Chapter 2

Review of Probability Theory

Contents

2.1 Random Experiment and Event

1. **Random Experiment**:

   Definition 2.1.1. *A random experiment (R.E.) is any experiment whose outcome is governed by chance.*

2. **Sample Space**:

   Definition 2.1.2. *The set of all possible outcomes of a random experiment is called the sample space, typically denoted as* Ω.

3. **Event**:

   Definition 2.1.3. *An event in* Ω *is a subset of* Ω.

   - An event is *certain* if it always occurs.
   - An event is *impossible* if it never occurs.
   - The *complementary event* of A is the event that occurs when A does not occur, and is denoted as $\overline{A}$.
   - The event $A \cup B$ occurs if A occurs or B occurs.
   - The event $A \cap B$ occurs if both A and B occur.
   - The event $A - B$ occurs if A occurs but not B.
   - Events are *incompatible (disjoint)* if $A \cap B = \emptyset$, meaning A and B cannot both happen.

Examples 2.1.1. *Random experiment: "Throwing a six-sided die" The sample space:*

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

- *Event A: "Getting the number 2"* $A = \{2\} \subset \Omega$
- *Event B: "Getting an even number"* $B = \{2, 4, 6\} \subset \Omega$

- *The complementary event of B ("Getting an odd number")* $\overline{B} = \{1,3,5\}$
- *Event C: "Getting a number less than 7" (a certain event)* $C = \{1,2,3,4,5,6\}$
- *Event D: "Getting a number greater than 8" (an impossible event)* $D = \emptyset$
- *The event* $B - A = \{4,6\}$
- *The event* $A \cup B = \{2,4,6\}$
- *The event* $A \cap B = \{2\}$

A and B are not incompatible because $A \cap B \neq \emptyset$.

2.2 Classical Definition of Probability

Definition 2.2.1. *For each event A in a random experiment, we define the probability of event A as:*

$$P(A) = \frac{\text{number of favorable outcomes for } A}{\text{total number of possible outcomes}} = \frac{\text{number of elements in } A}{\text{number of elements in } \Omega}$$

Examples 2.2.1. *Throwing a coin and observing the upper face is a random experiment. The sample space is:*

$$\Omega = \{Heads, Tails\}$$

- *Event A: "Getting heads"* $P(A) = \frac{1}{2}$
- *Event B: "Getting tails"* $P(B) = \frac{1}{2}$

2.3 Probability

Definition 2.3.1. *A probability is a function* $P : \Omega \to [0,1]$ *such that for any event* $A \in \Omega$*, we have:*

- $P(\Omega) = 1$
- *For any incompatible events A and B,* $P(A \cup B) = P(A) + P(B)$

2.3.1 Properties

- $P(\emptyset) = 0$
- $0 \leq P(A) \leq 1$
- $P(\overline{A}) = 1 - P(A)$
- If $A \subset B$, then $P(A) \leq P(B)$
- $P(A - B) = P(A) - P(A \cap B)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

2.4 Conditional Probabilities

Definition 2.4.1. *Let A and B be two events such that $P(B) \neq 0$. The probability of A given B, denoted $P(A/B)$ or $P_B(A)$, is given by:*

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

Examples 2.4.1. *A class consists of 17 students.*

- *8 students study English.*
- *7 students study German.*
- *2 students study both languages.*

We know that a student studies English. What is the probability that the student studies German?

$$P(German/English) = \frac{P(German \cap English)}{P(English)}$$

Given:

- $P(English) = \frac{8}{17}$
- $P(German) = \frac{7}{17}$
- $P(German \cap English) = \frac{2}{17}$

Thus:

$$P(German/English) = \frac{2}{17}/\frac{8}{17} = \frac{1}{4}$$

Remark 2.4.1. *A and B are independent events if and only if $P(A \cap B) = P(A)P(B)$.*

Remark 2.4.2. *If A and B are independent events, then:*

$$P(A/B) = P(A) \quad and \quad P(B/A) = P(B)$$

2.5 Total Probability Formula

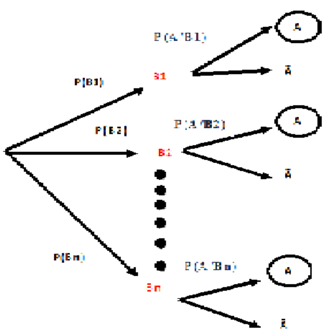
Definition 2.5.1. *Let E be a set. $B_1, B_2, \ldots, B_n$ form a partition of E if:*

- $\forall i \in \{1, \ldots, n\}$, $B_i \neq \emptyset$
- *For all $i \neq j$, $B_i \cap B_j = \emptyset$*
- $B_1 \cup B_2 \cup \cdots \cup B_n = E$

2.5.1 Bayes theorem

Theorem 2.5.1. *If events $B_1, B_2, \ldots, B_n$ form a partition of Ω and A is another event, then:*

$$P(A) = P(B_1)P(A/B_1) + P(B_2)P(A/B_2) + \cdots + P(B_n)P(A/B_n)$$



Examples 2.5.1. *We are given three boxes such that: - Box I contains* 10 *light bulbs,* 4 *of which are defective. - Box II contains* 6 *light bulbs,* 1 *of which is defective. - Box III contains* 8 *light bulbs,* 3 *of which are defective.*

A box is chosen at random, and a light bulb is randomly drawn from that box. What is the probability that the light bulb is defective?