon_0}$, we can write:

$$dV = K\frac{dq}{r}$$

Integrating over the entire charge distribution:

$$\int dV = \frac{K}{r}\int dq \implies V = \frac{Kq}{r} + C_\infty$$

From the figure, we can see that:

$$r = \sqrt{R^2 + y^2}$$

After substituting K and $q = \lambda \cdot 2\pi R$, we arrive at the expression:

$$V = \frac{\lambda}{2\epsilon_0} \cdot \frac{R}{\sqrt{R^2 + y^2}} + C_\infty$$

Now, to determine the magnitude of the electric field E, we differentiate the expression for V with respect to y, using the relation:

$$\vec{E} = -\frac{dV}{dy} \implies \vec{E} = \frac{\lambda R}{2\epsilon_0} \cdot \frac{y}{(R^2 + y^2)^{3/2}}\vec{u}$$

10.3.3 Field and Potential Created by a Disk:

Consider a disk with center O and radius R, uniformly charged on its surface. The surface charge density is σ ($\sigma > 0$) Figure 14.

1. Calculate the electric field and the potential created by this distribution at a point M on the axis (Oz).

To do this, we decompose the disk into rings of radius ρ and width $d\rho$. Let P be a point on the ring and P' the symmetric point of P with respect to O.

First, let's examine the symmetry of the problem: the distribution has a revolution symmetry around the axis OZ. Any plane containing the axis OZ is a plane of even symmetry for the distribution. Therefore, the electric field $\vec{E}$ at a point M on the axis OZ is directed along $\vec{k}$:

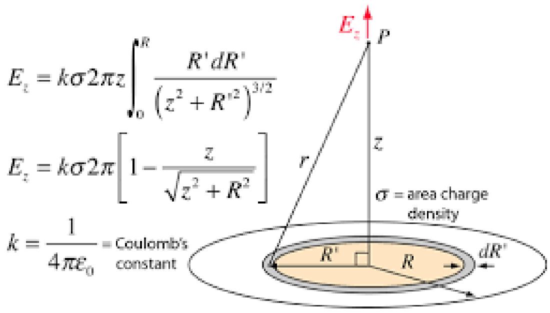

Figure 14: Potential Created by a Disk

$$\vec{E}(M) = E(0,0,Z) = E(Z)\vec{k}$$

A charge element $dq = \sigma\, ds$, centered at P (Figure II-9), creates at a point M on the axis of the disk an elementary field $d\vec{E}$ given by:

$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{dq}{r^2}\vec{u}$$

where $ds = \rho\, d\rho\, d\theta$ and $r = \sqrt{\rho^2 + Z^2}$.

The charged disk has a revolution symmetry around its axis, for example, the axis ZZ, so the field is directed along this axis. We have:

$$d\vec{E} = \frac{\sigma}{4\pi\epsilon_0}\frac{\rho\, d\rho\, d\theta}{\rho^2 + Z^2}\vec{u}$$

$$d\vec{E}_Z = d\vec{E}\cos\alpha = \frac{\sigma}{4\pi\epsilon_0}\frac{\rho\, d\rho\, d\theta \cos\alpha}{\rho^2 + Z^2}\vec{k}$$

The total electric field at point M is obtained by integrating over the entire disk:

$$E(M) = \frac{\sigma}{4\pi\epsilon_0}\int_0^R \int_0^{2\pi} \frac{\rho\, d\rho\, d\theta}{\rho^2 + Z^2}\cos\alpha$$

Since $\cos\alpha = \frac{Z}{r}$, we have:

$$E(M) = \frac{\sigma}{4\pi\epsilon_0}\int_0^R \int_0^{2\pi} \frac{\rho\, d\rho\, d\theta}{\rho^2 + Z^2}\frac{Z}{\sqrt{\rho^2 + Z^2}}$$

$$E(M) = \frac{\sigma}{2\epsilon_0}\left(\frac{Z}{|Z|} - \frac{Z}{\sqrt{R^2 + Z^2}}\right)\vec{k}$$

When Z is large, the field weakens. However, when $R \gg Z$, and M is very close to the disk, the field becomes:

$$E(M) = \pm\frac{\sigma}{2\epsilon_0}\vec{k}$$

The potential at point M is derived from the field by integration:

$$\vec{E}(M) = -\nabla V(M) = -\frac{dV}{dZ}\vec{k}$$

Thus,

$$V = \frac{\sigma}{2\epsilon_0}\left(Z - \sqrt{R^2 + Z^2}\right)$$

11. Electrostatic Energy

11.1 Energy of a Point Charge in an Electric Field

The work done to move a charge q from A to B in an electric field $\mathbf{E}$ is:

$$W_{AB} = q(V_A - V_B) \tag{11.20}$$

11.2 Energy of a System of Point Charges

The total electrostatic energy W of a system of point charges is given by:

$$W = \frac{1}{2}\sum_i q_i V_i \tag{11.21}$$

11.3 Energy of a Continuous Charge Distribution

$$W = \frac{1}{2}\iiint \rho V \, dV \tag{11.22}$$

12. Electric Dipole

12.1 Definition

An electric dipole consists of two equal and opposite charges separated by a small distance. The dipole moment $\mathbf{p}$ is given by:

$$\mathbf{p} = q\mathbf{a} \tag{12.23}$$

12.2 Potential Created by a Dipole

The potential at a point P due to a dipole is:

$$V = \frac{Kp\cos\theta}{r^2} \tag{12.24}$$

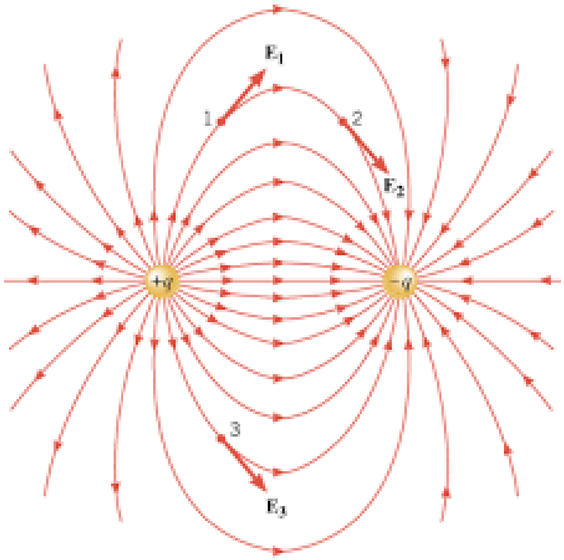

Figure 15: Electric Field of a Dipole

12.3 Electric Field of a Dipole

The radial and angular components of the electric field are:

$$E_r = \frac{Kp(2\cos\theta)}{r^3} \tag{12.25}$$

$$E_\theta = \frac{Kp\sin\theta}{r^3} \tag{12.26}$$

13. Gauss's Theorem

13.0.1 Objectives:

To be able to quickly provide the expression for the electrostatic field created by a source with a high degree of symmetry.

13.1 Prerequisites:

By drawing two networks of lines on any surface, the surface is decomposed into smaller areas bounded by these lines (see the figure).

If the lines are very numerous and evenly distributed, each of these areas has a very small surface. Consider a point P on the surface S. If the number of lines increases indefinitely, the small area around the point P decreases and tends to approach the portion of the tangent plane at P to the surface S. In the limit, its area dS becomes infinitely small and coincides with a portion of the plane. It is called the surface element surrounding the

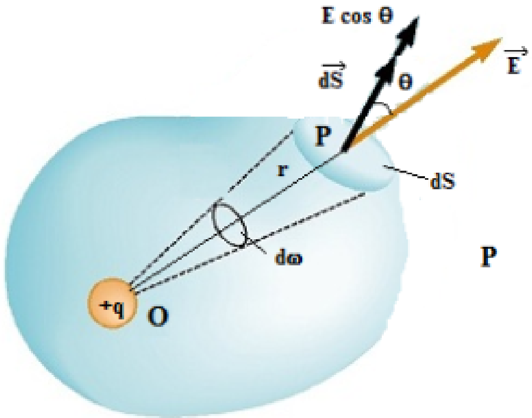

Figure 16: Gauss's Theorem

point P. Thus, any surface S can be considered as the juxtaposition of an infinite number of surface elements dS.

13.2 Surface Element:

Consider a surface element of area dS.

We associate with this element a vector called the "normal" vector $\vec{dS}$, defined as follows:

- Its origin is a point P on the element. - Its direction is normal to the surface. - Its magnitude is equal to the area dS.

The vector $\vec{dS}$ is therefore infinitely small. Its orientation is chosen arbitrarily (outward for closed surfaces). To orient $\vec{dS}$, one can also use the "corkscrew" rule. The contour C bounding the surface is oriented by arbitrarily choosing a positive direction of traversal. The vector $\vec{dS}$ is oriented according to the progression of a corkscrew turning in the direction of C.

13.3 Gauss's Theorem:

Gauss's theorem relies on the concept of the flux of a vector. This new concept is introduced in what follows. However, a good mastery of elementary vector operations, particularly the dot product, is necessary.

14. Concept of Flux

Let $\vec{E}$ denote the electric field vector at point P. Let $d\vec{S}$ be the surface element surrounding this point and the corresponding vector.

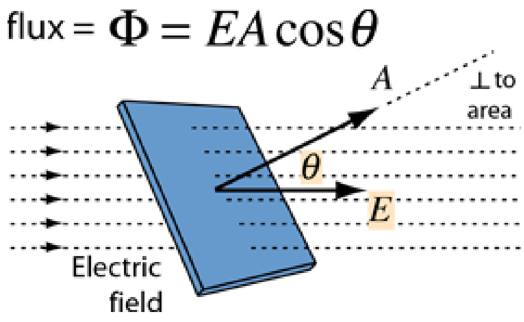

Figure 17: Concept of Flux

14.1 Definition:

By definition, the flux $d\Phi$ of the electric field $\vec{E}$ through the considered surface element $d\vec{S}$ is equal to the dot product:

$$d\Phi = \vec{E} \cdot d\vec{S}$$

This is called the elementary flux to indicate that it is relative to a surface element.

14.2 Sign of the Flux:

The sign of the flux depends on the direction of the vector $d\vec{S}$. Consider, for example, the two opposite vectors $d\vec{S}$ and $-d\vec{S}$, associated with a surface element.

If the vector $d\vec{S}$ makes an angle θ with the electric field $\vec{E}$, the vector $-d\vec{S}$ makes an angle $\pi - \theta$, and since $\cos(\pi - \theta) = -\cos(\theta)$, the dot products $\vec{E} \cdot d\vec{S}$ and $\vec{E} \cdot (-d\vec{S})$ have opposite values.

To calculate the algebraic flux of the electric field $\vec{E}$ through a surface element $d\vec{S}$, it is therefore necessary to choose, in accordance with the concept of positive flux, the direction of the vector $d\vec{S}$ associated with this element.

The flux of an electric field $\mathbf{E}$ through a closed surface S is given by:

$$\Phi = \oint \mathbf{E} \cdot d\mathbf{S} = \frac{Q_{\text{enc}}}{\epsilon_0} \tag{14.27}$$

where Q_{enc} is the total charge enclosed by the surface.

14.3 Flux Calculation

Consider the surface elements composing the surface S. For each of them, the elementary flux $d\Phi$ is calculated. The total flux Φ of the electric field through the surface S is obtained by summing the elementary fluxes. This sum is conventionally denoted by the notation:

$$\Phi = \iint_S \vec{E} \cdot d\vec{S}$$

To perform this calculation, the vectors $d\vec{S}$ associated with the surface elements are all oriented on the same side of the surface S.

14.4 Flux of a Point Charge

Let P be a point belonging to the surface element $d\vec{S}$. The field $\vec{E}$ created at P by the charge q is directed along $\vec{r}$ and oriented from q to P if $q > 0$; its magnitude is:

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$$

where r is the distance between q and P.

The elementary flux of this electric field through the surface element $d\vec{S}$ surrounding the point P is:

$$d\Phi = \vec{E}\cdot d\vec{S} = E\,dS\,\cos\theta$$

where θ is the angle between $\vec{E}$ and $d\vec{S}$.

However, $d\Omega = \frac{dS\cos\theta}{r^2}$ is the solid angle $d\Omega$ subtended by the contour of $d\vec{S}$ as seen from q (geometrically, it is a cone with vertex at q that is tangent to the surface element $d\vec{S}$).

Gauss's Theorem

Gauss's theorem is stated as follows:

14.5 Theorem:

The flux of the electric field through any closed surface S is equal to $\frac{1}{\epsilon_0}$ times the total algebraic charge contained within the volume bounded by this surface:

$$\Phi = \oint_S \vec{E}\cdot d\vec{S} = \frac{Q_{\text{int}}}{\epsilon_0}$$

14.5.1 Case of Charges Outside a Closed Surface S:

The elements $d\vec{S}_1$ and $d\vec{S}_2$ are seen under the same solid angle $d\Omega$ in absolute value. However, $\vec{E}_1$ and $d\vec{S}_1$ are collinear, while $\vec{E}_2$ and $d\vec{S}_2$ are opposite. Therefore, the fluxes $d\Phi_1$ and $d\Phi_2$ have opposite signs. The elementary fluxes cancel out in pairs, and the total flux of the field $\vec{E}$ created by the charge q outside the closed surface is zero.

14.5.2 Case of Charges Inside a Closed Surface S:

The sum of the elementary fluxes will not be zero because all the surface element vectors are, for example, all oriented outward from the surface. The total flux sent by q through S will be the sum of the elementary fluxes:

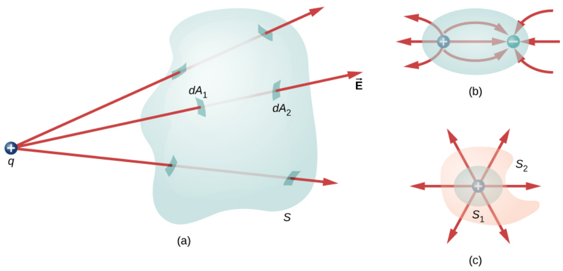

Figure 18: Case of Charges Inside a Closed Surface

$$\Phi = \oint_S \vec{E} \cdot d\vec{S} = \frac{q}{\epsilon_0}$$

The unit of solid angle is the angle that subtends a unit area on a sphere of unit radius. Since the surface area of a unit sphere is 4π, the solid angle that subtends the entire space from a point is 4π. The sum extends over the entire space, i.e., 4π.

If there are N charges q_i inside S:

$$\Phi = \oint_S \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} \sum_{i=1}^{N} q_i$$

By defining:

$$Q_{\text{int}} = \sum_{i=1}^{N} q_i$$

The flux of $\vec{E}$ through a closed surface is equal to $\frac{1}{\epsilon_0}$ times the sum of the interior charges, regardless of the exterior charges.

14.6 Application of Gauss's Theorem:

The application of Gauss's theorem is very useful in problems that exhibit a high degree of symmetry. Verify this property with the simple example of the field $\vec{E}$ created by a point charge q.

The following two simulations will allow you to apply Gauss's theorem in the case of two uniformly charged structures with axes of symmetry. You can demonstrate the simplicity with which Gauss's theorem allows the calculation of the electrostatic field created by these two charge distributions, which exhibit a high degree of symmetry.

14.7 Methodology

Gauss's theorem is a valuable tool for determining the electric field $\vec{E}$ at any point P when the source charges exhibit high symmetry. The steps for calculating $\vec{E}$ are as follows:

1. Determine the orientation of the field using symmetry considerations.

2. Choose a "Gaussian surface" S (imaginary, with no physical reality): - Passing through the point of interest P. - Most suitable for simplifying the expression of the flux of $\vec{E}$ through it. - Possessing the same symmetry properties as the source. - Not coinciding with a charged material surface.

3. Express the flux Φ through the closed surface S.

4. Determine the total charge Q_{int} enclosed within the volume bounded by S.

5. Apply Gauss's theorem:

$$\Phi = \oint_S \vec{E} \cdot d\vec{S} = \frac{Q_{\text{int}}}{\epsilon_0}$$

If the Gaussian surface is well chosen, the left-hand side of the equation is a simple function of $\vec{E}$ and the distance r. Thus, the expression for the field $\vec{E}$ can be obtained as a function of the distance r and the source charges.

14.7.1 Case of axial symmetry

A source charge distribution has axial symmetry if the charge density at a point is a function only of the distance from an axis.

Cylindrical Charge Cloud with Volume Density $\rho = f(r)$:

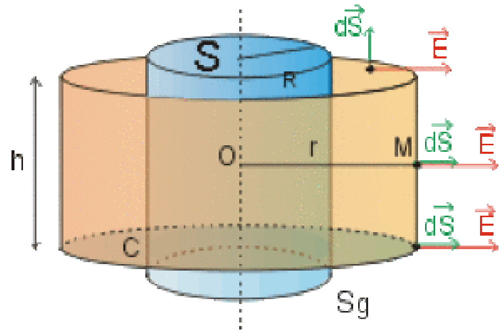

Figure 19: Case of axial symmetry

1. By symmetry, the electric field is radial (far from the edges of the source).

2. The most suitable Gaussian surface is a cylinder aligned with Δ and passing through the point of interest M (which can be inside or outside the source).

Point of Interest Outside the Source:

On the right sections S of the Gaussian cylinder S_g, the vectors $\vec{E}$ and $d\vec{S}$ are orthogonal, so the flux of $\vec{E}$ through S is zero. The flux of $\vec{E}$ through the closed Gaussian surface is reduced to the flux through the lateral surface.

$$\Phi = \iint_{S_g} \vec{E} \cdot d\vec{S} = \iint_{S_{\text{lat}}} \vec{E} \cdot d\vec{S} + \iint_{S} \vec{E} \cdot d\vec{S} = \iint_{S_{\text{lat}}} \vec{E} \cdot d\vec{S}$$

On the lateral surface, $\vec{E}$ and $d\vec{S}$ are collinear, so the flux reduces to:

$$\Phi = \iint_{S_{\text{lat}}} \vec{E} \cdot d\vec{S} = \iint_{S_{\text{lat}}} E \cdot dS$$

E is the same at every point on S_g and can therefore be taken out of the integral:

$$\Phi = \iint_{S_{\text{lat}}} E \cdot dS = E \iint_{S_{\text{lat}}} dS = ES_{\text{lat}}$$

The lateral surface area of the Gaussian surface is equal to $2\pi rh$:

$$\Phi = E \cdot 2\pi rh$$

Now, we only need to evaluate the charge Q_i inside the volume delimited by S_g according to the considered distribution. Gauss's theorem allows us to determine the field E by writing:

$$\Phi = E \cdot 2\pi rh = \frac{Q_i}{\epsilon_0}$$

15. Capacitance

The capacitance C of a conductor is defined as the ratio of the charge Q stored on the conductor to the electric potential V of the conductor:

$$C = \frac{Q}{V}$$

The unit of capacitance is the farad (F), where $1F = 1C/V$ (one farad equals one coulomb per volt).

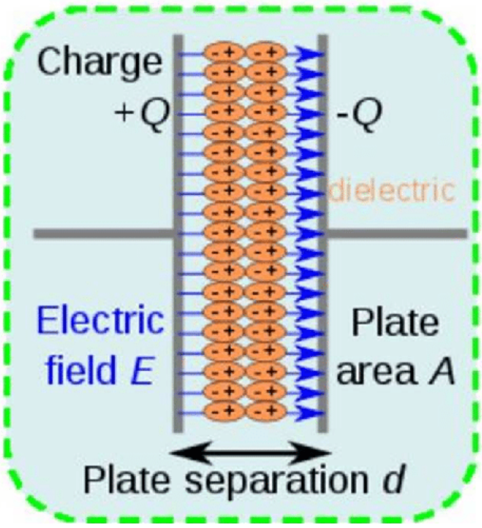

Figure 20: The Capacitance

$$C = \frac{Q}{V} \tag{15.28}$$

For a few common conductor geometries, the capacitance is given by:

1. Isolated Spherical Conductor of Radius R:

   $$C = 4\pi\varepsilon_0 R$$

   where ε_0 is the permittivity of free space.

2. Parallel Plate Capacitor with Plate Area A and Plate Separation d:

   $$C = \frac{\varepsilon A}{d}$$

   where $\varepsilon = \varepsilon_0\varepsilon_r$ is the permittivity of the dielectric medium.

3. Cylindrical Capacitor with Inner Radius R_1, Outer Radius R_2, and Length L:

   $$C = \frac{2\pi\varepsilon L}{\ln(R_2/R_1)}$$

The capacitance C of a conductor is defined as:
For a spherical conductor:

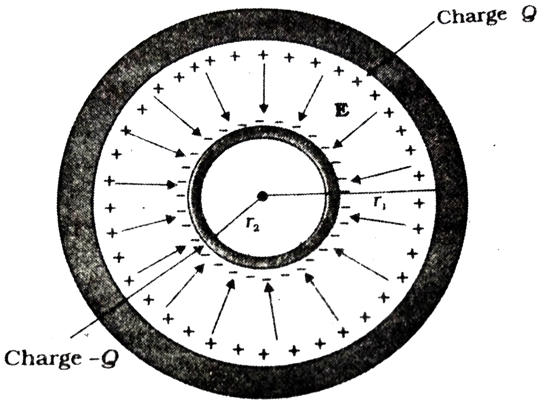

Figure 21: Spherical conductor

$$C = 4\pi\epsilon_0 R \tag{15.29}$$

For a cylindrical capacitor:

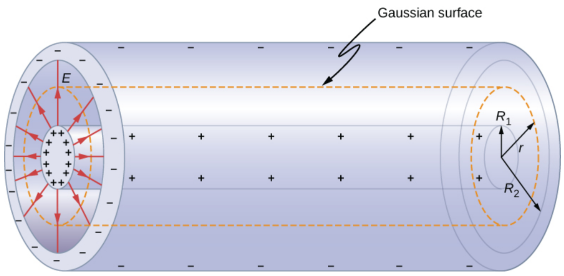

Figure 22: Cylindrical capacitor

$$C = \frac{2\pi\epsilon_0 h}{\ln(R_2/R_1)} \tag{15.30}$$

For a parallel plate capacitor:

$$C = \frac{\epsilon_0 S}{d} \tag{15.31}$$

15.1 Energy Stored in a Capacitor

The energy stored in a capacitor is given by:

$$W = \frac{1}{2} C V^2 \tag{15.32}$$