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First series of exercises : Mathematical Reminders and Electrostatics

1 Mathematical Reminders

1.1 Exercise 1:

Elements of Length, Surface, and Volume in Coordinate Systems.

1.2 Exercise 2:

Express the differential volume element dV in Cartesian, cylindrical, and spherical coordinates.

1.3 Exercise 3:

Derive the expression for the volume element dV in spherical coordinates (r, θ, ϕ) using the Jacobian determinant.

1.4 Exercise 4:

Compute the surface area of a torus with major radius R and minor radius r using cylindrical coordinates.

1.5 Exercise 5 :

Find the gradient of the function $\phi(x, y, z) = x^2y + yz^3$ in Cartesian coordinates.

1.6 Exercise 6:

Prove that the curl of the gradient of any scalar field f is zero, i.e., $\nabla \times (\nabla f) = 0$.

1.7 Exercise 7:

Compute a Double Integral. Evaluate the double integral:

$$I = \int_0^2 \int_0^x (x^2 + y)\, dy\, dx$$

2 Electrostatics

2.1 Exercise 1: Coulomb's Law

Two charges, $q_1 = 2\mu C$ and $q_2 = -3\mu C$, are separated by a distance of $r = 4$ cm in vacuum. Find the electrostatic force between them.

2.2 Exercise 2: Electric Dipole

An electric dipole consists of two charges $+q$ and $-q$ separated by a distance d. Derive the electric field at a point in the equatorial plane (perpendicular to the dipole axis).

2.3 Exercise 3: Gauss's Theorem

A spherical charge distribution has charge $Q = 5 \times 10^{-9} C$ within a sphere of radius $R = 10$ cm. Use Gauss's theorem to find the electric field at $r = 20$ cm.

2.4 Exercise 4: A solid sphere of radius

A solid sphere of radius R carries a volume charge density $\rho(r) = \rho_0 \left(1 - \frac{r}{R}\right)$. Use Gauss's theorem to find the electric field inside and outside the sphere.

2.5 Exercise 5: Capacitance of a Parallel Plate Capacitor

A capacitor has plates of area $A = 2$ m^2 separated by $d = 5$ mm. Calculate its capacitance.

2.6 Exercise 6: A cylindrical capacitor

A cylindrical capacitor consists of two concentric cylinders of radii a and b ($b > a$) and length L. Derive its capacitance.

3 Solution of Exercise series 1

3.1 Exercise 1: Elements of Length, Surface, and Volume in Coordinate Systems

Problem: Express the differential volume element dV in Cartesian, cylindrical, and spherical coordinates.

Solution:

1. **Cartesian Coordinates (x, y, z):** In Cartesian coordinates, the differential volume element is simply the product of the differential lengths in each direction:

$$dV = dx\, dy\, dz$$

2. **Cylindrical Coordinates (ρ, ϕ, z):** In cylindrical coordinates, the differential volume element is derived from the differential lengths in the radial (ρ), angular (ϕ), and vertical (z) directions:

$$dV = \rho\, d\rho\, d\phi\, dz$$

Here, ρ accounts for the "scaling" of the angular component as the radius increases.

3. **Spherical Coordinates (r, θ, ϕ):** In spherical coordinates, the differential volume element is derived from the differential lengths in the radial (r), polar (θ), and azimuthal (ϕ) directions:

$$dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

The term $r^2 \sin\theta$ arises from the Jacobian determinant when transforming from Cartesian to spherical coordinates.

3.2 Exercise 2: Express the Differential Volume Element dV in Cartesian, Cylindrical, and Spherical Coordinates

Problem: Express the differential volume element dV in Cartesian, cylindrical, and spherical coordinates.

Solution:

1. **Cartesian Coordinates (x, y, z):** The differential volume element is:

$$dV = dx\, dy\, dz$$

2. **Cylindrical Coordinates (ρ, ϕ, z):** The differential volume element is:

$$dV = \rho\, d\rho\, d\phi\, dz$$

3. **Spherical Coordinates (r, θ, ϕ):** The differential volume element is:

$$dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

Exercise 3: Derive the Expression for the Volume Element dV in Spherical Coordinates Using the Jacobian Determinant

Problem: Derive the expression for the volume element dV in spherical coordinates (r, θ, ϕ) using the Jacobian determinant.

Solution:

The transformation from Cartesian coordinates (x, y, z) to spherical coordinates (r, θ, ϕ) is given by:

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

The Jacobian matrix J is defined as:

$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix}$$

Substituting the partial derivatives:

$$J = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}$$

The determinant of the Jacobian matrix is:

$$|J| = \sin\theta\cos\phi\,(r\cos\theta\sin\phi\cdot 0 - (-r\sin\theta)\cdot r\sin\theta\cos\phi) - r\cos\theta\cos\phi\,(\sin\theta\sin\phi\cdot 0 - (-r\sin\theta)\cdot r\sin\theta\sin\phi) + (-$$

Simplifying, we get:

$$|J| = r^2\sin\theta$$

Thus, the differential volume element in spherical coordinates is:

$$dV = |J|\,dr\,d\theta\,d\phi = r^2\sin\theta\,dr\,d\theta\,d\phi$$

3.3 Exercise 4: Surface Area of a Torus

The surface area of a torus with major radius R and minor radius r can be computed using cylindrical coordinates. The parameterization of the torus is:

$$\begin{aligned} x &= (R + r\cos\theta)\cos\phi, \\ y &= (R + r\cos\theta)\sin\phi, \\ z &= r\sin\theta, \end{aligned}$$

where $\theta \in [0, 2\pi]$ and $\phi \in [0, 2\pi]$ are the angular parameters.

The surface element is given by:

$$dS = \left|\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \phi}\right| d\theta d\phi.$$

Computing the derivatives and the magnitude of the cross product, we obtain:

$$dS = r(R + r\cos\theta)d\theta d\phi.$$

Thus, the surface area of the torus is:

$$\begin{aligned} A &= \int_0^{2\pi}\int_0^{2\pi} r(R + r\cos\theta)d\theta d\phi. \\ &= \left(\int_0^{2\pi} r(R + r\cos\theta)d\theta\right)\left(\int_0^{2\pi} d\phi\right). \end{aligned}$$

Evaluating the integrals:

$$
\begin{aligned}
A &= 2\pi r \int_0^{2\pi} (R + r\cos\theta) d\theta \\
&= 2\pi r \left(R(2\pi) + r \int_0^{2\pi} \cos\theta d\theta \right).
\end{aligned}
$$

Since $\int_0^{2\pi} \cos\theta d\theta = 0$, we obtain:

$$
A = 4\pi^2 Rr.
$$

3.4 Exercise 5: Gradient of a Scalar Function

The function is given by:

$$
\phi(x, y, z) = x^2 y + yz^3.
$$

The gradient is computed as:

$$
\begin{aligned}
\nabla\phi &= \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z} \right). \\
\frac{\partial\phi}{\partial x} &= 2xy, \\
\frac{\partial\phi}{\partial y} &= x^2 + z^3, \\
\frac{\partial\phi}{\partial z} &= 3yz^2.
\end{aligned}
$$

Thus, the gradient is:

$$
\nabla\phi = (2xy, x^2 + z^3, 3yz^2).
$$

3.5 Exercise 6: Curl of the Gradient

We need to show that:

$$
\nabla \times (\nabla f) = 0.
$$

The curl operator is defined as:

$$
\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}.
$$

Expanding the determinant, we obtain:

$$
\nabla \times (\nabla f) = \left(\frac{\partial^2 f}{\partial y \partial x} - \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial z \partial y} - \frac{\partial^2 f}{\partial y \partial z}, \frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x} \right).
$$

Since the mixed partial derivatives of a smooth function are equal, each term is zero, proving:

$$
\nabla \times (\nabla f) = 0.
$$

3.6 Exercise 7: Evaluating a Double Integral

The given integral is:

$$I = \int_0^2 \int_0^x (x^2 + y)\,dy\,dx.$$

Evaluating the inner integral:

$$\begin{aligned}\int_0^x (x^2 + y)\,dy &= \left[x^2 y + \frac{y^2}{2}\right]_0^x \\ &= x^3 + \frac{x^2}{2}.\end{aligned}$$

Now, integrating over x:

$$I = \int_0^2 \left(x^3 + \frac{x^2}{2}\right) dx.$$

Evaluating term by term:

$$\int_0^2 x^3 dx = \frac{x^4}{4}\bigg|_0^2 = \frac{16}{4} = 4,$$
$$\int_0^2 \frac{x^2}{2} dx = \frac{1}{2} \cdot \frac{x^3}{3}\bigg|_0^2 = \frac{1}{2} \cdot \frac{8}{3} = \frac{4}{3}.$$

Thus, the final result is:

$$I = 4 + \frac{4}{3} = \frac{12}{3} + \frac{4}{3} = \frac{16}{3}.$$

4 Electrostatics

4.1 Exercise 1: Coulomb's Law

Two point charges, $q_1 = 2\mu C$ and $q_2 = -3\mu C$, are separated by a distance of $r = 4$ cm in vacuum. We use Coulomb's law:

$$F = k\frac{|q_1 q_2|}{r^2},$$

where $k = 9 \times 10^9$ Nm^2/C^2 is Coulomb's constant. Substituting values:

$$\begin{aligned}F &= 9 \times 10^9 \frac{(2 \times 10^{-6})(3 \times 10^{-6})}{(0.04)^2} \\ &= 9 \times 10^9 \frac{6 \times 10^{-12}}{1.6 \times 10^{-3}} \\ &= 33.75 \text{ N}.\end{aligned}$$

Since the charges are opposite, the force is attractive.

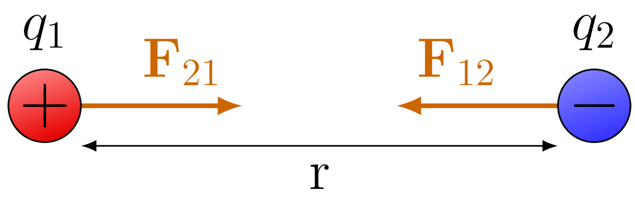

Figure 1: Coulomb$_{f}orce$

4.2 Exercise 2: Electric Dipole

An electric dipole consists of two charges $+q$ and $-q$ separated by a distance d. We derive the electric field at a point P on the equatorial plane (perpendicular to the dipole axis).

Using the electric field formula for a point charge:

$$E = k\frac{q}{r^2},$$

and resolving the components from both charges, the perpendicular components cancel, leaving the net field as:

$$E = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3},$$

where $p = qd$ is the dipole moment.

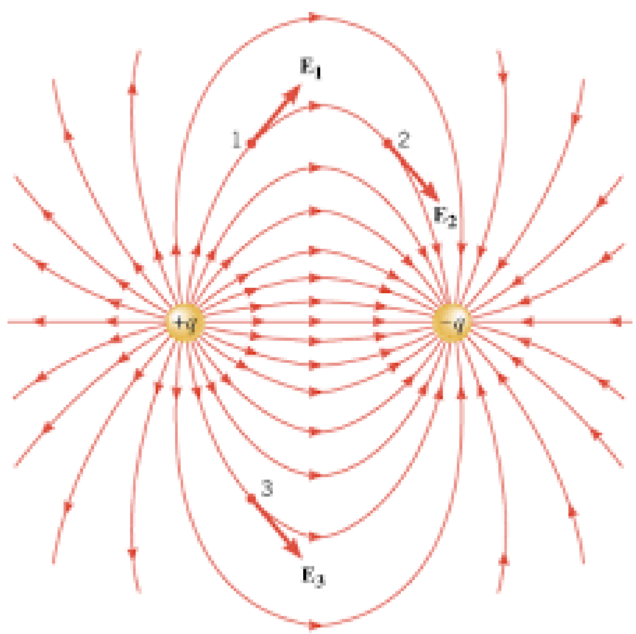

Figure 2: Dipole$_{f}ield$

4.3 Exercise 3: Gauss's Theorem

Given a spherical charge distribution with total charge $Q = 5 \times 10^{-9}C$ within a sphere of radius $R = 10$ cm, we find the electric field at $r = 20$ cm using Gauss's law:

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{Q}{\varepsilon_0}.$$

Since $r > R$, the charge acts as if concentrated at the center:

$$E(4\pi r^2) = \frac{Q}{\varepsilon_0}$$
$$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}.$$

Substituting values:

$$E = \left(9 \times 10^9\right)\frac{5 \times 10^{-9}}{(0.2)^2}$$
$$= 1.125 \times 10^3 \text{ N/C}.$$

4.4 Exercise 4: A Solid Sphere of Radius

A solid sphere of radius R carries a volume charge density

$$\rho(r) = \rho_0\left(1 - \frac{r}{R}\right).$$

Using Gauss's theorem, we determine the electric field inside and outside the sphere.

Solution:

Applying Gauss's law:

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{extenc}}{\epsilon_0}.$$

Inside the Sphere ($r < R$): The charge enclosed within a Gaussian sphere of radius r is:

$$Q_{extenc} = \int_0^r \rho(r')4\pi r'^2 dr'.$$

Substituting $\rho(r)$:

$$Q_{extenc} = 4\pi\rho_0 \int_0^r \left(1 - \frac{r'}{R}\right) r'^2 dr'.$$

Evaluating the integral:

$$Q_{extenc} = 4\pi\rho_0\left[\frac{r^3}{3} - \frac{r^4}{4R}\right].$$

Using Gauss's law:

$$E \cdot 4\pi r^2 = \frac{1}{\epsilon_0} 4\pi\rho_0 \left[\frac{r^3}{3} - \frac{r^4}{4R} \right].$$

Solving for E:

$$E = \frac{\rho_0}{\epsilon_0} \left(\frac{r}{3} - \frac{r^2}{4R} \right).$$

Outside the Sphere ($r > R$): The total charge in the sphere is:

$$Q = \int_0^R \rho(r) 4\pi r^2 dr.$$

Evaluating the integral:

$$Q = 4\pi\rho_0 \left[\frac{R^3}{3} - \frac{R^3}{4} \right] = 4\pi\rho_0 \frac{R^3}{12}.$$

Using Gauss's law:

$$E \cdot 4\pi r^2 = \frac{Q}{\epsilon_0},$$

$$E = \frac{\rho_0 R^3}{12\epsilon_0 r^2}.$$

4.5 Exercise 5: Capacitance of a Parallel Plate Capacitor

A capacitor with plates of area $A = 2$ m^2 separated by a distance $d = 5$ mm has capacitance given by:

$$C = \frac{\epsilon_0 A}{d}.$$

Substituting values:

$$C = \frac{(8.85 \times 10^{-12}) \times 2}{5 \times 10^{-3}}.$$

$$C = 3.54 \times 10^{-9} \text{ F} = 3.54 \text{ nF}.$$

4.6 Exercise 6: A Cylindrical Capacitor

A cylindrical capacitor consists of two concentric cylinders of radii a and b and length L. We derive its capacitance.

Solution:

Using Gauss's law, the electric field in the region between the cylinders is:

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{Q}{\epsilon_0}.$$

The Gaussian surface is a cylinder of radius r, length L, and surface area $2\pi rL$:

$$E(2\pi rL) = \frac{Q}{\epsilon_0}.$$

Solving for E:

$$E = \frac{Q}{2\pi\epsilon_0 Lr}.$$

The potential difference between the cylinders is:

$$V = \int_a^b E dr = \int_a^b \frac{Q}{2\pi\epsilon_0 Lr} dr.$$

Evaluating the integral:

$$V = \frac{Q}{2\pi\epsilon_0 L} \ln\left(\frac{b}{a}\right).$$

The capacitance is given by:

$$C = \frac{Q}{V} = \frac{2\pi\epsilon_0 L}{\ln(b/a)}.$$