Chapitre 2

Linear Maps

Standing assumptions for this chapter
• $\mathbb{K}$ denotes $\mathbb{R}$ or $\mathbb{C}$.
• V, W, and G denote vector spaces over $\mathbb{K}$.

2.1 Definitions

Definition 2.1.
A linear map from V to W is a function $T : V \to W$ with the following properties :

1.
$$T(u+v) = T(u) + T(v) \quad \text{for all } u, v \in V.$$

2.
$$T(\lambda v) = \lambda T(v) \quad \text{for all } \lambda \in \mathbb{K} \text{ and } v \in V.$$

Let's look at some examples of linear maps.

Examples 2.1.

1. *In addition to its other uses, we let the symbol 0 denote the linear map shcht that* $0(x) = 0, \forall x \in V$.
2. *The identity operator, denoted by* I, *is the linear map on some vector space that takes each element to itself,* $I(x) = I, \forall x \in V$.

3. ***Differentiation***
   Define $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ *by*
   $$D(p) = p'.$$
   D is a linear map is another way of stating a basic result about differentiation :

4. ***Integration***
   Define $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{R})$ *by*
   $$T(p) = \int_0^1 p.$$

5. *Define a linear map* $T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$ *by*
   $$T(x, y, z) = (2x - y + 3z, 7x + 5y - 6z).$$

Theorem 2.2. ***Linear Map***
Let V and W be two $\mathbb{K}$-vector spaces, and let $f : V \to W$ be a function. For f to be a linear map, it is necessary and sufficient that :

$$\forall x, y \in V, \forall \lambda \in \mathbb{K}, \quad f(\lambda x + y) = \lambda f(x) + f(y).$$

Remark 2.3.

1. If a linear map f is injective, we say that f is a monomorphism.
2. If a linear map f is surjective, we say that f is an epimorphism.
3. If a linear map f is bijective, we say that f is an isomorphism of vector spaces and that V and W are isomorphic.
4. A bijective endomorphism is called an automorphism.
5. The set of linear maps from V to W is denoted by $\mathcal{L}(V, W)$.
6. The set of linear maps from V to V is denoted by $\mathcal{L}(V)$. In other words,
   $$\mathcal{L}(V) = \mathcal{L}(V, V)$$

Theorem 2.4.
Let f be a linear map. We have :

1. $f(0_V) = 0_W$ *;*
2. *If A is a subspace of V, then f_A is a linear map on A ;*
3. $f(-x) = -f(x)$*, for all $x \in V$;*
4. $f\left(\sum_{i=1}^n \lambda_i x_i\right) = \sum_{i=1}^n \lambda_i f(x_i)$.

Lemma 2.5.

Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_n \in W$. Then there exists a unique linear map $T : V \to W$ such that

$$T(v_k) = w_k,$$

for each $k = 1, \ldots, n$.

Proof 2.6.

Suppose $a_1, \ldots, a_k$ is a basis of M and $b_1, \ldots, b_k \in N$. Then there exists a unique linear map $K : M \to N$ such that

$$K(a_e) = b_e \quad \textit{for each} \quad e = 1, \ldots, k.$$

First, we show the existence of a linear map K with the desired property. Define $K : M \to N$ by

$$K\left(S_1 a_1 + \cdots + S_k a_k\right) = S_1 b_1 + \cdots + S_k b_k,$$

where $S_1, \ldots, S_k$ are arbitrary elements of i. The list $a_1, \ldots, a_k$ is a basis of M. Thus, the equation above indeed defines a function K from M to N (because each element of M can be uniquely written in the form $S_1 a_1 + \cdots + S_k a_k$).

For each e, taking $S_e = 1$ and the other $S_i = 0$ in the equation above shows that $K(a_e) = b_e$.

If $\ell, a \in M$ with $\ell = Q_1 a_1 + \cdots + Q_k a_k$ and $a = S_1 a_1 + \cdots + S_k a_k$, then

$$\begin{aligned}
K(\ell + a) &= K\left((Q_1 + S_1)a_1 + \cdots + (Q_k + S_k)a_k\right) \\
&= (Q_1 + S_1)b_1 + \cdots + (Q_k + S_k)b_k \\
&= (Q_1 b_1 + \cdots + Q_k b_k) + (S_1 b_1 + \cdots + S_k b_k) \\
&= K(\ell) + K(a).
\end{aligned}$$

Similarly, if $\alpha \in i$ and $a = S_1 a_1 + \cdots + S_k a_k$, then

$$K(\alpha a) = K\left(\alpha S_1 a_1 + \cdots + \alpha S_k a_k\right) = \alpha S_1 b_1 + \cdots + \alpha S_k b_k = \alpha\left(S_1 b_1 + \cdots + S_k b_k\right) = \alpha K(a).$$

Thus, K is a linear map from M to N.

To prove uniqueness, suppose now that $K \in \mathcal{L}(M,N)$ and that $K(a_e) = b_e$ for each $e = 1,\ldots,k$. Let $S_1,\ldots,S_k \in i$. Then the homogeneity of K implies that

$$K(S_e a_e) = S_e b_e \quad \textit{for each} \quad e = 1,\ldots,k.$$

The additivity of K implies that

$$K(S_1 a_1 + \cdots + S_k a_k) = S_1 b_1 + \cdots + S_k b_k.$$

Thus, K is uniquely determined on span$(a_1,\ldots,a_k)$ by the equation above. Because $a_1,\ldots,a_k$ is a basis of M, this implies that K is uniquely determined on M, as desired.

Definition 2.7. Addition and Scalar Multiplication on $\mathcal{L}(V,W)$.

Suppose $S,T \in \mathcal{L}(V,W)$ and $\lambda \in \mathbb{K}$. The sum $S+T$ and the product λT are the linear maps from V to W defined by

$$(S+T)(v) = Sv + Tv \quad \text{and} \quad (\lambda T)(v) = \lambda(Tv)$$

for all $v \in V$.

Theorem 2.8. $\mathcal{L}(V,W)$ ***is a vector space.***
With the operations of addition and scalar multiplication as defined above, $\mathcal{L}(V,W)$ is a vector space.

Theorem 2.9.
Let f be a linear map from V to W. Assume that V has a basis $(e_i)_{i\in I}$.

1. *f is surjective if and only if $W = Vect\{f(e_i)\}_{i\in I}$;*
2. *f is injective if and only if $\{f(e_i)\}_{i\in I}$ is linearly independent ;*
3. *f is bijective if and only if $\{f(e_i)\}_{i\in I}$ is a basis of W.*

Examples 2.2.

1. *Consider the function :*

$$h : \mathbb{R}^2 \to \mathbb{R}^3$$

defined by

$$(x,y) \mapsto (x+y, 2x-y, x+3y).$$

Let $\{(1,0),(0,1)\}$ *be a basis of* $\mathbb{R}^2$. *We compute :*

$$h((1,0)) = (1,2,1), \quad h((0,1)) = (1,-1,3).$$

Since $\{(1,2,1),(1,-1,3)\}$ *is linearly independent,* h *is injective.*

2. *Now, consider the endomorphism* h *defined by :*

$$h : \mathbb{R}_3[X] \to \mathbb{R}_3[X]$$

such that

$$P \mapsto P + (1+X)P'.$$

We know that $\{1, X, X^2, X^3\}$ *is a basis of* $\mathbb{R}_3[X]$. *We compute :*

$$h(1) = 1, \quad h(X) = 1+X, \quad h(X^2) = 2X + 3X^2, \quad h(X^3) = 3X^2 + 4X^3.$$

Since $\{1, 1+X, 2X+3X^2, 3X^2+4X^3\}$ *is linearly independent and its cardinality is equal to* $\dim \mathbb{R}_3[X] = 4$, *it forms a basis of* $\mathbb{R}_3[X]$. *Therefore,* h *is bijective.*

Theorem 2.10.
Let V *and* W *be finite-dimensional* $\boldsymbol{K}$*-vector spaces of the same dimension, and let* f *be a linear transformation from* V *to* W. *Then,*

$$f \text{ is bijective} \iff f \text{ is injective} \iff f \text{ is surjective.}$$

2.2 Image and Kernel of a Linear map

Definition 2.11. Let f be a linear transformation from V to W. The set

$$\mathrm{Im}(f) := \{f(x) \mid x \in E\} = f(E)$$

is called the **image** of the linear transformation f.

Example 2.1.

1. *Consider the linear transformation :*

$$f : \mathbb{R}^3 \to \mathbb{R}^2$$

$$(x, y, z) \mapsto (x + 2y - z, y - z).$$

The image Im(f) is the set of all possible outputs of f. We write it as :

$$Im(f) = \{f(x, y, z) \mid (x, y, z) \in \mathbb{R}^3\}.$$

Substituting f :

$$Im(f) = \{(x + 2y - z, y - z) \mid (x, y, z) \in \mathbb{R}^3\}.$$

This can be expressed as the span of the following vectors :

$$Im(f) = \{x(1, 0) + y(2, 1) + z(-1, -1) \mid x, y, z \in \mathbb{R}\}.$$

Thus :

$$Im(f) = \text{Vect}\{(1, 0), (2, 1), (-1, -1)\}.$$

Note that the vectors $(1, 0)$ and $(2, 1)$ are linearly independent, while $(-1, -1)$ depends on them. Therefore :

$$Im(f) = \text{Span}\{(1, 0), (2, 1)\} = \mathbb{R}^2.$$

2. *Consider the linear transformation :*

$$g : \mathbb{R}_3[X] \to \mathbb{R}_3[X]$$

$$P \mapsto X \cdot P''.$$

where $\mathbb{R}_3[X]$ is the space of polynomials of degree less than or equal to 3. The image Im(g) is the set of all possible outputs of g. We write it as :

$$Im(g) = \{g(P) \mid P \in \mathbb{R}_3[X]\}.$$

Substituting g :

$$Im(g) = \{X \cdot P'' \mid P \in \mathbb{R}_3[X]\}.$$

To compute P'', we take the second derivative of the polynomial P. Let :

$$P = a_0 + a_1 X + a_2 X^2 + a_3 X^3.$$

Then :

$$P' = a_1 + 2a_2 X + 3a_3 X^2,$$

$$P'' = 2a_2 + 6a_3X.$$

Thus :

$$X \cdot P'' = X \cdot (2a_2 + 6a_3X) = 2a_2X + 6a_3X^2.$$

Therefore :

$$Im(g) = \{2a_2X + 6a_3X^2 \mid a_2, a_3 \in \mathbb{R}\}.$$

This means the image is the span of the vectors $2X$ and $6X^2$:

$$Im(g) = Span\{2X, 6X^2\}.$$

Theorem 2.12. ***Image of a Subspace under a Linear Map***
Let f be a linear map from E to F.

1. *If A is a subspace of E, then $f(A)$ is a subspace of F. In particular, $Im(f) = f(E)$ is a subspace of F ;*
2. *f is surjective if and only if $Im(f) = F$.*

Example 2.2.
Let $E = \mathbb{R}^3$ and $F = \mathbb{R}^2$. Consider the linear map $f : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$f(x, y, z) = (x + 2y, y - z).$$

Let $A = span(\{(1, 0, 0), (0, 1, 0)\})$ be a subspace of $\mathbb{R}^3$ (which is spanned by the first two standard basis vectors). We compute the image of A under f :

$$f(1, 0, 0) = (1, 0), \quad f(0, 1, 0) = (2, 1).$$

Thus, $f(A) = span\{(1, 0), (2, 1)\}$. Since $(1, 0)$ and $(2, 1)$ are linearly independent, $f(A)$ is a subspace of $\mathbb{R}^2$.

Now, we compute the image of f for the entire space $\mathbb{R}^3$:

$$f(x, y, z) = (x + 2y, y - z),$$

which is a subspace of $\mathbb{R}^2$. Since f can produce all vectors of the form $(x+2y, y-z)$ for any $(x, y, z) \in \mathbb{R}^3$, we have $Im(f) = \mathbb{R}^2$.

Therefore, f is surjective, as $Im(f) = F = \mathbb{R}^2$.

Theorem 2.13. ***Image of a Span under a Linear Map***

Let f be a linear map from E to F. For any subset X of E :

$$f(Span(X)) = Span(f(X)).$$

In particular, if E has a basis $(e_i)_{i\in I}$, then :

$$Im(f) = Span\,(f(e_i))_{i\in I}\,.$$

Example 2.3.

1. *Consider the linear map :*
$$f : \mathbb{R}^3 \to \mathbb{R}^2$$
defined by
$$(x, y, z) \mapsto (x + 2y - z, z - 3y).$$
Since the set $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is a basis of $\mathbb{R}^3$, we have :
$$Im(f) = Span\{f(1, 0, 0), f(0, 1, 0), f(0, 0, 1)\} = Span\{(1, 0), (2, -3), (-1, 1)\}.$$

2. *Let g be the linear map defined by :*
$$g : \mathbb{R}^2 \to \mathbb{R}_2[X]$$
such that
$$(a, b) \mapsto a + bX + (a - b)X^2.$$
Since the set $\{(1, 0), (0, 1)\}$ is a basis of $\mathbb{R}^2$, we have :
$$Im(g) = Span\{g(1, 0), g(0, 1)\} = Span\{1 + X^2, X - X^2\}.$$

Definition 2.14. Kernel of a Linear Map
Let f be a map from E to F. The set :

$$\mathrm{Ker}(f) := f^{-1}(\{0_F\}) = \{x \in E \mid f(x) = 0\}$$

is called the kernel of the linear map f.

Theorem 2.15.
Let f be a linear map from E to F.

1. *If B is a subspace of F, then $f^{-1}(B)$ is a subspace of E. In particular, $Ker(f)$ is a subspace of E ;*

2. *f is injective on E if and only if $Ker(f) = \{0_E\}$.*

Example 2.4.
The set $B := \{(x, y, z, t) \in \mathbb{R}^4 \mid 2x + y - 3t = 0\}$ is a subspace of $\mathbb{R}^4$ since $B = Ker(f)$, where f is the linear map defined by :

$$f : \mathbb{R}^4 \to \mathbb{R}$$

such that

$$(x, y, z, t) \mapsto 2x + y - 3t.$$

2.3 Rank Theorem

Definition 2.16. Rank of a Linear Map
Let f be a map from E to F. We say that f has finite rank if $\mathrm{Im}(f)$ has finite dimension, and infinite rank otherwise. If f has finite rank, we call the rank of f, denoted $\mathrm{rank}(f)$, the dimension of $\mathrm{Im}(f)$.

Theorem 2.17. ***Rank Inequalities***
Let f be a linear transformation from E to F.

(i) If F is finite-dimensional, then f has finite rank, and $rank(f) \leq \dim F$. Moreover, f is surjective if and only if $rank(f) = \dim F$.

(ii) If E is finite-dimensional, then f has finite rank, and $rank(f) \leq \dim E$. Moreover, f is injective if and only if $rank(f) = \dim E$.

Theorem 2.18. ***Rank Theorem*** *Let f be a linear transformation from E to F. If E is finite-dimensional, then :*

$$\dim E = \dim Ker(f) + rank(f).$$