Contents

Chapter 1

Taylor's Formula and Limited Development

The limited development $LD_n(x_0)$ is useful in many areas of mathematics and physics, including solving differential equations, computing integrals, evaluating limits, and analyzing the local behavior of a function along with its polynomial approximation.

1.1 Taylor's Formula

1.1.1 Taylor's Formula

The Taylor formula allows the approximation of a function that is differentiable multiple times in the neighborhood of a point by a polynomial whose coefficients depend only on the derivatives of the function at that point.

Definition 1.1.1. *A function that is continuous on $[a,b]$ and differentiable at $x_0 \in]a,b[$ can be expressed in a neighborhood of x_0 as follows*

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + R(x),$$

where $R(x) = \varepsilon(x)(x - x_0)$, and $\lim_{x \to x_0} \varepsilon(x) = 0$, This shows that if f is differentiable, then f can be approximated by a polynomial of degree 1 (a line).

Example 1.1.2. *Consider the function $f = e^x$, and $x_0 = 0$. f can be written as*

$$f(x) \simeq f(0) + (x-0)f'(0) = x+1$$

Taylor's formula generalizes this result by showing that functions that are $n-$times differentiable can be approximated in a neighborhood of x_0 by polynomials of degree n, that is to say

$$\begin{aligned} f(x) &= \underbrace{\sum_{k=0}^{n} \frac{f^{(k)(x_0)}}{k!}(x-x_0)^k}_{p_n(x)} + R_n(x), \\ &= f(x_0) + (x-x_0)f'(x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(x), \end{aligned}$$

where $R_n(x)$ is the remainder of order n, such that

$$\begin{aligned} R_n(x) &= \frac{f^{(n+1)}(x_0)}{(n+1)!}(x-x_0)^{n+1} + \cdots \\ &= \varepsilon(x)(x-x_0)^n \qquad \textit{and} \qquad \lim_{x\to x_0} \varepsilon(x) = 0. \end{aligned}$$

1.1.2 Taylor's Theorem

Let f and $g : [a,b] \longrightarrow \mathbb{R}$ be two functions satisfying the following conditions:

1. $f \in C^n([a,b])$, and $f^{(n)}$ is differentiable on $]a,b[$.
2. The function $g \in C([a,b])$ and it is differentiable on $]a,b[$ and $\forall x \in]a,b[\ ,\ g'(x) \neq 0$, for $x_0 \in [a,b]$: then $\forall x \in [a,b],\quad x \neq x_0$ we have:

$$\begin{aligned} f(x) &= f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(x_0,x).....(*) \\ &= \textstyle\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + R_n(x_0,x) \end{aligned}$$

such that $R_n(x_0,x) = \dfrac{f^{(n+1)}(c)(x-c)^n\ g(x)g(x_0)}{n!\ g'(c)},\quad c \in]x,x_0[.....(**)$

- The expression (*) is called the Taylor formula with generalized remainder (**).
- he choice of different functions g satisfying condition (**) leads to different forms of the remainder $R_n(x_0,x)$.

1.1.3 Taylor's three formulas

Notation 1.1.3. *Let $I = [a, b]$ be an interval of $\mathbb{R}$, x_0 be an interior point of I, and let $f : I \longrightarrow R$ be a function. We fix a natural number n.*

We say that a function is of class C^n on I if it is n times differentiable on I, and its n-th derivative is continuous on I.

Taylor-Lagrange

Theorem 1.1.4. *Let f be of class C^{n+1} on I, and $x_0 \in [a, b]$. For all $x \in [a, b]$, $x \neq x_0$, we have:*

$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}, \quad c \in]x, x_0[.$

The term $\frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$ is called the Lagrange remainder.

Example 1.1.5.

1. *Consider the function* $\sin(x)$. *The* ***Taylor-Lagrange*** *formula up to order* 3 *in the neighborhood of* 0 *is written as follows*

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^4}{4!}\cos(c).$$

2. *Consider again $x \longrightarrow e^x$. The* ***Taylor-Lagrange*** *formula up to order 4 in the neighborhood of 0 is writtenas follows*

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}\ e^c.$$

Taylor-Maclaurin

If we set $x_0 = 0$ in Taylor-Lagrange's formula, we obtain Maclaurin's formula

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}, \quad 0 < \theta < 1.$$

Taylor-Young

Theorem 1.1.6. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function, and let $x_0 \in [a, b]$, suppose that $f^{(n)}(x_0)$ exists and finite, then $\forall x_0 \in [a, b]$*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + o(x - x_0)^n,$$

the remainder $R_n(x_0, x) = o(x - x_0)^n$ is called Young's remainder, and it satisfies the following property:

$$\lim_{x \to x_0} \frac{R_n(x_0, x)}{(x - x_0)^n} = 0.$$

By setting $\varepsilon(x) = \dfrac{R_n(x_0, x)}{(x - x_0)^n}$ for $x \neq x_0$ and $\varepsilon(x_0) = 0$ we obtain the Young remainder in the following form: $R_n(x_0, x) = \varepsilon(x)(x - x_0)^n$ where $\lim\limits_{x \to x_0} \varepsilon(x) = 0$
If $x_0 = 0$, we obtain the Maclaurin-Young formula

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + x^n \varepsilon(x) \quad where \quad \lim_{x \to 0} \varepsilon(x) = 0$$

Example 1.1.7. *The Taylor-Young formula for the function $\sin(x)$ up to order $2n + 1$ at 0 is written as follows*

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n + 1)!} + x^{2n+1}\varepsilon(x).$$

Indeed, we must calculate the successive derivatives of $\sin(x)$ at 0. We have

$$\sin(0) = 0, \ \sin'(0) = \cos(0) = 1, \quad \sin''(0) = -\sin(0) = 0, \cdots$$

More generally, for all $k \in \mathbb{N}$ we have

$$\sin^{(2k)}(0) = 0, \quad and \quad \sin^{(2k+1)}(0) = (-1)^k \cos(0) = (-1)^k$$

hence the result.

1.2 Limited Development

1.2.1 Developments limited to the neighborhood of 0

Definition 1.2.1. *Let f be a function defined in the neighborhood of $x = 0$, possibly except at 0. We say that f admits a limited development of order n in the neighborhood of 0 if there exist real numbers $a_0,\ a_1, a_2, \cdots, a_n \in \mathbb{R}$ and a function ε such that for any non-zero element x of an interval I of $\mathbb{R}$:*

$$\begin{aligned} f(x) &= a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + x^n\varepsilon(x) \\ &= P_n(x) + x^n\varepsilon(x) \end{aligned},$$

such that $\lim\limits_{x\to 0}\varepsilon(x) = 0$.

Remark 1.2.2. *The polynomial $P_n(x)$ is called regular part of the limited development and $x^n\varepsilon(x)$ is remainder or complementary part.*

Example 1.2.3. *Let $f = \dfrac{1}{1-x}$. f admits $LD_n(0)$, indeed:*
Since $1 - x^{n+1} = (1-x)(1 + x + x^2 + \cdots + x^n)$, we have

$$\frac{1}{1-x} - \frac{x^{n+1}}{1-x} = \frac{1-x^{n+1}}{1-x} = \frac{(1-x)(1+x+\cdots+x^n)}{1-x} = 1 + x + \cdots + x^n,$$

where

$$\frac{1}{1-x} = 1 + x + \cdots + x^n + \frac{x^{n+1}}{1-x} = 1 + x + \cdots + x^n\frac{x}{1-x},$$

Therefore, the function $f(x) = \dfrac{1}{1-x},\ \ x \neq 1$ admits a limited development of order n at $x = 0$, with $\varepsilon(x) = \dfrac{x}{1-x}$, where $\lim\limits_{x\to 0}\varepsilon(x) = 0$.

1.2.2 Properties of Limited Development

- If f admits a $LD_n(x_0)$, then $\lim\limits_{x\to x_0} f(x)$ exists, is finite, and is equal to a_0. This criterion is generally used to show that a function does not admit $LD_n(x_0)$.

  For example the function $\ln(x)$ does not admit $LD_n(0)$, because $\lim\limits_{x\to 0}\ln(x) = -\infty$.

- A function does not necessarily have an $LD_n(x_0)$, but if it does, it is unique.
- **Parity**
  - **Even function** The $LD_n(x_0)$ of an even function has a main part that contains only monomials of even degree. That is to say the coefficients $a_{2k+1} = 0$.
  - **Odd function** The $LD_n(x_0)$ of an odd function has a main part that contains only monomials of odd degree. That is to say the coefficients $a_{2k} = 0$.
- The $LD_n(x_0)$ of a polynomial of degree n is the polynomial itself.

1.2.3 Obtaining Limited Development Using the Taylor-Young Formula

Theorem 1.2.4. *If f is of class C^{n-1} in a neighborhood of a and $f^{(n)}(a)$ exists, then f has a limited expansion of order n in a neighborhood of a, given by the Taylor-Young formula*

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + (x-a)^n\varepsilon(x)$$

where $\lim_{x\to a}\varepsilon(x) + 0$.

particular case: If $f^{(n)} = (0)$ exists then f has the following limited development

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + x^n\varepsilon(x), \quad \textit{such that} \quad \lim_{x\to 0}\varepsilon(x) + 0.$$

Corollary 1.2.5. *If $f^{(n)}(0)$ exists and if f admits a limited expansion of order n*

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + x^n\varepsilon(x),$$

then $f(0) = a_0, \quad \frac{f'(0)}{1!} = a_1, \quad \frac{f''(0)}{2!} = a_2, \cdots, \frac{f^{(n)}(0)}{n!} = a_n.$

1.2.4 Limited Development of usual Functions

Below, we show some well-known limited development of common function of order n, at $x = 0$ using Maclaurin's formula :

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + o(x^n)$$

$$\ln(x1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1}\frac{x^n}{n} + o(x^n)$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + o(x^n)$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} - \cdots + (-1)^{n-1}\frac{1 \times 3 \times 5 \times \cdots \times (2n-3)}{2^n n!}x^n + o(x^n)$$

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3x^2}{8} - \cdots + (-1)^n\frac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{2^n n!}x^n + o(x^n)$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + o(x^n)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + o(x^{2n+1})$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2})$$

Remark 1.2.6. *We will often work with $x_0 = 0$, based on changes of variables:*

1. *If $x_0 \in \mathbb{R}^*$, we put $t = x - x_0$, and then $t \longrightarrow 0$ when $x \longrightarrow x_0$.*
2. *If $x_0 \longrightarrow \infty$, we put $t = \frac{1}{x}$, and then $t \longrightarrow 0$ when $x \longrightarrow \infty$.*

Example 1.2.7. *Find $LD_3(\frac{\pi}{4})$ for the function $x \longrightarrow \sin(x)$.*
We put $t = x - \frac{\pi}{4}$, then $t \to 0$ when $x \to \frac{\pi}{4}$. Thus, $x = t + \frac{\pi}{4}$.
So

$$\begin{aligned} f(x) &= \sin x = \sin(t+\frac{\pi}{4}) = \sin(t)\cos(\frac{\pi}{4}) + \cos(t)\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}\sin(t) + \frac{\sqrt{2}}{2}\cos(t). \\ &= \frac{\sqrt{2}}{2}\left(t - \frac{t^3}{6} + o(t^3)\right) + \frac{\sqrt{2}}{2}\left(1 - \frac{t^2}{2} + o(t^3)\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}t - \frac{\sqrt{2}}{4}t^2 - \frac{\sqrt{2}}{12}t^3 + o(t^3) \ . \\ &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3 + o\left(\left(x - \frac{\pi}{4}\right)^3\right) \end{aligned}$$

Example 1.2.8. *Find $LD_n(1)$ for the function $x \longrightarrow e^x$.*

We put $t = x - 1$, then $t \to 0$ when $x \to 1$. Thus, $x = t + 1$. Thus

$$\begin{aligned} e^x &= e\left(1 + y + \frac{y^2}{2!} + \cdots + \frac{y^n}{n!} + o(y^n)\right) \\ &= e\left(1 + (x-1) + \frac{(x-1)^2}{2!} + \cdots + \frac{(x-1)^n}{n!} + o((x-1)^n)\right) \end{aligned}$$

1.2.5 Operation on Limited Development

- **Sum:** If f admits a $LD_n(0)$: $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + o(x^n)$,

  and g also admits a $LD_n(0)$: $g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n + o(x^n)$.

  Then $f + g$ admits a $LD_n(0)$, given by the sum of the two expansions:

$$(f+g)(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots + (a_n + b_n)x^n + o(x^n).$$

Example 1.2.9. *Find the $LD_4(0)$ of $ln(1 + x) + e^x$.*

As

$$\begin{aligned} \ln(x+1) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + o(x^4) \\ e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + o(x^4) \end{aligned}$$

Hence: $ln(1 + x) + e^x = 1 + 2x + \frac{x^3}{2} - \frac{5x^4}{24} + o(x^4)$

- **Product:** If f admits a $LD_n(0)$: $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + o(x^n)$,

  and g also admits a $LD_n(0)$: $g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n + o(x^n)$.

  Then the product fg admits a $LD_n(0)$, obtained by retaining only the monomials of degree at most n in the expansion of the product:

$$(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)\,(b_0 + b_1x + b_2x^2 + \cdots + b_nx^n).$$

Example 1.2.10. *Find $LD_3(0)$ of $x \longrightarrow \sin(x)\cos(x)$.*

We have

$$\begin{aligned} \cos(x) &= 1 - \frac{x^2}{2} + o(x^3) \\ \sin(x) &= x - \frac{x^3}{6} + o(x^3) \end{aligned}$$

Then, we develop the product, only considering terms of order 3 or less:

$$\begin{aligned} \cos(x)\sin(x) &= \left(1 - \frac{x^2}{2} + o(x^3)\right)\left(x - \frac{x^3}{6} + o(x^3)\right) \\ &= x - \frac{2x^3}{3} + o(x^3) \end{aligned}$$

- **Quotient:** If f admits a $LD_n(0)$: $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + o(x^n)$,

  and g also admits a $LD_n(0)$: $g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n + o(x^n)$, with $b_0 \neq 0$.

  Then $\frac{f}{g}$ admits a $LD_n(0)$, obtained by performing the division according to increasing degrees, up to order n, of the polynomial $(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)$ by the polynomial $(b_0 + b_1x + b_2x^2 + \cdots + b_nx^n)$.

Example 1.2.11. *Let us compute $LD_5(0)$ for $x \longrightarrow \tan(x) = \dfrac{\sin(x)}{\cos(x)}$*

We have

$$\begin{aligned} \sin(x) &= x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5) \\ \cos(x) &= 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5) \end{aligned}$$

Thus,

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)}{1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5)}$$

Then, we develop the division according to increasing degrees up to order 5:

$x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)$	$1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5)$
$x - \frac{x^3}{2} + \frac{x^5}{24} + o(x^5)$	$x + \frac{x^3}{3} + \frac{2x^5}{15}$
$\frac{x^3}{3} - \frac{x^5}{30} + o(x^5)$	
$\frac{x^3}{3} - \frac{x^5}{6} + o(x^5)$	
$\frac{2x^5}{15} + o(x^5)$	
$\frac{2x^5}{15} + o(x^5)$	
$o(x^5)$	

x.png

Therefore, $\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + o(x^5)$

- **Composition** If f admits a $LD_n(g(0))$:

$$f(x) = a_0 + a_1(x - g(0)) + a_2(x - g(0))^2 + \cdots + a_n(x - g(0))^n + (x - g(0))^n\varepsilon(x),$$

and g also admits a $LD_n(0)$: $g(x) = b_0 + b_1x + \cdots + b_nx^n + x^n\varepsilon(x)$.

Then, $f \circ g$ admits a $LD_n(0)$, obtained by substituting the limited development of g into that of f and keeping only the monomials of degree n or less.

Example 1.2.12. *Let us compute* $LD_3(0)$ *for* $x \longrightarrow \sin\left(\frac{1}{1-x} - 1\right)$.

Since,

$$\begin{aligned} \frac{1}{1-x} - 1 &= -x + x^2 - x^3 + o(x^3) \\ \sin(x) &= x - \frac{x^3}{6} + o(x^3) \end{aligned}$$

Then, we compose, considering only terms of order 3 or less:

$$\begin{aligned}\sin\left(\frac{1}{1-x}-1\right) &= -x+x^2-x^3-\frac{1}{6}(-x^3)+o(x^3)\\ &= -x+x^2-\frac{5x^3}{6}+o(x^3)\end{aligned}$$

- **Differentiability:** If $f : I \longrightarrow \mathbb{R}$ admits a $LD_{n+1}(0)$ and f is differentiated at least $n+1$ times, then f' admits a $LD_n(0)$, obtained by differentiating the limited development of f.

  Example 1.2.13. *compute $LD_3(0)$ for $x \longrightarrow \dfrac{1}{(1-x)^2}$.*
  Since $\dfrac{1}{(1-x)^2} = \left(\dfrac{1}{1-x}\right)'$, and $\dfrac{1}{1-x} = 1+x+x^2+x^3+x^4+o(x^4)$. Derive the $LD_4(0)$ of $\dfrac{1}{1-x}$, we obtain $LD_3(0)$ for $\dfrac{1}{(1-x)^2}$:

  $$\frac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+o(x^3).$$

- **Integration:** If $f : I \longrightarrow \mathbb{R}$ admits a $LD_n(0)$, and f is integrable on I, then f admits a $LD_{n+1}(0)$, obtained by integrating the limited development of f. i.e: if

  $$f(x) = a_0 + a_1x + \cdots + a_nx^n + x^n\varepsilon(x), \text{ where } \lim_{x\to 0}\varepsilon(x) = 0$$

  then

  $$F(x) = \textstyle\int_0^x f(t)dt = a_0x + \dfrac{a_1}{2}x^2 + \cdots + \dfrac{a_n}{n+1}x^{n+1} + x^{n+1}\tau(x), \text{ where } \lim_{x\to 0}\tau(x) = 0.$$

1.2.6 Generalized LD

Let f be a function defined in the neighborhood of 0 except possibly at 0. Suppose that f does not admit a limited expansion in the neighborhood of 0 but the function $x^\alpha f(x)$ (α positive real) admits a limited development in the neighborhood of 0 then for $\alpha \neq 0$

$$x^\alpha f(x) = a_0 + a_1x + \cdots + a_nx^n + o(x^n)$$

Hence $f(x) = \dfrac{1}{x^\alpha}\left(a_0 + a_1x + \cdots + a_nx^n + o(x^n)\right)$.
this expression is called the generalized limited development in the neighborhood of 0.

Example 1.2.14. *Consider the function* $f(x) = \dfrac{1}{x - x^2}$*. The function* f *does not admit a* $LD(0)$ *because* $\lim_{x \to 0} f(x) = +\infty$*. But*

$$xf(x) = x.\frac{1}{x - x^2} = \frac{1}{1 - x} = 1 + x + x^2 + \cdots + x^n + o(x^n).$$

The generalized limited expansion of f *is*

$$\begin{aligned} f(x) &= \frac{1}{x}\left(1 + x + x^2 + \cdots + x^n + o(x^n)\right) \\ &= \frac{1}{x} + 1 + x + \cdots + x^{n-1} + o(x^{n-1}) \end{aligned}$$

Exercise: Find the $LD_4(0)$ of the following functions:

1. $f(x) = \dfrac{x}{\sin(x)}$.
2. $g(x) = \dfrac{1}{\cos(x)}$.
3. $h(x) = \dfrac{\ln(1 + x)}{1 + x}$.
4. $k(x) = e^{\cos(x)}$