Chapitre 1

Vector spaces

In this course, the field $(\mathbb{K}, +, \times)$ denotes $\mathbb{R}$, $\mathbb{C}$ or any commutative field.

1.1 Definition

Definition 1.1.
Let $(E, +)$ be an abelian group and $\mathbb{K}$ a field. We say that E is a vector space over $\mathbb{K}$ or a $\mathbb{K}$-vector space if there exists a map :

$$\begin{aligned} \mathbb{K} \times E &\longrightarrow E \\ (\lambda, x) &\longmapsto \lambda \bullet x \end{aligned}$$

called the external multiplication law, and it must satisfy the following properties :

1. $\forall x, y \in E, \forall \lambda \in \mathbb{K},\ \lambda \bullet (x + y) = \lambda \bullet x + \lambda \bullet y$,
2. $\forall x \in E, \forall \lambda, \mu \in \mathbb{K},\ (\lambda + \mu) \bullet x = \lambda \bullet x + \mu \bullet x$,
3. $\forall x \in E, \forall \lambda, \mu \in \mathbb{K},\ (\lambda \times \mu) \bullet x = \lambda \bullet (\mu \bullet x)$,
4. $\forall x \in E,\ 1_{\mathbb{K}} \bullet x = x$.

The elements of E are called vectors and those of $\mathbb{K}$ are called scalars. The neutral element of the group $(E, +)$ is denoted 0_E or 0 and is called the zero vector of E.

Rules of Calculation :

1. $\beta \bullet x = 0_E \Leftrightarrow \beta = 0_{\mathbb{K}} \quad or \quad x = 0_E$.
2. $\forall x \in E; \forall \beta \in K :\ -(\beta \cdot x) = (-\beta) \cdot x = \beta \cdot (-x)$.

3. $\forall x \in E \setminus \{0\}$, $\forall \beta, \gamma \in K$, $\beta \cdot x = \gamma \cdot x \Rightarrow \beta = \gamma$.
4. $\forall x \in E$, $\forall \beta_1, \dots, \beta_n \in K$: $\sum_{k=1}^{n} (\beta_k \cdot x) = (\sum_{k=1}^{n} \beta_k) \cdot x$.
5. $\forall x_1, \dots, x_n \in E$, $\forall \beta \in K$: $\sum_{k=1}^{n} \beta \cdot x_k = \beta \cdot (\sum_{k=1}^{n} x_k)$.

Example 1.1. ***Examples of Vector Spaces :***

1. ***The Vector Space*** $\mathbb{R}^n$ ***:***
   (a) *Description : The space* $\mathbb{R}^n$ *consists of vectors with* n *real components.*
   (b) *Example : The space* $\mathbb{R}^3$ *consists of vectors with three real components, such as :*
   $$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad v_1, v_2, v_3 \in \mathbb{R}$$
   (c) *Operations :*
      i. *Vector Addition :*
      $$\mathbf{v} + \mathbf{w} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{pmatrix}$$
      ii. *Scalar Multiplication :*
      $$\alpha \cdot \mathbf{v} = \alpha \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} \alpha v_1 \\ \alpha v_2 \\ \alpha v_3 \end{pmatrix}$$
2. ***The Vector Space*** $\mathbb{C}^n$ ***:***
   (a) *Description : The space* $\mathbb{C}^n$ *consists of vectors with* n *complex components.*
   (b) *Example : The space* $\mathbb{C}^2$ *consists of vectors with two complex components, such as :*
   $$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad v_1, v_2 \in \mathbb{C}$$
   (c) *Operations :*
      i. *Vector Addition :*
      $$\mathbf{v} + \mathbf{w} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix}$$

ii. *Scalar Multiplication :*

$$\alpha \cdot \mathbf{v} = \alpha \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \alpha v_1 \\ \alpha v_2 \end{pmatrix}$$

3. ***The Vector Space of Continuous Functions*** $C([a,b])$ ***:***

   (a) *Description : The space* $C([a,b])$ *consists of all continuous functions defined on the interval* $[a,b]$.

   (b) *Example : If* $f(x)$ *and* $g(x)$ *are continuous functions on* $[a,b]$, *then any linear combination* $h(x) = \alpha f(x) + \beta g(x)$ *where* $\alpha, \beta \in \mathbb{R}$ *will also be a continuous function on* $[a,b]$.

   (c) *Operations :*

      i. *Function Addition : If* $f(x)$ *and* $g(x)$ *are continuous functions, then their sum* $h(x) = f(x) + g(x)$ *is also continuous.*

      ii. *Scalar Multiplication : If* $f(x)$ *is a continuous function and* $\alpha \in \mathbb{R}$, *then the product* $\alpha \cdot f(x)$ *is continuous.*

4. ***The Vector Space of Polynomials*** $\mathbb{R}[x]$ ***:***

   (a) *Description : The space* $\mathbb{R}[x]$ *consists of all polynomials with real coefficients.*

   (b) *Operations :*

      i. *Polynomial Addition : If* $f(x) = 3x^2 + 2x + 1$ *and* $g(x) = x^3 - x$, *their sum is :*

$$f(x) + g(x) = 3x^2 + 2x + 1 + x^3 - x = x^3 + 3x^2 + x + 1$$

      ii. *Scalar Multiplication : If* $f(x)$ *is a polynomial and* $\alpha \in \mathbb{R}$, *then :*

$$\alpha \cdot f(x) = \alpha(3x^2 + 2x + 1) = 3\alpha x^2 + 2\alpha x + \alpha$$

Definition 1.2. Linear Combinations
Let E be a $\mathbb{R}$-vector space, and let $\{x_1, x_2, \ldots, x_n\}$ be a family of n vectors in E. A *linear combination* of the family $\{x_1, x_2, \ldots, x_n\}$ is any vector of the form :

$$\sum_{i=1}^{n} \lambda_i x_i$$

where $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n$ are scalars in the field $\mathbb{R}$.

Example 1.2. ***Example : Linear Combination***
Consider the vector space $\mathbb{R}^2$, which is the space of all 2-dimensional real vectors. Let the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

be two vectors in $\mathbb{R}^2$.

A linear combination of these two vectors is any vector of the form :

$$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 = \lambda_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

where λ_1 and λ_2 are scalars in $\mathbb{R}$.

Let's choose $\lambda_1 = 2$ and $\lambda_2 = -1$. The linear combination becomes :

$$2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + (-1) \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + \begin{pmatrix} -3 \\ -4 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Thus, the linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ with scalars 2 and -1 results in the vector $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

1.2 Subspaces of Vector Spaces

1.2.1 Definition

Definition 1.3.
Let E be a vector space over $\mathbb{K}$, and let F be a subset of E. For F to be a subspace of E, the following conditions must be satisfied :

1. F is a subgroup of E.
2. $\forall x \in F, \forall \lambda \in \mathbb{K}, \lambda x \in F$.

or equivalently :

1. $F \neq \emptyset$.
2. $\forall x, y \in F, x - y \in F$.

3. $\forall x \in F, \forall \lambda \in \mathbb{K}, \lambda x \in F$.

or equivalently :

1. $F \neq \emptyset$.
2. $\forall x, y \in F, \forall \lambda, \mu \in \mathbb{K}; \lambda x + \mu y \in F$.

Example 1.3. *1. $\{0\}$ is a vector subspace of the vector space E. It is the smallest subspace of E.*

2. E itself is a vector subspace of E. It is the largest subspace of E.

3.

$$F := \left\{(x, y, z) \in \mathbb{R}^3 \mid 2x + y - z = 0\right\}.$$

is a subspace of $\mathbb{R}^3$.

4. $F_1 = \{(x, y, z) \in \mathbb{R}^3 \mid 2x - y + 3z = 1\}$ is not subspace of $\mathbb{R}^3$.

5. Let $\mathbb{K}_n[X] = \{P \in \mathbb{K}[X] \mid \deg(P) \leq n\}$ be the set of polynomials in $\mathbb{K}[X]$ with degree at most n. This set is a vector subspace of $\mathbb{K}[X]$.

Proposition 1.4. *Let F be a subspace of a $\mathbb{K}$-vector space E, and let $\{f_i\}_{i \in I} \subset F$. Then, any linear combination of the $\{f_i\}_{i \in I}$ belongs to F.*

1.2.2 Operations on vector subspaces

Recall that if F and G are sets, their intersection is the set of elements in F that are also in G. Also, the union of F and G is the set of elements that belong to either F or G (or both).
The union of two vector subspaces is not always a vector subspace.

Example 1.4. *Let $E_1 = \{(x, 0) \mid x \in \mathbb{R}\}$ and $E_2 = \{(0, y) \mid y \in \mathbb{R}\}$, which are two subspaces of $\mathbb{R}^2$. The union $E_1 \cup E_2$ is not a vector space.*

Reasoning :

The sets E_1 and E_2 are both subspaces of $\mathbb{R}^2$, but their union is not closed under vector addition. Consider the elements $(1, 0) \in E_1$ and $(0, 1) \in E_2$. The sum of these two vectors is :

$$(1, 0) + (0, 1) = (1, 1).$$

However, $(1,1) \notin E_1 \cup E_2$, since neither $(1,1)$ is in E_1 nor in E_2. Therefore, $E_1 \cup E_2$ is not closed under addition, and hence it is not a subspace of $\mathbb{R}^2$.

Theorem 1.5. *Let E_1 and E_2 be two vector subspaces of E. Then, $E_1 \cup E_2$ is a vector subspace if and only if $E_1 \subset E_2$ or $E_2 \subset E_1$.*

Démonstration. We will show that if $E_1 \cup E_2$ is a subspace, then $E_1 \subset E_2$ or $E_2 \subset E_1$. To do this, we prove the contrapositive : if $E_1 \not\subset E_2$ and $E_2 \not\subset E_1$, then $E_1 \cup E_2$ is not a subspace. Since $E_1 \not\subset E_2$ and $E_2 \not\subset E_1$, there exist elements $x \in E_1$ such that $x \notin E_2$, and $y \in E_2$ such that $y \notin E_1$. Then, $x + y \notin E_1 \cup E_2$. (Suppose that $x+y \in E_1$, since $x \in E_1$, we would have $x+y-x \in E_1$, which contradicts the assumption that $y \notin E_1$). Thus, $x + y \notin E_1 \cup E_2$, and since addition is not closed in $E_1 \cup E_2$, it follows that $E_1 \cup E_2$ is not a subspace. The converse is obvious. □

Proposition 1.6. *The arbitrary intersection of vector subspaces is a vector subspace.*

Definition 1.7. (Sum)
Let E be a $\mathbb{K}$-vector space, and let F and G be two vector subspaces of E. The sum of F and G, denoted $F + G$, is the set

$$F + G = \{u + v \mid u \in F \text{ and } v \in G\}.$$

Proposition 1.8. *Let E_1 and E_2 be two vector subspaces of E, the set*

$$E_1 + E_2 = \{x_1 + x_2 \mid x_1 \in E_1, x_2 \in E_2\}$$

has the following properties :

1. *$E_1 + E_2$ is a vector subspace of E.*
2. *$E_1 \cup E_2 \subseteq E_1 + E_2$.*
3. *$E_1 + E_2$ is the smallest vector subspace of E that contains $E_1 \cup E_2$.*

Remark 1.9.
Let F, G, and H be three vector subspaces of E. We have the following :

1. $F + G = G + F$; $F + (G + H) = (F + G) + H$; $F + \{0\} = F$; $F + E = F$; $F + F = F$.
2. If $H = F + G$, the expression $F = H - G$ does not make sense.
3. If $F + G = F + H$, we cannot immediately conclude that $G = H$.

Definition 1.10. Let E_1 and E_2 be two vector subspaces of E. We say that E_1 and E_2 are in direct sum in E (or that E_1 is complementary to E_2 in E) if one of the following equivalent assertions is satisfied :

1. $$E_1 \cap E_2 = \{0_E\} and \quad E_1 + E_2 = E.$$
2. For every $w \in E$, there exists a unique pair of vectors $(u, v) \in E_1 \times E_2$ such that $w = u + v$.

Example 1.5. *The vector subspaces of* $\mathbb{R}^3$ *:*

$$F = \{(a, a, a) \in \mathbb{R}^3 \mid a \in \mathbb{R}\} \quad and \quad G = \{(x, y, z) \in \mathbb{R}^3 \mid x = y - 2z\}$$

are in direct sum, since if $(a, a, a) \in G$*, then* $a = a - 2a$*, hence* $a = 0$*, and therefore* $F \cap G = \{(0, 0, 0)\}$*.*

Example 1.6. *Let* $F_2 = \{(x, y, z) \in \mathbb{R}^3 \mid 2x - y = 0\}$ *and* $F_3 = \{(x, y, z) \in \mathbb{R}^3 \mid x - 4y - z = 0\}$*. Are the subspaces* F_2 *and* F_3 *complementary in* $\mathbb{R}^3$ *?*

Proposition 1.11. *(Existence of Complementary Vector Subspaces) Let* E *be a vector space over the field* K*. Every vector subspace of* E *has at least one complementary vector subspace.*

1.2.3 Generated subspace

Theorem 1.12. *Let* $v_1, \ldots, v_n$ *be a finite set of vectors in a vector space* E *over a field* $\mathbb{K}$*. Then :*
– *The set of linear combinations of the vectors* $v_1, \ldots, v_n$ *is a subspace of* E*.*
– *It is the smallest subspace of* E *(in the sense of inclusion) containing the vectors* $v_1, \ldots, v_n$*.*

Notation. This subspace is called the subspace generated by $v_1, \ldots, v_n$, and is denoted $\mathrm{Span}(v_1, \ldots, v_n)$. Therefore, we have :

$$u \in \mathrm{Span}(v_1, \ldots, v_n) \iff \exists \lambda_1, \ldots, \lambda_n \in K \text{ such that } u = \lambda_1 v_1 + \cdots + \lambda_n v_n$$

Remark 1.13. 1. Saying that $\mathrm{Span}(v_1, \ldots, v_n)$ is the smallest subspace of E containing the vectors $v_1, \ldots, v_n$ means that if F is a subspace of E containing the vectors $v_1, \ldots, v_n$, then $\mathrm{Span}(v_1, \ldots, v_n) \subseteq F$.

2. More generally, we can define the subspace spanned by any subset V (not necessarily finite) of a vector space : $\text{Span}(V)$ is the smallest subspace containing V.

Example 1.7.

1. The vector subspace of $\mathbb{R}^3$ generated by the set $A = \{(-2,0,1),(3,1,1)\}$ is

$$Span(A) = \{a(-2,0,1) + b(3,1,1) \mid a,b \in \mathbb{R}\} = \{(-2a+3b, b, a+b) \mid a,b \in \mathbb{R}\}.$$

2. In $\mathbb{C}^2[X]$, let the vectors $u = iX - X^2$ and $v = 1 + i + X$. Then

$$Span(u,v) = \{\alpha(iX-X^2)+\beta(1+i+X) \mid \alpha,\beta \in \mathbb{C}\} = \{\beta(1+i)+(\alpha i+\beta)X-\beta X^2 \mid \alpha,\beta \in \mathbb{C}\}.$$

Theorem 1.14. *1. Si un vecteur w est combinaison linéaire des vecteurs $(v_1,\dots,v_n)$, alors*

$$Span(v_1,\dots,v_n,w) = Span(v_1,\dots,v_n)$$

2. Si un vecteur w est combinaison linéaire des vecteurs $(v_1,\dots,v_{k-1},v_{k+1},\dots,v_n)$, alors

$$Span(v_1,\dots,v_{k-1},v_k+w,v_{k+1},\dots,v_n) = Span(v_1,\dots,v_n)$$

Example 1.8. *Let $v_1 = (1,0,0), v_2 = (0,1,0), v_3 = (0,0,1)$ in $\mathbb{R}^3$, and suppose that $w = v_1 + v_2 = (1,1,0)$.*

1. ***First part of the theorem :*** *Since $w = v_1 + v_2$, w is a linear combination of v_1 and v_2. Therefore,*

$$Span(v_1,v_2,v_3,w) = Span(v_1,v_2,v_3)$$

This means that adding w to the set (v_1,v_2,v_3) does not change the space spanned by v_1,v_2,v_3.

2. ***Second part of the theorem :*** *Now suppose we replace v_1 by $v_1 + w$. We have :*

$$v_1 + w = (1,0,0) + (1,1,0) = (2,1,0)$$

And $w = v_1 + v_2$, which is already a linear combination of the vectors v_1 and v_2. Therefore,

$$Span(v_1+w,v_2,v_3) = Span(v_1,v_2,v_3)$$

Thus, replacing v_1 by $v_1 + w$ does not affect the space spanned by v_1, v_2, v_3.

Theorem 1.15.
Let A and B be two subsets of a $\mathbb{K}$-vector space E. Then :

1. *$A \subseteq B$ implies $Span(A) \subseteq Span(B)$*
2. *$Span(A \cup B) = Span(A) + Span(B)$*

1.2.4 Generating, free and dependent families

Definition 1.16. (Generating familie)
Let E be a $\mathbb{K}$-vector space and $X = \{x_i \mid i \in I\}$ a subset of E. We say that the subset X is generating E or generates E if every element of E is a linear combination of elements of X, i.e., $E = \text{Span}(X)$.

Example 1.9. 1. *Consider the set $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, where :*

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

*This set S is called a **generating set** for the vector space $\mathbb{R}^3$ because every vector in $\mathbb{R}^3$ can be written as a linear combination of the vectors in S. Specifically, any vector $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$ can be written as :*

$$\mathbf{v} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$$

Thus, the set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ spans $\mathbb{R}^3$, meaning it generates the entire vector space $\mathbb{R}^3$.

2. *In $\mathbb{C}$ viewed as an $\mathbb{R}$-vector space, the family $\{1, i\}$ is generating.*
3. *Consider the vector space of polynomials of degree at most 4, denoted by $\mathbb{R}_2[X]$, which consists of all polynomials of the form :*

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$$

A generating set for this space is the set :

$$\{1, x, x^2, x^3, x^4\}$$

*This set is called a **basis** for P_4, because every polynomial of degree at most 4 can be expressed as a linear combination of these elements. For instance, the polynomial $p(x) = 2 + 3x - 4x^2 + x^3$ can be written as :*

$$p(x) = 2 \cdot 1 + 3 \cdot x - 4 \cdot x^2 + 1 \cdot x^3 + 0 \cdot x^4$$

Thus, the set $\{1, x, x^2, x^3, x^4\}$ generates P_4.

4. *The family $\{X^k\}_{k\in\mathbb{N}}$ generates $\mathbb{R}[X]$, and for all $n \in \mathbb{N}$, the set $\{1, X, \ldots, X^n\}$ generates $\mathbb{R}_n[X]$.*

Remark 1.17. The generating set of a $\mathbb{K}$-vector space is not unique.

Definition 1.18. (Free familie)
Let E be a $\mathbb{K}$-vector space and let $\{v_1, \ldots, v_n\}$ be a family of vectors in E. We say that the family $\{v_1, \ldots, v_n\}$ is free or linearly independent in E if :
for all $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$, $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0_E \Rightarrow \lambda_1 = \cdots = \lambda_n = 0_K$.

Example 1.10. *Let $E = \mathbb{R}^3$ be the 3-dimensional Euclidean space, and consider the set of vectors :*

$$S = \left\{ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

*This set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is **linearly independent** in $\mathbb{R}^3$.*

Example 1.11. *Consider the vector space $\mathbb{R}_3[X]$, which is the space of all polynomials of degree at most 3. A general element in P_3 is of the form :*

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

Now, consider the set of polynomials :

$$S = \{1, x, x^2\}$$

*We will show that this set is **linearly independent** in P_3. To verify this, suppose that a linear combination of the elements in S equals the zero polynomial :*

$$c_1 \cdot 1 + c_2 \cdot x + c_3 \cdot x^2 = 0$$

This means that the polynomial :

$$c_1 + c_2x + c_3x^2 = 0$$

is the zero polynomial. For this to hold for all x, the coefficients of each power of x must be zero. Therefore, we have the system of equations :

$$c_1 = 0$$
$$c_2 = 0$$
$$c_3 = 0$$

Since the only solution to this system is $c_1 = c_2 = c_3 = 0$, the set $\{1, x, x^2\}$ is linearly independent in P_3.

Definition 1.19. (dependent familie)
Let E be a $\mathbb{K}$-vector space and let $\{v_1, \ldots, v_n\}$ be a family of vectors in E. We say that the family $\{v_1, \ldots, v_n\}$ is **dependent** if it is not free, which means there exist $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$ such that $(\lambda_1, \ldots, \lambda_n) \neq (0, \ldots, 0)$ and $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0_E$.

Example 1.12. *Consider the following two vectors :*

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

These vectors are linearly dependent because $\mathbf{v}_2 - 2 \cdot \mathbf{v}_1 = 0$.

Theorem 1.20.
Let $n \geq 2$. The family $\{v_1, v_2, \ldots, v_n\}$ is linearly dependent if and only if one of the vectors $v_1, v_2, \ldots, v_n$ is a linear combination of the others.

1.2.5 Base

Definition 1.21.
We say that the family $\{x_i\}_{i \in I}$ is a basis of E if $\{x_i\}_{i \in I}$ is a linearly independent and generating family of E.

Example 1.13.

1. *$\{1, i\}$ is a basis of the $\mathbb{R}$-vector space $\mathbb{C}$.*
2. *An example of a basis for $\mathbb{R}^2$ is the set of standard unit vectors :*

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

.

3. *An example of a basis for the space of polynomials of degree at most 3, denoted $\mathbb{R}_3[X]$, is the set :*

$$\{1, x, x^2, x^3\}$$

Remark 1.22.

1. The empty set is a basis for the zero vector space.
2. There is no uniqueness of the basis for a given vector space.

Theorem 1.23.
Let $B = \{e_1, e_2, \ldots, e_n\}$ be a basis of a $\mathbb{K}$-vector space E. Any element v of E can be uniquely written as a linear combination of the e_i's :

$$v = \sum_{i=1}^{n} \alpha_i e_i \quad \textit{where} \quad \alpha_i \in \mathbb{K}.$$

The scalars α_i are called the coordinates of v with respect to the basis B.

Proposition 1.24. ***Properties :***

1. *$\{x\}$ is a linearly independent set if and only if $x \neq 0$.*
2. *Any family that contains a generating set is itself a generating set.*
3. *Any subfamily of a linearly independent family is linearly independent.*
4. *Any family that contains a dependent family is itself dependent.*
5. *Any family $\{v_1, v_2, \ldots, v_p\}$ where one of the vectors v_i is the zero vector is dependent.*

Theorem 1.25. ***Basis of a Direct Sum of Two Subspaces***
Let E be a K-vector space, and let F and G be two non-trivial subspaces of E (i.e., neither is reduced to $\{0\}$). If B_F is a basis of F and B_G is a basis of G, then $B_F \cup B_G$ is a generating set for $F + G$ (the sum of the subspaces). Moreover, if F and G are in direct sum, then $B_F \cup B_G$ is a basis for $F \oplus G$ (the direct sum of the subspaces).

Example 1.14. ***Example :***
Consider the following vector subspaces in $\mathbb{R}^4$ *:*

$$F = Span\{(1,-1,0,2)\} \quad and \quad G = Span\{(-2,5,3,1),(1,1,-2,-2)\}.$$

Since $(1,-1,0,2) \neq 0$ *in* $\mathbb{R}^4$ *and the vectors* $(-2,5,3,1)$ *and* $(1,1,-2,-2)$ *are not collinear, it follows that* $\{(1,-1,0,2)\}$ *is a basis for* F *and* $\{(-2,5,3,1),(1,1,-2,-2)\}$ *is a basis for* G*. Therefore, the set* $\{(1,-1,0,2),(-2,5,3,1),(1,1,-2,-2)\}$ *is a generating set for* $F+G$*.*

1.3 Finite-Dimensional Vector Space

Definition 1.26.

1. Let $\{x_i\}_{i\in I}$ be a family of elements of E. The *cardinality* of S is the number of elements in S.
2. E is a finite-dimensional vector space if E has a generating set with finite cardinality. Otherwise, E is an infinite-dimensional vector space.

Example 1.15. ***Examples :***

1. *The vector spaces* $\mathbb{R}^n$ *for* $n \in \mathbb{N}$ *and* $\mathbb{R}_n[X]$ *for* $n \in \mathbb{N}$ *are finite-dimensional.*
2. $\mathbb{R}[X]$ *is infinite-dimensional vector space.*

Theorem 1.27.
All bases of the same vector space E *have the same cardinality. This common number is called the dimension of* E*. We denote it by* $\dim E$*.*

Theorem 1.28.
Every vector space E *that is non-trivial (i.e.,* $E \neq \{\mathbf{0}\}$*) and finite-dimensional admits a basis.*

Corollary 1.29.
In a vector space of dimension n*, we have the following :*

1. *Every linearly independent set has at most* n *elements.*
2. *Every generating set has at least* n *elements.*

Remark 1.30.
If $\dim E = n$, to show that a set of n elements is a basis of E, it is sufficient to prove that it is either linearly independent or generating.

Theorem 1.31. ***Incomplete Basis Theorem :***
Let E be a finite-dimensional vector space and L a linearly independent set in E. Then there exists a basis B of finite cardinality that contains L.

Theorem 1.32. ***Theorem 1.15 : Grassmann's Formula***
Let E be a finite-dimensional vector space and F, G be two subspaces of E. Then,

$$\dim(F+G) = \dim(F) + \dim(G) - \dim(F \cap G).$$

In particular, F and G are in direct sum if and only if

$$\dim(F+G) = \dim(F) + \dim(G).$$

Theorem 1.33. ***Theorem 1.16***
If E is a finite-dimensional vector space and F is a subspace of E, then F is also finite-dimensional, and we have

$$\dim(F) \leq \dim(E).$$

Furthermore,

$$\dim(F) = \dim(E) \quad \textit{if and only if} \quad F = E.$$

Remark 1.34. **Remark :**
The dimension of the zero vector space is 0.

Example 1.16. ***Examples :***

1. $\dim(\mathbb{K}^n) = n$.
2. $\dim(\mathbb{K}_n[X]) = n+1$.
3. $\dim(\mathbb{K}[X])$ *is infinite.*
4. $\dim_{\mathbb{C}} \mathbb{C} = 1 \quad \textit{and} \quad \dim_{\mathbb{R}} \mathbb{C} = 2$.

Theorem 1.35.
Let F and G be two subspaces of a finite-dimensional vector space E. Then F and G are supplementary in E if and only if at least two of the following three conditions are true :

1. $\dim F + \dim G = \dim E$,
2. $F \cap G = \{\mathbf{0}\}$,
3. $F + G = E$.