Introduction to Physics 2: electrostatics and electrokinetics and magnetism

This course is intended for first-year students in Mathematical Sciences

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1. Introduction: Mathematics Revision

This lesson provides a mathematical revision covering fundamental concepts essential for physics and engineering applications. We will review the elements of length, surface area, and volume in different coordinate systems, as well as important mathematical operators and calculus techniques.

1.1 Elements of Length, Surface Area, and Volume in Different Coordinate Systems

We will analyze the fundamental differential elements in the three primary coordinate systems: Cartesian, cylindrical, and spherical. Understanding these elements is crucial for evaluating integrals in physics and engineering.

1.1.1 Cartesian Coordinate System

A Cartesian coordinate system is defined by an origin point O and three mutually perpendicular axes (Ox, Oy, Oz). The unit vectors along these axes are $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Any point M in space is represented by the position vector:

$$\overrightarrow{R} = \overrightarrow{OM} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

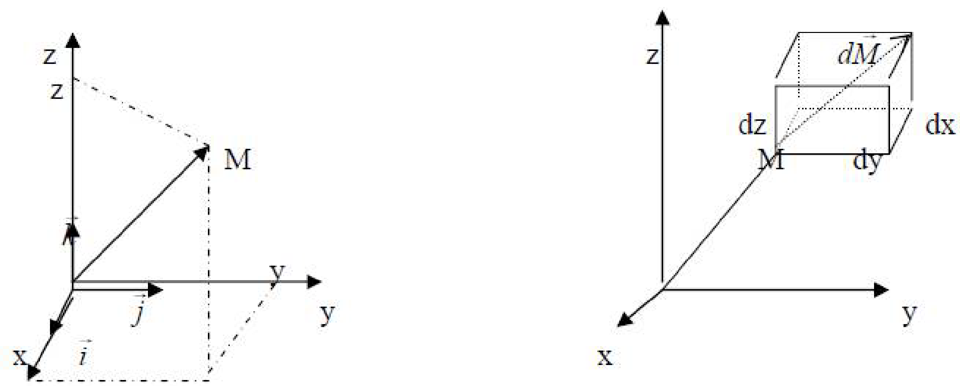

Figure 1: Cartesian basis (a) Position vector and (b) elementary displacement and volume

Example: Consider a straight-line motion along the x-axis where $x = 2t$, $y = 3$, and $z = 0$. The velocity vector is given by:

$$\mathbf{v} = \frac{d\overrightarrow{R}}{dt} = \frac{d}{dt}(2t\mathbf{i} + 3\mathbf{j} + 0\mathbf{k}) = 2\mathbf{i}$$

Differential Length Element: The differential displacement is given by:

$$d\overrightarrow{OM} = d\,\overrightarrow{l} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$$

Differential Surface Element: The surface element depends on the plane of integration:

$$dS_x = dydz, \quad dS_y = dxdz, \quad dS_z = dxdy$$

Differential Volume Element: The elementary volume is given by:

$$dV = dx\,dy\,dz$$

1.1.2 Cylindrical Coordinate System

In the cylindrical coordinate system (r, θ, z), a point is represented as:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

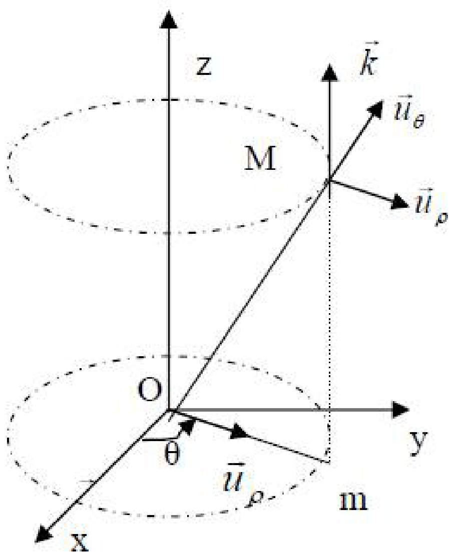

Figure 2: Cylindrical base

It should also be noted that we can write:

$$\mathbf{u}_p = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

and derive this vector with respect to θ:

We obtain:

$$d\mathbf{u}_p = d\theta(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}),$$

knowing that:

$$\cos\left(\theta + \frac{\pi}{2}\right) = -\sin\theta \quad \text{and} \quad \sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta.$$

Thus:

$\dfrac{d\mathbf{u}_p}{d\theta}$ can be obtained by rotating $\mathbf{u}_p$ by an angle of $\dfrac{\pi}{2}$, and we can write:

$$\frac{d\mathbf{u}_p}{d\theta} = \mathbf{u}_\theta.$$

The position vector **DM** is written as:

$$\mathbf{DM} = \rho\mathbf{u}_p + z\mathbf{k} = (x\mathbf{i} + y\mathbf{j}) + z\mathbf{k},$$

where x and y are the Cartesian coordinates of the point M in the Oxy plane, given by:

$$x = \rho\cos\theta, \quad y = \rho\sin\theta, \quad \text{and} \quad z = z.$$

The expression for the elementary displacement is:

$$d\mathbf{DM} = d\rho\,\mathbf{u}_p + \rho\,d\theta\,\mathbf{u}_\theta + dz\,\mathbf{k}.$$

The expression for the elementary surface is:

$$ds = \rho\,d\rho\,d\theta.$$

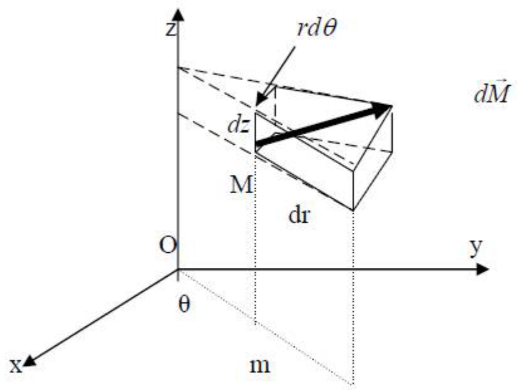

Figure 3: Cylindrical coordinates

Example: Find the velocity vector for a particle moving in a circular path where $r = 2$, $\theta = t^2$, and $z = 4t$. The velocity components are:

$$v_r = \frac{dr}{dt} = 0, \quad v_\theta = r\frac{d\theta}{dt} = 2(2t), \quad v_z = \frac{dz}{dt} = 4$$

Thus, the velocity vector is:

$$\mathbf{v} = 0\mathbf{e_r} + 4t\mathbf{e}_\theta + 4\mathbf{e_z}$$

Differential Length Element:

$$d\overrightarrow{l} = dr\mathbf{e_r} + rd\theta\mathbf{e}_\theta + dz\mathbf{e_z}$$

Differential Surface Element:

$$dS_r = rd\theta dz, \quad dS_\theta = drdz, \quad dS_z = rdrd\theta$$

Differential Volume Element:

$$dV = rdrd\theta dz$$

1.1.3 Spherical Coordinate System

In the spherical coordinate system (r, θ, ϕ), a point is represented as:

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

The position vector of point M in spherical coordinates, meaning in the spherical basis, is written as:

$$\overrightarrow{OM} = r\overrightarrow{u_r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

From the figure, we can express x, y, z in terms of r, θ, ϕ:

$$X = OM\cos\varphi = r\sin\theta\cos\varphi,$$

$$Y = OM\sin\varphi = r\sin\theta\sin\varphi,$$

$$Z = OM\cos\theta = r\cos\theta.$$

Thus, we deduce:

$$\overrightarrow{u_r} = \sin\theta\cos\varphi\,\mathbf{i} + \sin\theta\sin\varphi\,\mathbf{j} + \cos\theta\,\mathbf{k}.$$

The unit vector $\overrightarrow{u_\varphi}$ at OM is written as:

$$\overrightarrow{u_\varphi} = \cos\varphi\,\mathbf{i} + \sin\varphi\,\mathbf{j}.$$

This vector $\overrightarrow{u_\varphi}$ can be obtained either by replacing φ with $\varphi + 2\pi$ or by differentiating $\overrightarrow{u_r}$ with respect to φ:

$$\overrightarrow{u_\varphi} = -\sin\varphi\,\mathbf{i} + \cos\varphi\,\mathbf{j}.$$

This basis vector can also be expressed as the derivative of $\overrightarrow{u_r}$ with respect to φ:

$$\overrightarrow{u_\varphi} = \frac{1}{\sin\theta}\frac{\partial\overrightarrow{u_r}}{\partial\varphi}.$$

The third basis vector in the spherical coordinate system is given by:

$$\overrightarrow{u_\theta} = \frac{\partial\overrightarrow{u_r}}{\partial\theta}.$$

1.1.4 Elementary Displacement:

$$d\overrightarrow{M} = d(r\overrightarrow{u_r}) = dr\overrightarrow{u_r} + rd\overrightarrow{u_r} + r\frac{\partial\overrightarrow{u_r}}{\partial\theta}d\theta + r\frac{\partial\overrightarrow{u_r}}{\partial\varphi}d\varphi.$$

$$= dr\overrightarrow{u_r} + rd\theta\overrightarrow{u_\theta} + r(\sin\theta d\varphi)\overrightarrow{u_\varphi}.$$

1.1.5 Elementary Surface and Volume:

$$dS = r^2 \sin\theta\, d\theta\, d\varphi.$$

$$dV = r^2 \sin\theta\, dr\, d\theta\, d\varphi.$$

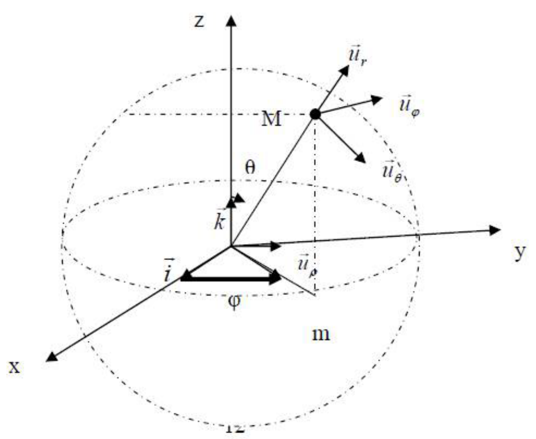

Figure 4: Spherical base

Example: Find the length of an infinitesimal arc in spherical coordinates for a small change in θ while keeping r and ϕ constant.

$$dl = rd\theta$$

Differential Length Element:

$$d\vec{l} = dr\mathbf{e_r} + rd\theta\mathbf{e}_\theta + r\sin\theta d\phi\mathbf{e}_\phi$$

Differential Surface Element:

$$dS_r = r^2\sin\theta d\theta d\phi, \quad dS_\theta = r\sin\theta dr d\phi, \quad dS_\phi = r dr d\theta$$

Differential Volume Element:

$$dV = r^2\sin\theta dr d\theta d\phi$$

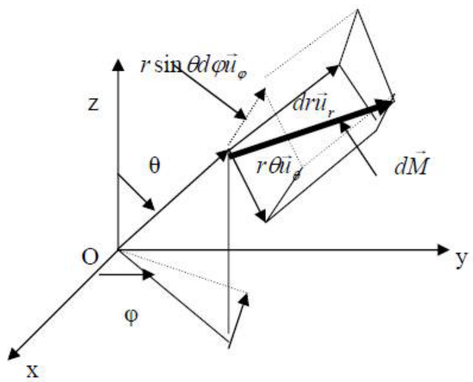

Figure 5: elementary volumes in spherical coordinates

1.1.6 Solid Angles

A solid angle $d\Omega$ in spherical coordinates is given by:

$$d\Omega = \sin\theta d\theta d\phi$$

The total solid angle in three-dimensional space is:

$$\Omega = \int_0^{2\pi}\int_0^{\pi}\sin\theta d\theta d\phi = 4\pi$$

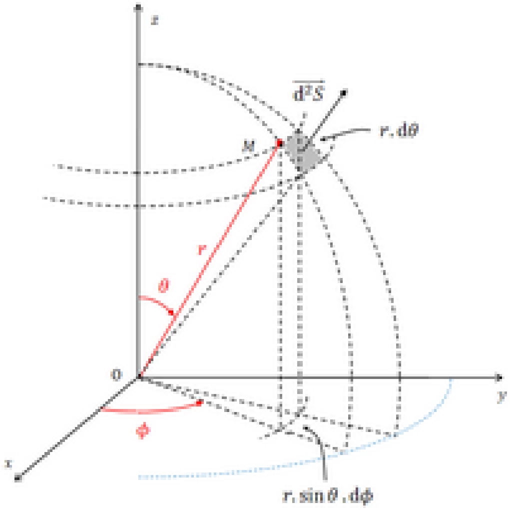

Figure 6: Solid Angles

1.2 Operators in Vector Calculus

Example: Compute the gradient of the scalar function $f(x,y,z) = x^2 + y^2 + z^2$:

$$\nabla f = (2x, 2y, 2z)$$

Example: Compute the divergence of the vector field $\mathbf{A} = (x^2, y^2, z^2)$:

$$\nabla \cdot \mathbf{A} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2) = 2x + 2y + 2z$$

These examples reinforce the mathematical concepts necessary for physics applications.

1.3 Applications:

1.3.1 Calculate the perimeter of a circle C with radius R (simple integral).

Solution:

We have $dl = R\,d\theta$, hence:

$$C = \int_0^{2\pi} R\,d\theta = 2\pi R.$$

1.3.2 Calculate the area of a disk D with radius R (double surface integral).

We use the differential surface element $dS = dp\,p\,d\theta$, hence:

Solution:

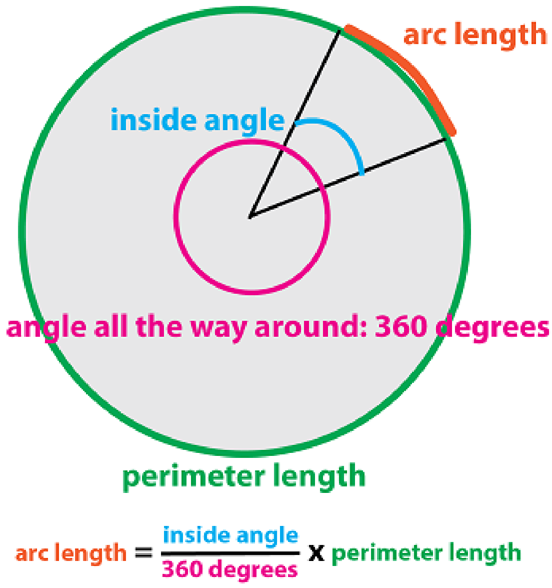

Figure 7: Perimeter of a circle

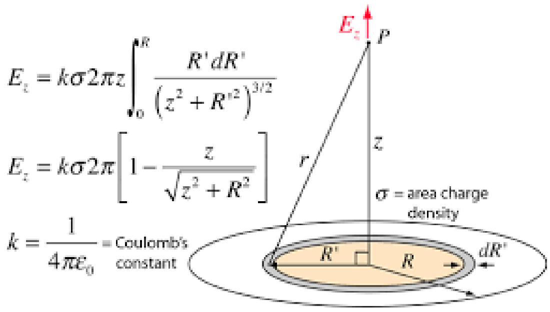

Figure 8: Area of a disk

$$D = \iint_S dp\,d\theta = \int_0^R \int_0^{2\pi} \rho\,d\rho\,d\theta.$$

Evaluating the integral:

$$D = \int_0^{2\pi} d\theta \int_0^R \rho\,d\rho = 2\pi \times \frac{R^2}{2} = \pi R^2.$$

1.3.3 Calculate the volume of a cylinder V with radius R and height H (triple volume integral).

We use the differential volume element $dV = dp\,p\,d\theta\,dz$, hence:

Solution:

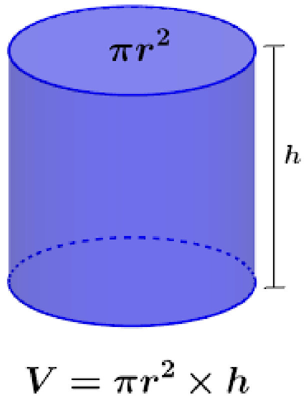

Figure 9: Volume of a cylinder

$$V = \iiint_V d\rho\, d\theta\, dz = \int_0^R \rho\, d\rho \int_0^{2\pi} d\theta \int_0^H dz.$$

Evaluating the integral:

$$V = \int_0^H dz \int_0^{2\pi} d\theta \int_0^R \rho\, d\rho.$$

$$V = H \times 2\pi \times \frac{R^2}{2} = \pi R^2 H.$$

1.3.4 Calculate the surface area of a hemisphere D with radius R (excluding the horizontal disk) (double surface integral).

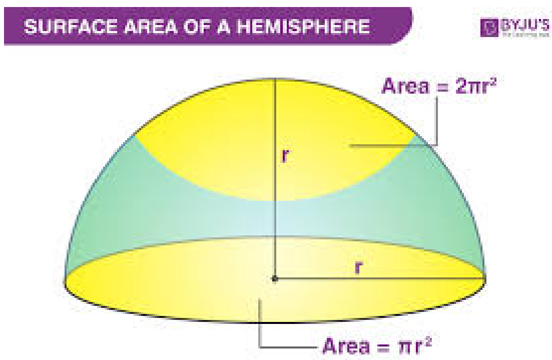

Figure 10: Surface area of a hemisphere

We use the differential surface element $dS = R^2 \sin\theta\, d\theta\, d\phi$, hence:
Solution:

$$D = \iint_S R^2 \sin\theta \, d\theta \, d\phi.$$

Evaluating the integral:

$$D = R^2 \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi.$$

$$D = R^2(-\cos\theta\Big|_0^{\pi}) \times (2\pi) = R^2(1+1) \times 2\pi = 2\pi R^2.$$

1.3.5 Calculate the volume of a sphere V with radius R (triple volume integral).

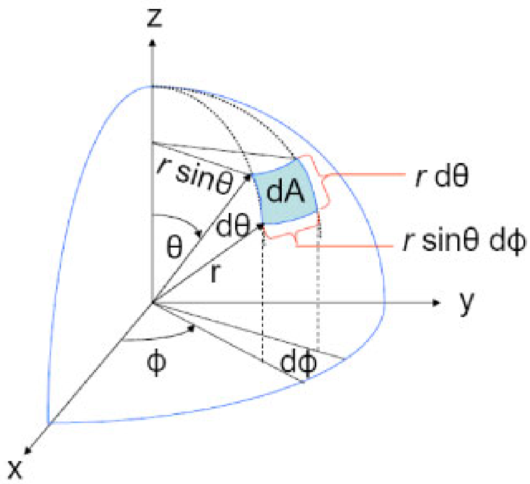

Figure 11: Volume of a sphere

We use the differential volume element $dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$, hence:
Solution:

$$V = \iiint_V r^2 \sin\theta \, dr \, d\theta \, d\phi.$$

Evaluating the integral:

$$V = \int_0^R r^2 \, dr \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi.$$

$$V = \left(\frac{R^3}{3}\right) \times (-\cos\theta\Big|_0^{\pi}) \times 2\pi.$$

$$V = \frac{R^3}{3} \times (1+1) \times 2\pi = \frac{R^3}{3} \times 2 \times 2\pi = \frac{4}{3}\pi R^3.$$