Gauss' Law

Carl Friedrich Gauss
1777 – 1855

One of the primary goals of physics is to find simple ways of solving such labor-intensive problems. One of the main tools in reaching this goal is the use of symmetry. We discuss a relationship between charge and electric field that allows us, in certain symmetric situations, to find the electric field of an extended charged object with a few lines of algebra. The relationship is called **Gauss' law,** which was developed by German mathematician and physicist Carl Friedrich Gauss (1775-1855) Guass' law relates the electric field at points on a (closed) Gaussian surface to the net charge enclosed by that

Figure.1 shows a particle with charge *Q* that is surrounded by an imaginary concentric sphere. At points on the sphere (said to be a *Gaussian surface*), the electric field vectors have a moderate magnitude (given by $E = kQ/r^2$) and point radially away from the particle (because it is positively charged). The electric field lines are also outward and have a moderate density (which, recall, is related to the field magnitude).We say that the field vectors and the field lines *pierce* the surface.

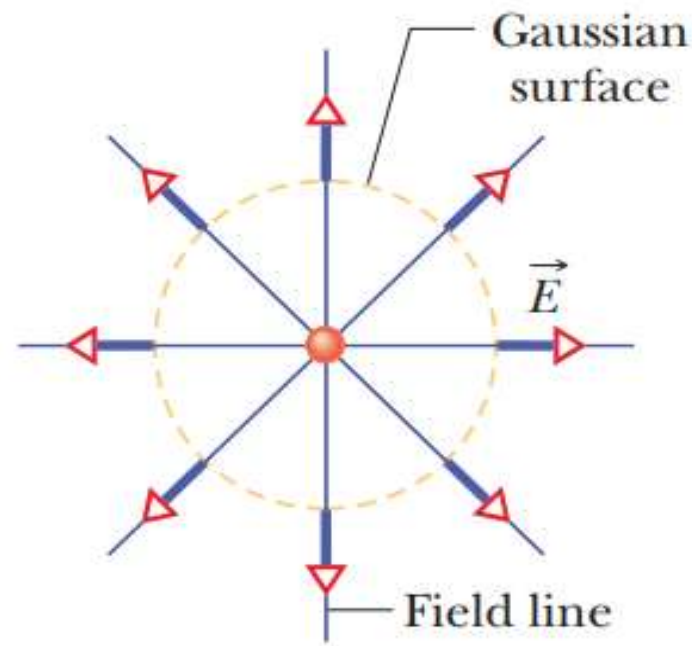


Figure 1. Electric field vectors and field lines pierce an imaginary, spherical Gaussian surface that encloses a particle with charge +*Q*.

Figure.2 is similar except that the enclosed particle has charge 2Q. Because the enclosed charge is now twice as much, the magnitude of the field vectors piercing outward through the (same) Gaussian surface is twice as much as in Fig.1 and the density of the field lines is also twice as much.

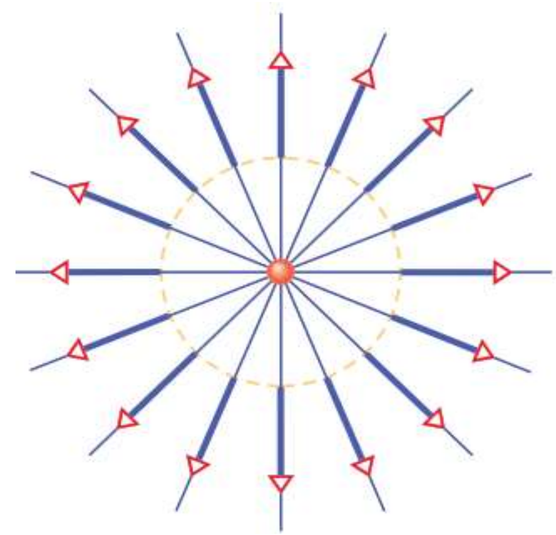


Figure 2. Now the enclosed particle has charge +2Q.

Guass' law relates the electric field at points on a (closed) Gaussian surface to the net charge enclosed by that surface.

Can you tell what the enclosed charge is now? Figure 3.
From the fact that the density of field lines is half that of Fig.1, we also see that the charge must be 0.5Q.

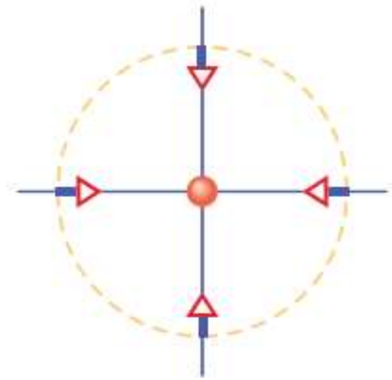


Figure 3.

Definitions

- ***Symmetry***:

The balanced structure of an object, the halves of which are alike.

- ***Closed surface:***

A surface that divides space into an inside and outside region, so one can't move from one region to another without crossing the surface

.

- ***Gaussian surface:***

A hypothetical closed surface that has the same symmetry as the problem we are working on note this is not a real surface it is just an

Electric Flux

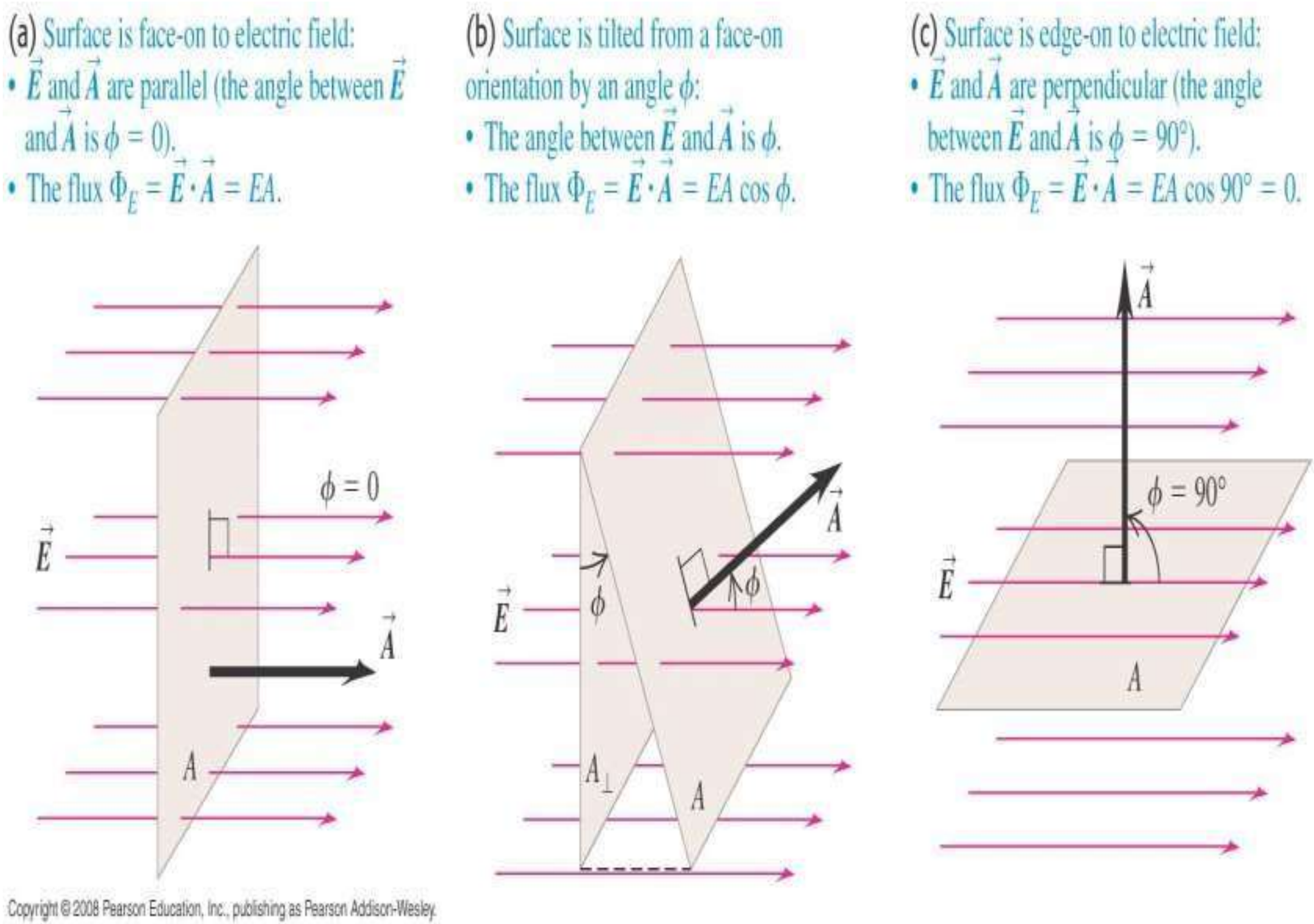
•***Flux :*** The rate of flow through an **area** or **volume**.

•***Electric Flux:*** The rate of flow of an electric field through an **area** or **volume** represented by the number of E field lines penetrating a surface.

The *amount* of electric field piercing the patch is defined to be the **electric flux** Δ $\Delta\Phi = \vec{E} \cdot \Delta\vec{A},$

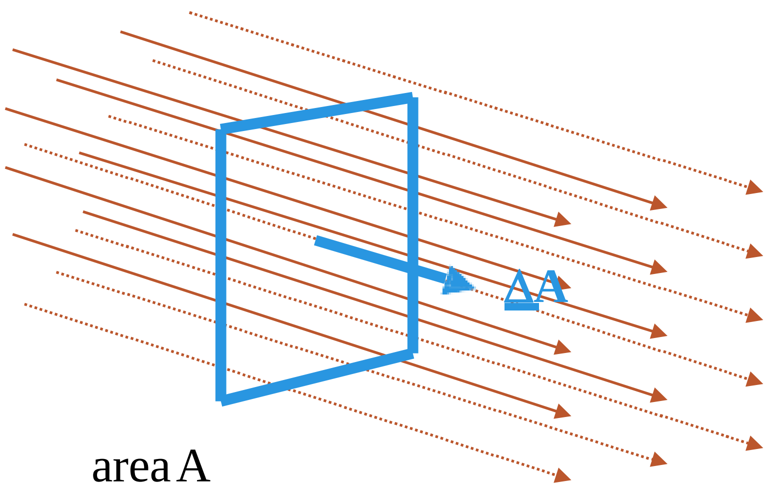
Electric flux has SI units of volt metres (V m), or, equivalently, newton metres squared per coulomb ($N\ m^2\ C^{-1}$). Thus, the SI baseunits of electric flux are $kg \cdot m^3 \cdot s^{-3} \cdot A^{-1}$.

Flux of a Uniform Electric Field

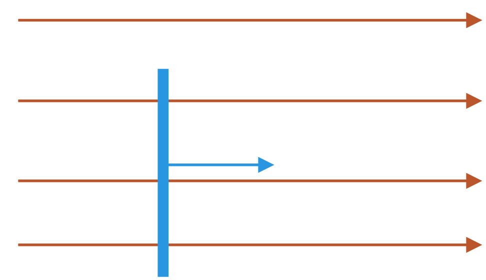


Electric Flux

We define the electric flux $\Delta\Phi$, of the electric field E, through the surface ΔA, as:



E



Normal to surface, magnitude ΔA

$$\Delta\Phi = \vec{E} \cdot \Delta\vec{A},$$

$\underline{\Delta A} \equiv \rightarrow$

Figure 4.

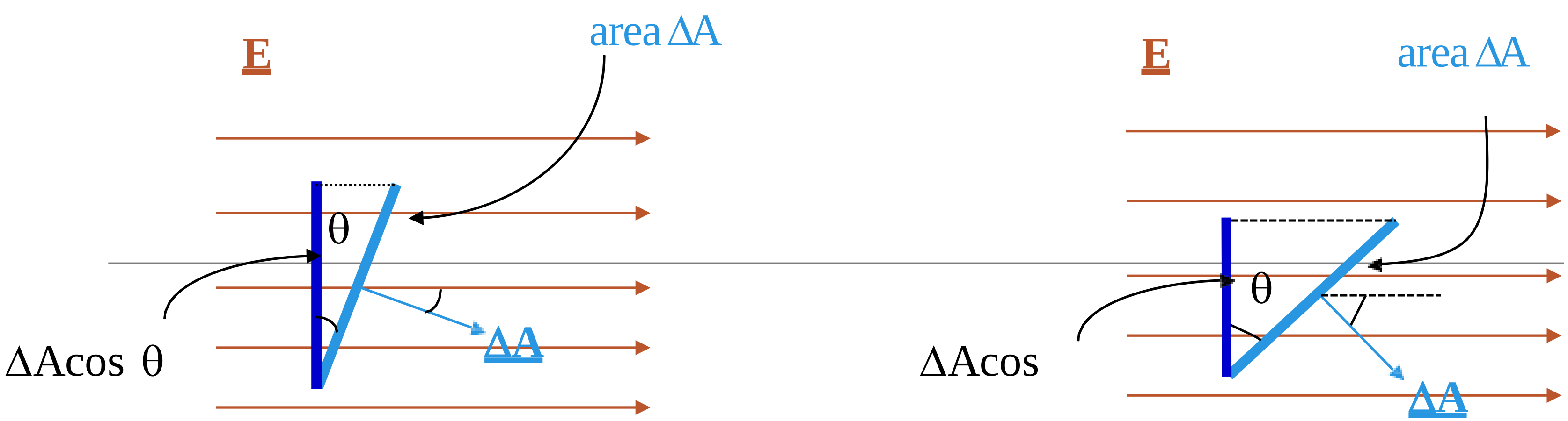
Where:

$\underline{\Delta A}$ is a vector normal to the surface (magnitude A, and direction normal to the surface).

The flux also depends on orientation

$$\Delta\Phi = \underline{E} \cdot \Delta\underline{A} = E\,\Delta A \cos\theta$$

θ is the angle between $\underline{E}$ and $\Delta\underline{A}$

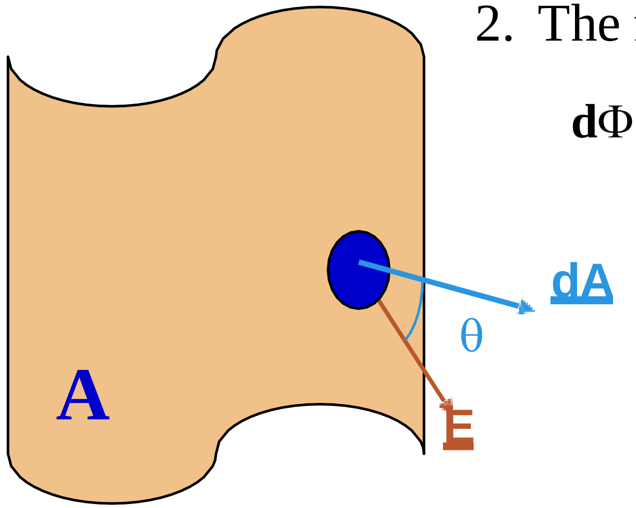


The number of field lines through the tilted surface / equals the number through its projection. Hence, the flux through the tilted surface is simply given by the flux through its projection: E (ΔA cos θ).

What if the surface is curved, or the field varies with position ??

$\Phi = \underline{\mathbf{E}} \cdot \underline{\mathbf{A}}$

1. We divide the surface into small regions with area dA



2. The flux through dA is $d\Phi = E\, dA \cos\theta$

$d\Phi = \underline{\mathbf{E}} \cdot d\underline{\mathbf{A}}$

3. To obtain the total flux we need

to integrate over the surface A

$$\Phi = \int d\Phi = \int \underline{\mathbf{E}} \cdot \underline{d\mathbf{A}}$$

Question

How does the flux Φ_E through the entire surface change when the charge +Q is moved from position 1 to position 2?

a) Φ_E increases

b) Φ_E decreases

c) Φ_E doesn't change

Just depends on charge not position

Example of Constant Field

$$\mathbf{E} = 4\hat{i} \text{ N/C}$$

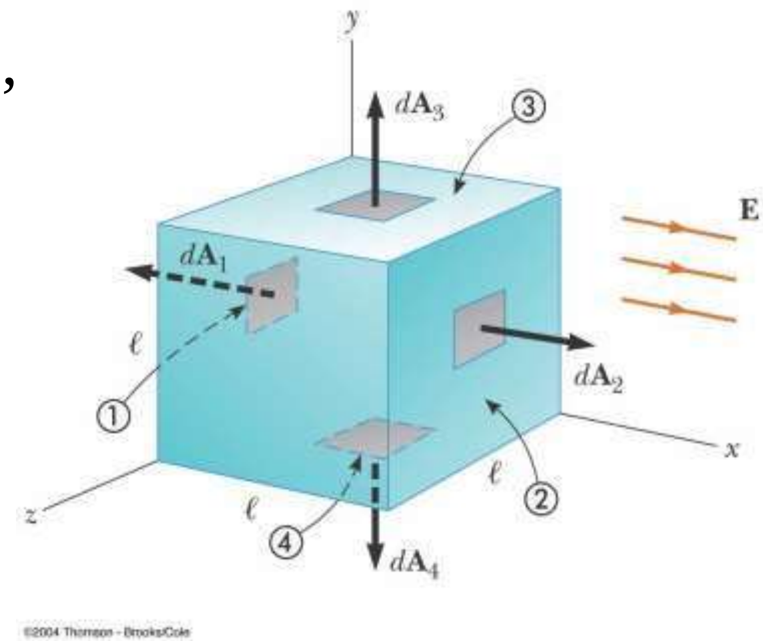
$$\mathbf{A} = (2\hat{i} + 3\hat{j}) \text{ m}^2$$

$$\Phi_E \equiv \mathbf{E} \cdot \mathbf{A} = 4\hat{i} \cdot (2\hat{i} + 3\hat{j}) = 8 \text{ Nm}^2/\text{C}$$

Flux Through a Cube

- Uniform electric field **E** = (E, 0 ,0)

- The flux through the surfaces 3, 4, 5, and 6
  equal zero => Side
  1: $\Phi = -E\, l^2$ Side
  2: $\Phi = E\, l^2$
  For the other sides, $\Phi = 0$
  Therefore, $\Phi_{total} = 0$



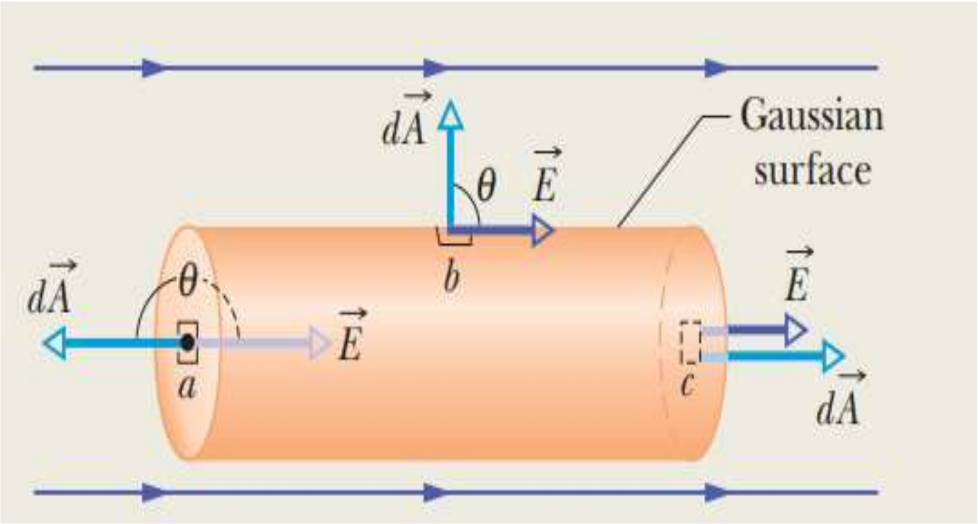
- The net flux over all six faces is

$$\Phi_E = -El^2 + El^2 + 0 + 0 + 0 + 0 = 0$$

Example Flux Through a Cylınder

❖ Assume E is constant, to the right

- Flux through left face?
- Flux through right face?
- Flux through curved side?
- Total flux through cylinder?



$$\Phi = \oint \vec{E} \cdot d\vec{A} = \int_a \vec{E} \cdot d\vec{A} + \int_b \vec{E} \cdot d\vec{A} + \int_c \vec{E} \cdot d\vec{A}.$$

$$\int_a \vec{E} \cdot d\vec{A} = \int E(\cos 180°)\, dA = -E \int dA = -EA,$$

$$\int_c \vec{E} \cdot d\vec{A} = \int E(\cos 0)\, dA = EA.$$

$$\int_b \vec{E} \cdot d\vec{A} = \int E(\cos 90°)\, dA = 0.$$

$$\Phi = -EA + 0 + EA = 0.$$

Figure 6 A cylindrical Gaussian surface, closed by end caps, is immersed in a uniform electric field. The cylinder axis is parallel to the field direction.

➔General statement of Gauss' law

$$\Phi = \oint_S \vec{E} \cdot d\vec{A} = \frac{q_{enclosed}}{\varepsilon_0}$$

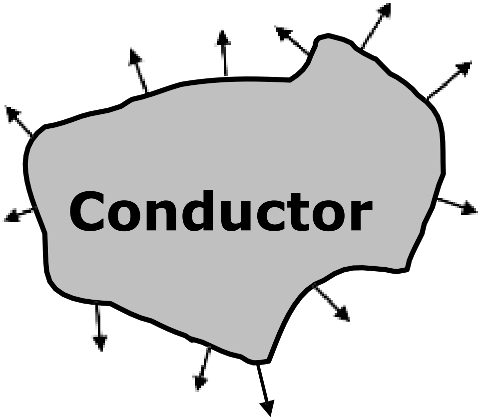
$$1/\varepsilon_0 = 4 \pi k_e$$

Integration over closed surface $q_{enclosed}$ is charge inside the surface
Charges outside surface have no effect

(This does not mean they do not contribute to **E**.)

➔Can be used to calculate E fields. But remember

- ◆ Outward E field, flux > 0
- ◆ Inward E field, flux < 0



➔Consequences of Gauss' law (as we shall see)

- ◆ Excess charge on conductor is always on surface
- ◆ E is always normal to surface on conductor

(Excess charge distributes on surface in such a way)

- $\varepsilon_o = 8.85 \times 10^{-12}\ C^2/Nm^2$: permittivity of free space
- The area in Φ is an imaginary Gaussian surface (does not have to coincide with the surface of a physical object).

Gauss' Law

Gauss' Law depends on the enclosed charge only

$$\Phi = \oint \vec{E} \bullet d\vec{A} = \frac{q_{enc}}{\varepsilon_o}$$

1. If there is a positive net flux there is a net positive charge enclosed
2. If there is a negative net flux there is a net negative charge enclosed
3. If there is a zero net flux there is no net charge enclosed

- Gauss' Law works in cases of symmetry

Power of Gauss' Law: Calculating E Fields

- Valuable for cases with high symmetry
  - E = constant, ⊥ surface
  - E || surface

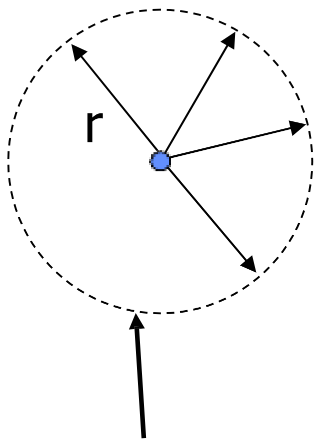
$$\int_S \mathbf{E} \cdot d\mathbf{A} = \pm EA$$

$$\int_S \mathbf{E} \cdot d\mathbf{A} = 0$$

- Spherical symmetry
  - E field vs r for point charge
  - E field vs r inside uniformly charged sphere
  - Charges on concentric spherical conducting shells
- Cylindrical symmetry
  - E field vs r for line charge
  - E field vs r inside uniformly charged cylinder
- Rectangular symmetry
  - E field for charged plane
  - E field between conductors, e.g. capacitors

Derive Coulomb's Law From Gauss' Law

- ➔Charge +Q at a point
  - By symmetry, E must be radially symmetric
- ➔Draw Gaussian' surface around point
  - Sphere of radius r
  - E field has constant mag., ⊥ to Gaussian surface



$$\oint_S \mathbf{E} \cdot d\mathbf{A} = E(4\pi r^2) = \frac{Q}{\varepsilon_0}$$

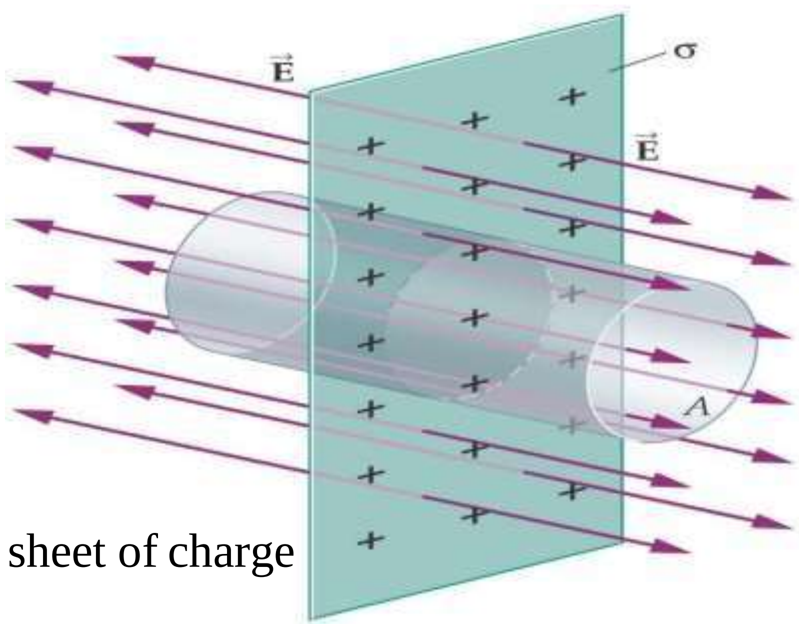
Gauss' Law

$$E = \frac{Q}{4\pi\varepsilon_0 r^2} = \frac{kQ}{r^2}$$

Solve for E

Applying Gauss's Law: Field Due to a Plane of Charge

Gauss's law is useful only when the electric field is constant on a given surface



Infinite sheet of charge

- The uniform field must be perpendicular to the sheet and directed either toward or away from the sheet
- Use a cylindrical Gaussian surface
- The flux through the ends is EA and there is no field through the curved part of the surface
- Surface charge density $\sigma = Q / A$

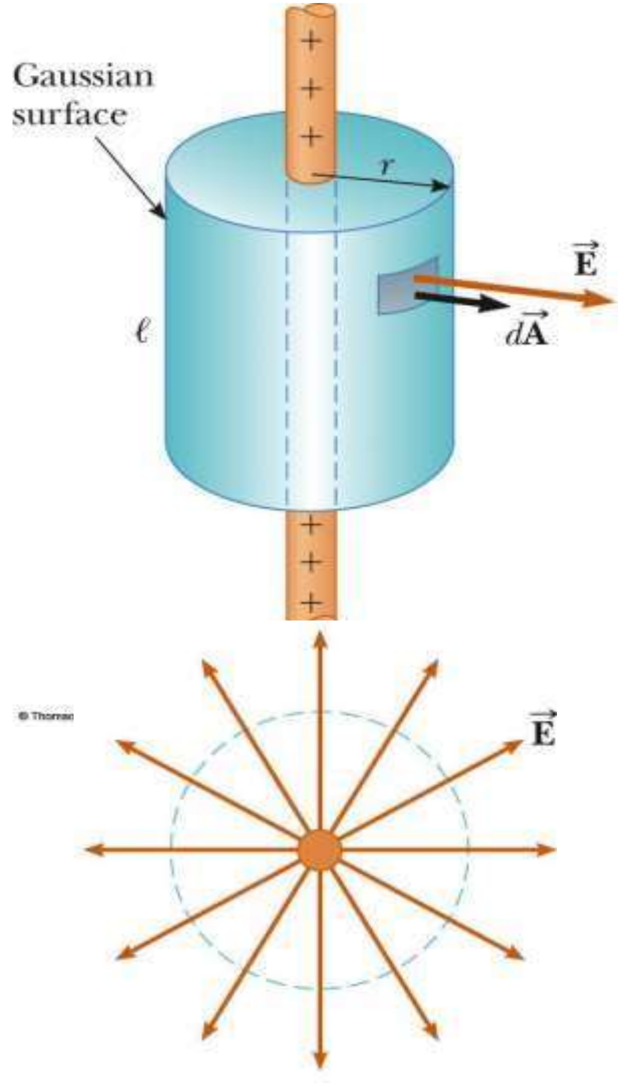
1. Select Gauss surface In this case a cylindrical
2. Calculate the flux of the electric field through the Gauss surface
   $\Phi = 2 E A$
3. Equate $\Phi = q_{encl}/\varepsilon_0$
   $2EA = q_{encl}/\varepsilon_0$
4. Solve for E
   $E = q_{encl} / 2 A \varepsilon_0 = \sigma / 2 \varepsilon_0$
   with $\sigma = (q_{encl} / A)$

Field Due to a Line of Charge

$$\Phi_E = \oint \vec{E} \cdot dA = E \oint dA = \frac{Q}{\varepsilon_0}$$

$$E(2\pi r l) = \frac{\lambda l}{\varepsilon_0}$$

$$E = \frac{\lambda}{2\pi\varepsilon_0 r} = 2k\frac{\lambda}{r}$$



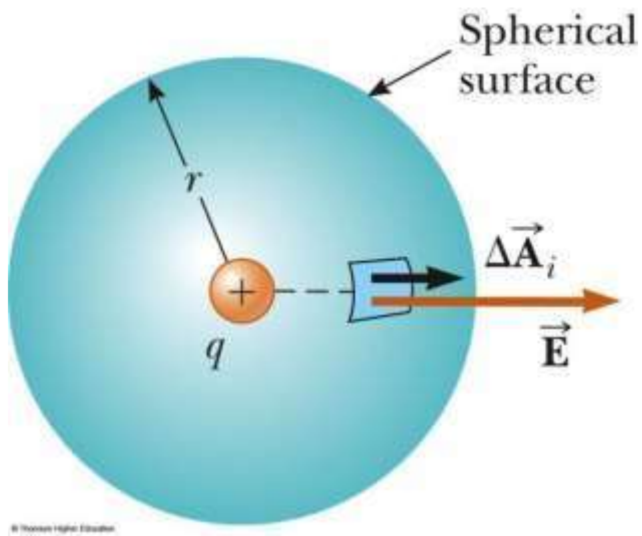
(b)

Electric field produced by a point charge

- A positive point charge q is located at the center of a sphere of radius r
- The magnitude of the electric field everywhere on the surface of the sphere is $E = k_e q / r^2$
- $A_{\text{sphere}} = 4\pi r^2$

$$\Phi_E = \oint \vec{E} \cdot dA = E \oint dA$$

$$= k_e \frac{q}{r^2} \cdot 4\pi r^2 = 4\pi k_e q = \frac{q}{\varepsilon_0}$$



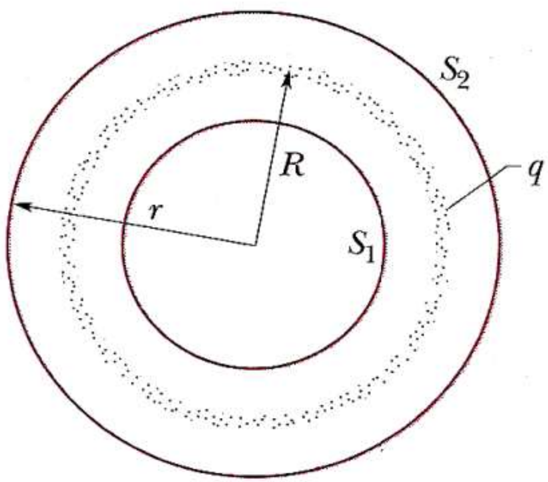
Gauss' Law

Applying Gauss' Law: Spherical Symmetry

Here we use Gauss' law to prove the two shell theorems presented without proof

A shell of uniform charge attracts or repels a charged particle that is outside the shell as if all the shell's charge were concentrated at the center of the shell.

If a charged particle is located inside a shell of uniform charge, there is no electrostatic force on the particle from the shell.



A thin, uniformly charged, spherical shell with total charge q, in cross section. Two Gaussian surfaces S_1 and S_2 are also shown in cross section. Surface S_2 encloses the shell, and S_1 encloses only the empty interior of the shell.

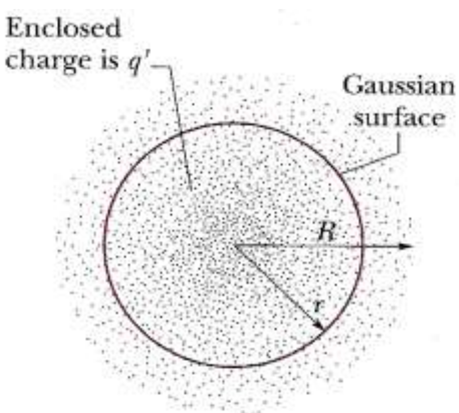
Gauss' Law

Figure. Shows a charged spherical shell of total charge *q* and radius *R* and two concentric spherical Gaussian surfaces S_1 and S_2 .as we applied Gauss's law to surface S_2, for which $r \geq R$, we would find that

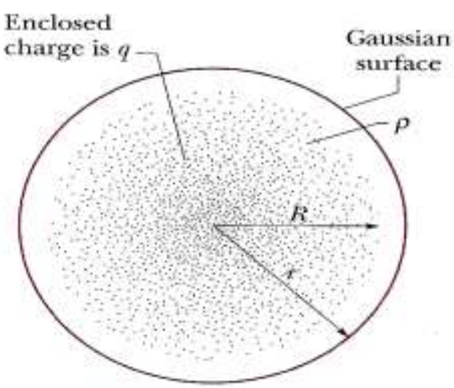
$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \qquad \text{(spherical shell, field at } r \geq R\text{)}.$$

This field is the same as set up by a point charge *q* at the center of the shell of charge.Thus, the force produced by a shell pf charge *q* on a charged particle placed outside the shell is the force produced by a point charge *q* located at the center of the shell. This proves the first shell theorem. Applying Gauss s law to surface S_1, for which $r<R$, leads directly to.

$$E = 0 \qquad \text{(spherical shell, field at } r < R\text{)},$$



(b)



(a)

Gauss' Law

- **Equation says that the charge lying *outside* the Gaussian surface does not set up a net electric field on the Gaussian surface.**

- **Equation says that the charge *enclosed* by the surface sets up an electric field as if that enclosed charge were concentrated at the center.**

- **q' represent that enclosed charge, we can then rewrite As**

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q'}{r^2} \qquad \text{(spherical distribution, field at } r \le R\text{).}$$

If the full charge *q* enclosed within radius *R* is uniform, then *q'* enclosed within radius r is proportional to *q*:

$$\frac{\begin{pmatrix}\text{charge enclosed by}\\ \text{sphere of radius } r\end{pmatrix}}{\begin{pmatrix}\text{volume enclosed by}\\ \text{sphere of radius } r\end{pmatrix}} = \frac{\text{full charge}}{\text{full volume}}$$

Or

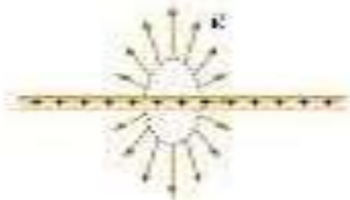
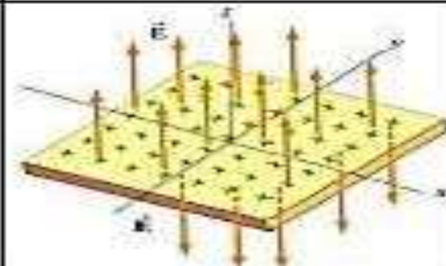
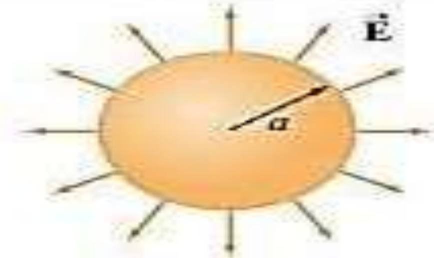
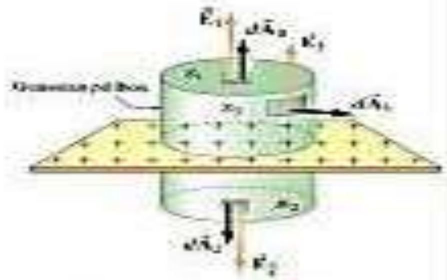
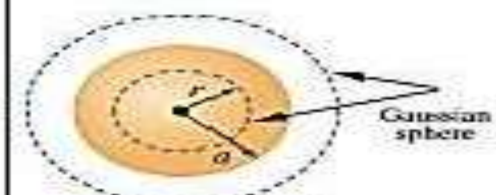
That give us

$$\frac{q'}{\frac{4}{3}\pi r^3} = \frac{q}{\frac{4}{3}\pi R^3}.$$

Substituting this in last equ

$$q' = q\frac{r^3}{R^3}.$$

$$E = \left(\frac{q}{4\pi\varepsilon_0 R^3}\right)r \qquad \text{(uniform charge, field at } r \le R\text{).}$$

System	Infinite line of charge	Infinite plane of charge	Uniformly charged solid sphere
Figure			
Identify the symmetry	Cylindrical	Planar	Spherical
Determine the direction of $\vec{E}$			
Divide the space into different regions	$r > 0$	$z > 0$ and $z < 0$	$r \le a$ and $r \ge a$
Choose Gaussian surface	 Coaxial cylinder	 Gaussian pillbox	 Concentric sphere
Calculate electric flux	$\Phi_E = E(2\pi r l)$	$\Phi_E = EA + EA = 2EA$	$\Phi_E = E(4\pi r^2)$
Calculate enclosed charge q_{in}	$q_{enc} = \lambda l$	$q_{enc} = \sigma A$	$q_{enc} = \begin{cases} Q(r/a)^3 & r \le a \\ Q & r \ge a \end{cases}$
Apply Gauss's law $\Phi_E = q_{in} / \varepsilon_0$ to find E	$E = \dfrac{\lambda}{2\pi\varepsilon_0 r}$	$E = \dfrac{\sigma}{2\varepsilon_0}$	$E = \begin{cases} \dfrac{Qr}{4\pi\varepsilon_0 a^3}, & r \le a \\ \dfrac{Q}{4\pi\varepsilon_0 r^2}, & r \ge a \end{cases}$