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Algebra II, Worksheet 1

Exercise n°1 Let X be a non-empty set, and let $(E,+,\cdot)$ be a $K-$vector space. Consider the set $\mathcal{F}(X,E)=E^X$ of all mappings defined on X with values in E. We define an addition operation (internal operation) +

$$\begin{array}{rlcl} +: & \mathcal{F}(X,E)\times\mathcal{F}(X,E) & \longrightarrow & \mathcal{F}(X,E) \\ & (f,g) & \longmapsto & f+g, \end{array}$$

defined as

$$\forall x\in X:(f+g)(x)=f(x)+g(x),$$

and a scalar multiplication operation (external operation) $\cdot$

$$\begin{array}{rlcl} \cdot: & K\times\mathcal{F}(X,E) & \longrightarrow & \mathcal{F}(X,E) \\ & (\alpha,f) & \longmapsto & \alpha\cdot f, \end{array}$$

defined as

$$\forall x\in X:(\alpha\cdot f)(x)=\alpha\cdot f(x)$$

1. Assume that $(E^X,+)$ is a commutative group with the zero mapping $0_{E^X}=0$ as neutral element. Show that $(E^X,+,\cdot)$ forms a $K-$vector space.

Exercise n°2 : Let $(E,+,\cdot)$ be a $K-$vector space. Prove the following properties :

(1) $\forall x\in E,\forall\alpha\in K:\alpha\cdot 0_E=0_E$ and $0_K\cdot x=0_E$.
(2) $\forall x\in E,\forall\alpha\in K:\alpha\cdot x=0_E\Longleftrightarrow\alpha=0_K$ or $x=0_E$.
(3) $\forall x,y\in E,\forall\alpha\in K:\alpha\cdot(x-y)=\alpha\cdot x-\alpha\cdot y$.
(4) $\forall x\in E,\forall\alpha,\beta\in K:(\alpha-\beta)\cdot x=\alpha\cdot x-\beta\cdot x$.

Exercise n°3 : Determine whether $\mathbb{R}^2$, equipped with the following internal and external operations, forms a $\mathbb{R}-$vector space :

$$(a_1)\ \ \forall(a,b),(c,d)\in\mathbb{R}^2,\forall\alpha\in\mathbb{R}:\begin{cases}(a,b)+(c,d)=(a+b,b+d)\\ \alpha\cdot(a,b)=(a,\alpha b)\end{cases}$$

$$(a_2)\ \ \forall(a,b),(c,d)\in\mathbb{R}^2,\forall\alpha\in\mathbb{R}:\begin{cases}(a,b)+(c,d)=(a+c,b+d)\\ \alpha\cdot(a,b)=(\alpha^2a,\alpha^2b)\end{cases}$$

$$(a_3)\ \ \forall(a,b),(c,d)\in\mathbb{R}^2,\forall\alpha\in\mathbb{R}:\begin{cases}(a,b)+(c,d)=(c,d)\\ \alpha\cdot(a,b)=(\alpha a,\alpha b)\end{cases}$$

Exercise n°4 : Determine whether the following sets are vector subspaces of E over K :

$$F_1=\{(x,y,z)\in E:x+y+z=a,a\in\mathbb{R}\},\quad E=\mathbb{R}^3,K=\mathbb{R}$$
$$F_2=\mathcal{P}(\mathbb{R},\mathbb{R})=\{f\in E,\forall x\in\mathbb{R}:f(-x)=f(x)\},\quad E=\mathcal{F}(\mathbb{R},\mathbb{R}),K=\mathbb{R}$$
$$F_3=\mathfrak{I}(\mathbb{R},\mathbb{R})=\{f\in E,\forall x\in\mathbb{R}:f(-x)=-f(x)\},\quad E=\mathcal{F}(\mathbb{R},\mathbb{R}),K=\mathbb{R}$$
$$F_4=\{P\in E:P(0)=P(1)=0\},\quad E=\mathbb{R}[X],K=\mathbb{R}$$
$$F_5=\mathbb{R}_n[X]=\{P\in E:\deg(P)\leq n\},n\in\mathbb{N},\quad E=\mathbb{R}[X],K=\mathbb{R}$$
$$F_6=\{f\in E:f\text{ is bounded}\},\quad E=\mathcal{F}(\mathbb{R},\mathbb{R}),K=\mathbb{R}$$

where :
$\mathcal{P}(\mathbb{R},\mathbb{R})$ (resp. $\mathfrak{I}(\mathbb{R},\mathbb{R})$) denotes the set of even (resp. odd) mappings from $\mathbb{R}$ to $\mathbb{R}$.
$\mathbb{R}_n[X]$ the set of all polynomials with the indeterminate X of degree less than or equal to n with coefficients in the field $\mathbb{R}$ of real numbers.