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Algebra II, Worksheet 3

Exercise No. 1 :
1. Let E, F, G be three $\mathbb{K}-$vector spaces. Consider two linear mappings $f \in \mathcal{L}_{\mathbb{K}}(E, F)$ and $g \in \mathcal{L}_{\mathbb{K}}(F, G)$. Show that the composition $g \circ f$ is a linear mapping, that is, $g \circ f \in \mathcal{L}_{\mathbb{K}}(E, G)$.
2. Let $B = \{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$ be the canonical basis of $\mathbb{R}^3$. Consider the linear mapping $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by

$$f(e_1) = \frac{1}{3}(-e_1 + 2e_2 + 2e_3), f(e_2) = \frac{1}{3}(2e_1 - e_2 + 2e_3), f(e_3) = \frac{1}{3}(2e_1 + 2e_2 - e_3).$$

(a) Determine the explicit expression of the linear mapping f.
(b) Consider the subspaces of $\mathbb{R}^3$ defined by

$$F_1 = \left\{v \in \mathbb{R}^3 : f(v) = -v\right\} \text{ and } F_2 = \left\{v \in \mathbb{R}^3 : f(v) = v\right\}.$$

(i) Show that F_1 and F_2 are vector subspaces of $\mathbb{R}^3$.
(j) Verify that $v_1 = e_1 - e_2$ and $v_2 = e_1 - e_3$ belong to F_1, and that $v_3 = e_1 + e_2 + e_3$ belongs to F_2.
(k) Show that the family $B' = \{v_1 = e_1 - e_2, v_2 = e_1 - e_3, v_3 = e_1 + e_2 + e_3\}$ forms a new basis of $\mathbb{R}^3$.
(l) Compute $f^2 = f \circ f$ and deduce that f is bijective, and determine its inverse f^{-1}.

Exercise No. 2 : Let $\mathbb{R}_2[X] = \{P \in \mathbb{R}[X] : \deg(P) \leqslant 2\}$ be the vector space of polynomials with real coefficients of degree less than or equal to 2. We define the mapping f on $\mathbb{R}_2[X]$ by

$$\forall P \in \mathbb{R}_2[X] : f(P) = -\frac{(X+1)^2}{2}P^{(2)} + (X+1)P^{(1)},$$

where $P^{(1)}$ and $P^{(2)}$ denote the first and second derivatives of P, respectively.
1. Prove that f is an endomorphism of $\mathbb{R}_2[X]$ and show that it satisfies $f \circ f = f$.
2. Construct a basis for each of the vector subspaces $\ker(f)$ and $Im(f)$. Deduce the rank of f.

Exercise No. 3 : Let E and F be two vector spaces over a field $\mathbb{K}$, and let $f \in \mathcal{L}_{\mathbb{K}}(E, F)$ be a linear mapping. Consider a family of vectors $L = \{x_1, x_2, ..., x_n\}$ in E. Prove the following :
1. If $f(L)$ is linearly independent in F, then L is also linearly independent in E.
2. If L is linearly independent in E and f is injective, then $f(L)$ is linearly independent in F.
3. If L is linearly dependent in E, then $f(L)$ is also linearly dependent in F.

Exercise No. 4 : (Supplementary Exercise) Let E be a vector space over a field $\mathbb{K}$. Consider an endomorphism $f : E \longrightarrow E$ such that $f^2 = f \circ f = Id_E$. Define the subspaces $F_1 = \ker(f - Id_E)$ and $F_2 = \ker(f + Id_E)$.
1. Compute $f(x_1)$ and $f(x_2)$ for any $x_1 \in F_1$ and any $x_2 \in F_2$.
2. Prove that $E = F_1 \oplus F_2$.

Exercise No. 5 : (Supplementary Exercise) Let $\mathbb{R}_2[X] = \{P \in \mathbb{R}[X] : \deg(P) \leqslant 2\}$ be the vector space of polynomials with real coefficients of degree less than or equal to 2. We define the mapping $f : \mathbb{R}_2[X] \longrightarrow \mathbb{R}$ by

$$\forall P = aX^2 + bX + c \in \mathbb{R}_2[X] : f(P) = a + b\sqrt{2}$$

1. Prove that f is a linear mapping.
2. Determine $\ker(f)$ the kernel of f and show that $L = \left\{P_1 = 1, P_2 = \sqrt{2}X^2 - X\right\}$ forms a basis of $\ker(f)$.
3. Prove that $B' = \left\{Q_1 = 1, Q_2 = \sqrt{2}X^2 - X, Q_3 = \frac{1}{\sqrt{2}}X\right\}$ is a basis of $\mathbb{R}_2[X]$.
4. Consider the linear mapping $g : \mathbb{R}_2[X] \longrightarrow \mathbb{R}_2[X]$ defined by

$$g(Q_1) = g(Q_2) = 0, g(Q_3) = Q_3$$

(a) Show that $g \circ g = g$ and determine $\ker(g)$ the kernel of g.