Democratic and Popular Republic of Algeria

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Analysis 2

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Exercises Serie N° 1

Note: questions marked (∗) left to the students.

Exercise 1

1. *Show that:*
   - $\arcsin x < \dfrac{x}{\sqrt{1-x^2}}, \qquad \forall x \in]0,1[.$
   - $\arctan x > \dfrac{x}{1+x^2}, \qquad \forall x > 0.$
   - $\arcsin x + \arccos x = \dfrac{\pi}{2}, \qquad \forall x \in]-1,1[. \qquad (*)$
   - $\arctan x + \arctan \dfrac{1}{x} = \dfrac{\pi}{2}, \qquad \forall x > 0. \qquad (*)$
2. *Calculate the derivatives of the following functions:*
   - $f(x) = \arctan(\sqrt{x^2-1})$.
   - $g(x) = \arctan(\arctan x)$.

Exercise 2

1. *Prove that:* $ch(x-y) = ch(x)ch(y) - sh(x)sh(y)$. (∗)
2. *Solve the equation:* $2ch(2x) + 10sh(2x) = 5$.

Exercise 3

Using Taylor-Lagrange formula, prove that:

1. $\left[0, \dfrac{\pi}{2}\right], \quad x - \dfrac{x^3}{6} \leq \sin x \leq x - \dfrac{x^3}{6} - \dfrac{x^5}{125}.$
2. $\forall x \in \mathbb{R}_+, \ 1 + \dfrac{x}{3} - \dfrac{x^2}{9} \leq \sqrt[3]{1+x} \leq 1 + \dfrac{x}{3} - \dfrac{x^2}{9} + \dfrac{5x^3}{91}. \qquad (*)$

Exercise 4

1. *compute* $LD_3(0)$ *for:*
   - $x \longrightarrow \sin\left(\dfrac{1}{1-x} - 1\right).$
   - $x \longrightarrow \dfrac{1}{(1-x)^2}.$
2. *Using the Maclaurin expansion of the function* $\ln(x+1)$*, to show that:*

$$\forall x > 0, \quad x - \frac{x^2}{2} < \ln(x+1) < x.$$

Exercise 5

For fixed real a *we define the function* f_a *by* $f_a(x) = \arctan\left(\dfrac{x+a}{1-ax}\right)$.

1. *Let n be an integer. Determine the limited expansion at 0 up to order $2n-1$ of the derivative function f'_a.*
2. *Deduce the limited expansion of f_a up to order $2n$.*

Exercise 6

1. *Write the Maclaurin formula with the Lagrange remainder for the function $f(x) = e^x$ to order n.*
2. *Show that:*

$$1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} < e < 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{e}{(n+1)!}.$$

3. *Deduce the limit of the sequence $u_n = \sum_{k=0}^{n} \frac{1}{k!}$.*

Exercise 7

Find the limit of the sequence (u_n) defined by:

$$\prod_{k=1}^{n} \left(1 + \frac{k}{n^2}\right), \quad \forall n \in \mathbb{N}^*.$$

Exercise 8

Compute the limited developments up to order n in the neighborhood of x_0 of the following functions:

1. $f(x) = x(ch\ x)^{\frac{1}{x}}, \quad x_0 = 0, \quad n = 3.$
2. $f(x) = \ln(1 + \sin x), \quad x_0 = 0, \quad n = 3.$
3. $f(x) = \tan x, \quad x_0 = 0, \quad n = 5.$
4. $f(x) = \dfrac{\ln(1+x)}{1+x}, \quad x_0 = 0, \quad n = 3.$
5. $f(x) = e^{3x} \sin(2x), \quad x_0 = 0, \quad n = 4.$
6. $f(x) = e^{\sqrt{x}}, \quad x_0 = 1, \quad n = 3.$
7. $f(x) = \dfrac{x^3+2}{x-1}, \quad x_0 = +\infty, \quad n = 3.$

Exercise 9

Calculate the following limits:

1. $\lim\limits_{x \to 0} \left(\dfrac{1}{\ln(1+x)} - \dfrac{1}{x} \right).$
2. $\lim\limits_{x \to 0} \left(\dfrac{e^{3x} \sin 3x}{sh\ (-2x)} \right).$
3. $\lim\limits_{x \to +\infty} \left(1 + \dfrac{1}{x}\right)^x.$
4. $\lim\limits_{x \to 0} \dfrac{x - \sin x}{x^3}.$
5. $\lim\limits_{x \to +\infty} x^2 \left(e^{\frac{1}{x}} - e^{\frac{1}{1+x}}\right).$