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1st Year Mathematics (Algebra 2)

Series of Tutorial No. 1
Vector space

Exercise 1.

1. *Consider in* $\mathbb{R}^*$ *the internal operation* $\oplus$ *and the external operation* $\cdot$ *defined by:*

$$\forall x, y \in \mathbb{R}^*, \quad x \oplus y = x + y$$

and

$$\forall l \in \mathbb{R}, \quad \forall x \in \mathbb{R}^*, \quad l \cdot x = x^l.$$

Show that $(\mathbb{R}^*, \oplus, \cdot)$ *is a vector space over* $\mathbb{R}$.

2. *Let* $\mathbb{R}^2$ *be equipped with the internal operation defined by:*

$$\forall (x_1, x_2), (y_1, y_2) \in \mathbb{R}^2, \quad (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$

and an external operation defined by:

$$\forall l \in \mathbb{R}, \quad \forall (x_1, x_2) \in \mathbb{R}^2, \quad l \cdot (x_1, x_2) = (l \cdot x_1, l \cdot x_2).$$

Is $(\mathbb{R}^2, +, \cdot)$ *a vector space over* $\mathbb{R}$*?*

Exercise 2.
Among the following sets, determine which are, or are not, vector subspaces:

1. $E_1 = \{(x, y, z) \in \mathbb{R}^3 \mid x + 2y + z = 0\}$
2. $E_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 5\}$
3. $E_3 = \{(x, y, z, t) \in \mathbb{R}^4 \mid x = 2y = 3z = 4t\}$
4. $E_4 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$
5. $E_5 = \{(x, y) \in \mathbb{R}^2 \mid y = x^3\}$
6. $E_6 = \{(x, y, z) \in \mathbb{R}^3 \mid x + 2y + 3z = 0\} \setminus \{(x, y, z) \in \mathbb{R}^3 \mid x - y + z = 1\}$
7. $E_7 = \{(x, y, z) \in \mathbb{R}^3 \mid x + 2y + 3z = 0\} \cup \{(x, y, z) \in \mathbb{R}^3 \mid x - y + z = 0\}$
8. $E_8 = \{P \in \mathbb{R}[X] \mid P(0) = P(1)\}$
9. $E_9 = \{P \in \mathbb{R}[X] \mid \deg(P) \leq 2\}$
10. $E_{10} = \{P \in \mathbb{R}[X] \mid P'(X) \text{ divides } P(X)\}$
11. $E_{11} = \{f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) \mid f \text{ is bounded}\}$
12. $E_{12} = \{f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) \mid f \text{ is bounded below}\}$

13. $E_{13} = \{f \in C^\infty(\mathbb{R}, \mathbb{R}) \mid f' + 3f = 0\}$

14. $E_{14} = \{f \in C^\infty([a, b], \mathbb{R}) \mid \int_a^b f(t)dt = 0\}$

Exercise 3.

1. *Let the vector* $t = (2, 1, 0, -3)$*. Does this vector belong to the subspace of* $\mathbb{R}^4$ *spanned by the vectors:*
$$u = (2, 3, 1, 0), \quad v = (1, -1, 2, 3), \quad w = (0, 1, 3, -1)?$$

2. *Let* G *be the subspace of* $\mathbb{R}^3$ *spanned by the vectors* $(3, 1, -2)$ *and* $(1, -1, 4)$*.*

   Prove that there exist real numbers a, b, c *such that:*
$$G = \{(x, y, z) \in \mathbb{R}^3 \mid ax + by + cz = 0\}.$$

Exercise 4.
Let E *be the set of all convergent real sequences,* F *the set of all real sequences converging to zero, and* G *the set of all constant real sequences.*

1. *Prove that* E*,* F*, and* G *are subspaces of* $\mathbb{R}^{\mathbb{N}}$*.*

2. *Prove that* $E = F \oplus G$*.*

Exercise 5.
Are the following families linearly independent?

1. $\{(1, -2, 3), (0, 1, -1), (2, 3, 5)\}$ *in* $\mathbb{R}^3$*.*

2. $\{(3, -1, 4), (1, 0, -2), (5, -3, 10)\}$ *in* $\mathbb{R}^3$*.*

3. $\{(1, 0, 0), (0, 2, 0), (1, 2, 3), (2, -1, -1)\}$ *in* $\mathbb{R}^3$*.*

4. $\{(2, -1, 0, 3), (0, 1, -2, -1), (5, -2, 4, -3)\}$ *in* $\mathbb{R}^4$*.*

5. $\{1, X, 1 + X^2\}$ *in* $\mathbb{R}_2[X]$*.*

6. $\{1, X^2 - X, X^3 + 2X\}$ *in* $\mathbb{R}_3[X]$*.*

Exercise 6.
Let F*,* G*, and* H *be three subspaces of a* K*-vector space* E *such that:*
$$F + G = F + H, \quad F \cap G = F \cap H, \quad \text{and} \quad G \subseteq H.$$
Prove that $G = H$*.*

Exercise 7.
Show that the vectors $v_1 = (0, 1, 1)$*,* $v_2 = (1, 0, 1)$*, and* $v_3 = (1, 1, 0)$ *form a basis of* $\mathbb{R}^3$*. Find the components of the vector* $w = (1, 1, 1)$ *in this basis* (v_1, v_2, v_3)*.*

Exercise 8.
Let
$$E = \{(x, y, z) \in \mathbb{R}^3 \mid x + y - z = 0 \text{ and } x - y - z = 0\}$$
and
$$F = \{(x, y, z) \in \mathbb{R}^3 \mid x + y - 2z = 0\}$$
be two subsets of $\mathbb{R}^3$*. Assume that* F *is a subspace of* $\mathbb{R}^3$*. Let* $a = (1, 0, 1)$*,* $b = (1, 1, 1)$*, and* $c = (0, 2, 1)$*.*

1. *Show that E is a subspace of $\mathbb{R}^3$.*

2. *Determine a generating family of E and show that this family is a basis.*

3. *Show that $\{b, c\}$ is a basis of F.*

4. *Show that $\{a, b, c\}$ is a linearly independent family in $\mathbb{R}^3$.*

5. *Prove that $E \oplus F = \mathbb{R}^3$.*

6. *Let $u = (x, y, z)$. Express u in the basis $\{a, b, c\}$.*

Exercise 9.
Let $E =$ Vect(a, b, c, d) be a subspace of $\mathbb{R}^3$, where:

$$a = (2, -1, -1), \quad b = (-1, 2, 3), \quad c = (1, 4, 7), \quad d = (1, 1, 2).$$

1. *Is (a, b, c, d) a basis of $\mathbb{R}^3$?*

2. *Show that (a, b) is a basis of E.*

3. *Determine one or more equations characterizing E.*

4. *Complete a basis of E to form a basis of $\mathbb{R}^3$.*

1 Additional exercises

Exercise 10.
We consider the subset F of $\mathbb{R}^4$ defined by:

$$F = \{(x, y, z, t) \in \mathbb{R}^4 \mid x + y = 0 \text{ and } x + z = 0\}.$$

1. *Show that F is a subspace of $\mathbb{R}^4$ and provide a basis for F.*

2. *Complete the basis found in part (1) to form a basis of $\mathbb{R}^4$.*

3. *Let $u_1 = (1, 1, 1, 1)$, $u_2 = (1, 2, 3, 4)$, and $u_3 = (-1, 0, -1, 0)$. Is the family $\{u_1, u_2, u_3\}$ linearly independent?*

4. *Let G be the vector space spanned by u_1, u_2, and u_3. What is the dimension of G?*

5. *Provide a basis for $F \cap G$. Deduce that $F + G = \mathbb{R}^4$.*

6. *Can every vector in $\mathbb{R}^4$ be written uniquely as the sum of a vector from F and a vector from G?*

Exercise 11.
Let

$$E = \{(x, y, z, t) \in \mathbb{R}^4 \mid x + y + z - t = 0,\ x - 2y + 2z + t = 0,\ x - y + z = 0\}$$

and assume that E is a vector space. Let $F = \{(x, y, z, t) \in \mathbb{R}^4 \mid 2x + 6y + 7z - t = 0\}$. Let $a = (2, 1, -1, 2)$, $b = (1, 1, -1, 1)$, $c = (-1, -2, 3, 7)$, and $d = (4, 4, -5, -3)$ be four vectors in $\mathbb{R}^4$.

1. (a) *Determine a basis for E and deduce its dimension.*

   (b) *Complete this basis to form a basis of $\mathbb{R}^4$.*

2. (a) *Show that F is a subspace of $\mathbb{R}^4$.*
   (b) *Determine a basis for F.*
   (c) *Is it true that $E \oplus F = \mathbb{R}^4$?*
3. (a) *Show that $F = Vect(b, c, d)$.*
   (b) *Let $u = (x, y, z, t) \in F$. Express u as a linear combination of b, c, and d.*

Exercise 12.
Let

$$E = \{(x, y, z) \in \mathbb{R}^3 \mid x + y - z = 0 \text{ and } x - y - z = 0\}$$

and

$$F = \{(x, y, z) \in \mathbb{R}^3 \mid x + y - 2z = 0\}$$

be two subsets of $\mathbb{R}^3$. Assume that F is a subspace of $\mathbb{R}^3$. Let $a = (1, 0, 1)$, $b = (1, 1, 1)$, and $c = (0, 2, 1)$.

1. *Show that E is a subspace of $\mathbb{R}^3$.*
2. *Determine a generating family for E and show that this family forms a basis.*
3. *Show that $\{b, c\}$ is a basis for F.*
4. *Show that $\{a, b, c\}$ is a linearly independent set in $\mathbb{R}^3$.*
5. *Prove that $E \oplus F = \mathbb{R}^3$.*
6. *Let $u = (x, y, z)$. Express u in the basis $\{a, b, c\}$.*

Exercise 13.
Let $u_1 = (1, -1, 2)$, $u_2 = (1, 1, -1)$, and $u_3 = (-1, -5, -7)$. Let $E = Vect(u_1, u_2, u_3)$. Let $F = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$.

1. *Provide a basis for E.*
2. *Show that F is a subspace of $\mathbb{R}^3$.*
3. *Provide a basis for F.*
4. *Provide a basis for $E \cap F$.*