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Algebra II, Worksheet 2

Exercise n°1 :
(1) Find the value of $a \in \mathbb{R}$ for which the vector $x = (1, 1, 2)$ in $\mathbb{R}^3$ can be written as a linear combination of the vectors $x_1 = (1, -1, 1)$ and $x_2 = (0, 1, a)$.
(2) Let E be a $\mathbb{K}-$vector space. Suppose the family of vectors $A = \{v_1, v_2, v_3, v_4\}$ in E is linearly independent. Show that the family of vectors $B = \{e_1 = v_1, e_2 = v_1 + v_2, e_3 = v_1 + v_2 + v_3, e_4 = v_1 + v_2 + v_3 + v_4\}$ is also linearly independent.
(3) Let E be a $\mathbb{K}-$vector space, and let F_1 and F_2 be two subspaces of E. Their sum is defined as

$$F_1 + F_2 = \{x \in E, \exists x_1 \in F_1, \exists x_2 \in F_1 : x = x_1 + x_2\}$$

(a) Prove that $F_1 + F_2$ is a subspace of E.

Exercise n°2 : Let $E = \mathcal{F}(\mathbb{R}, \mathbb{R})$ be the space of all mappings from $\mathbb{R}$ to $\mathbb{R}$. Define $F = span(B) =< B >$, where $B = \{f_1, f_2\}$ and f_1, f_2 are mappings defined by

$$\forall x \in \mathbb{R} : f_1(x) = \cos x, f_2(x) = \sin x$$

(1). Show that $B = \{f_1, f_2\}$ is a linearly independent family in E.
(2). Let α and β be fixed real numbers. Define the mappings f and f as follows

$$\forall x \in \mathbb{R} : f(x) = \cos(x + \alpha), g(x) = \sin(x + \beta)$$

(a). Prove that f and g belong to F, and determine their coordinates in the basis B of F.

Exercise n°3 : We consider the following subspaces of $\mathbb{R}^4$

$$F = \left\{(x, y, z, t) \in \mathbb{R}^4 : 2x + y = 0 \text{ and } -x + 3z - t = 0\right\}, G = \left\{(x, y, z, t) \in \mathbb{R}^4 : x + 2y + 3z + t = 0\right\}$$

1. Find a basis for each of the subspaces F and G.
2. Determine the intersection $F \cap G$ and deduce a basis for $F \cap G$.
3. Compute the dimensions $\dim_{\mathbb{R}} F$, $\dim_{\mathbb{R}} G$, $\dim_{\mathbb{R}} F + G$.

Exercise n°4 : Let $\mathcal{F}(\mathbb{R}, \mathbb{R})$ denote the vector space over $\mathbb{R}$ of all mappings from $\mathbb{R}$ to $\mathbb{R}$. Let $EV(\mathbb{R}, \mathbb{R})$ and $OD(\mathbb{R}, \mathbb{R})$ be the subspaces of $\mathcal{F}(\mathbb{R}, \mathbb{R})$ consisting of even and odd mappings from $\mathbb{R}$ to $\mathbb{R}$, respectively, defined by

$$EV(\mathbb{R}, \mathbb{R}) = \{f \in \mathcal{F}(\mathbb{R}, \mathbb{R}), \forall x \in \mathbb{R} : f(-x) = f(x)\}, OD(\mathbb{R}, \mathbb{R}) = \{f \in \mathcal{F}(\mathbb{R}, \mathbb{R}), \forall x \in \mathbb{R} : f(-x) = -f(x)\}$$

1. Let $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$. Define two mappings f_1 and f_2 in $\mathcal{F}(\mathbb{R}, \mathbb{R})$ as follows

$$\forall x \in \mathbb{R} : f_1(x) = \frac{1}{2}[f(x) + f(-x)] \quad \text{and} \quad f_2(x) = \frac{1}{2}[f(x) - f(-x)].$$

(a) Prove that f_1 is even and f_2 is odd. Deduce that $\mathcal{F}(\mathbb{R}, \mathbb{R}) = EV(\mathbb{R}, \mathbb{R}) + OD(\mathbb{R}, \mathbb{R})$.
(b) Show that $EV(\mathbb{R}, \mathbb{R})$ and $OD(\mathbb{R}, \mathbb{R})$ are complementary in $\mathcal{F}(\mathbb{R}, \mathbb{R})$, i.e., $\mathcal{F}(\mathbb{R}, \mathbb{R}) = EV(\mathbb{R}, \mathbb{R}) \oplus OD(\mathbb{R}, \mathbb{R})$.

Exercise n°5 : **(Supplementary Exercise)** Let $F_1 = span(L_1) =< L_1 >$ and $F_2 = span(L_2) =< L_2 >$ be the vector subspaces of $\mathbb{R}^3$ generated (or spanned) respectively by the families $L_1 = \{a_1 = (0, -1, 1), a_2 = (1, 1, 1)\}$ and $L_2 = \{b_1 = (1, -1, 1), b_2 = (1, -1, -1)\}$.
1. Determine a generating family L_3 for $F_1 + F_2$.
2. Find the dimensions of $F_1, F_2, F_1 + F_2$ and $F_1 \cap F_2$.