Chapter 4: Common functions

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Plan

References

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Theorem

A function f that is continuous and strictly monotonic from an interval $I \subseteq \mathbb{R}$ to $\mathbb{R}$ is bijective on I to $f(I)$. Its inverse function $f^{-1} : f(I) \to I$ exists, is continuous, and follows the monotonicity of f.

Proof:

The function f is surjective from I to $f(I)$. Since f is strictly monotonic, it is also injective, making f bijective on $f(I)$. Thus, f^{-1} exists and follows the monotonicity of f. For example, if f is strictly increasing, let $y_1, y_2 \in f(I)$ such that $y_1 < y_2$. Then:

$$y_1 \neq y_2 \implies f^{-1}(y_1) \neq f^{-1}(y_2)$$

and there exist $x_1, x_2 \in I$ such that:

$$f^{-1}(y_1) = x_1, \quad f^{-1}(y_2) = x_2.$$

If we assume $x_1 > x_2$, then since f is strictly increasing, we get:

$$f(x_1) > f(x_2),$$

which contradicts $y_1 < y_2$. Hence:

$$x_1 < x_2 \iff f^{-1}(y_1) < f^{-1}(y_2).$$

Thus, f^{-1} is strictly increasing. Moreover, since f is continuous on I, $f(I)$ is an interval, and f^{-1} is continuous.

Definition: The exponential function is defined as $f(x) = a^x$, where $a > 0$ and $a \neq 1$. The exponential function, denoted by $\exp$, is the unique differentiable function on $\mathbb{R}$ that is equal to its derivative and satisfies $\exp(0) = 1$.

Properties:

1. $\exp(x) > 0, \forall x \in \mathbb{R}$.
2. $\exp(x+y) = \exp(x)\exp(y), \forall x, y \in \mathbb{R}$.
3. Using Euler's notation: $\exp(x) = e^x$, where $e \approx 2.718$.
4. The function $\exp$ is strictly increasing on $\mathbb{R}$.
5. $e^x = e^y \iff x = y, \quad e^x < e^y \iff x < y.$
6. $\exp$ is a bijection from $\mathbb{R}$ to $\mathbb{R}_*^+$.

Logarithmic Functions

Definition: The logarithmic function is given by $f(x) = \log_a x$, where $a > 0$ and $a \neq 1$. This function is the inverse of the exponential function. When $a = e \approx 2.71828$, the function becomes $f(x) = \ln x$, called the *natural logarithm*.
The natural logarithm function is defined on $(0, +\infty)$ to $\mathbb{R}$ such that:

$$\forall x > 0 : x = e^y \iff y = \ln x.$$

Properties:

1. $\ln 1 = 0, \quad \ln e = 1.$
2. $\ln(e^x) = x, \quad e^{\ln x} = x, \quad \forall x > 0.$
3. $\ln$ is strictly increasing on $(0, +\infty)$.
4. $\ln(xy) = \ln x + \ln y, \quad \ln\left(\frac{1}{y}\right) = -\ln y.$
5. $\ln(x^n) = n \ln x, \quad \forall n \in \mathbb{N}.$

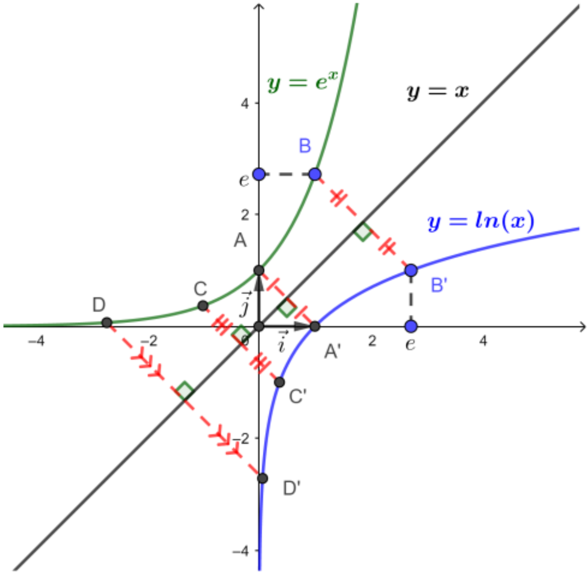


Figure: Source: BOUHARIS Epouse, OUDJDI DAMERDJI Amel, Cours et exercices corrigés d'Analyse 1, Première année Licence MI

Trigonometric Functions

The standard trigonometric functions include:

$$\sin x, \quad \cos x, \quad \tan x = \frac{\sin x}{\cos x}, \quad \csc x = \frac{1}{\sin x},$$

$$\sec x = \frac{1}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}.$$

- The variable x is generally expressed in radians (π radians = 180°). - For real values of x, $\sin x$ and $\cos x$ lie in the range $[-1, 1]$.

Key Properties of Trigonometric Functions

1. **Pythagorean Identities:**

$$\sin^2 x + \cos^2 x = 1, \quad 1 + \tan^2 x = \sec^2 x, \quad 1 + \cot^2 x = \csc^2 x.$$

2. **Angle Addition and Subtraction Formulas:**

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y, \quad \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y.$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}.$$

3. **Sign Change Properties:**

$$\sin(-x) = -\sin x, \quad \cos(-x) = \cos x, \quad \tan(-x) = -\tan x.$$

Function arcsin

The function:

$$f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1], \quad x \mapsto f(x) = \sin x$$

is continuous and strictly increasing on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Therefore, f is bijective, and its inverse function exists. It is continuous and strictly increasing. We have:

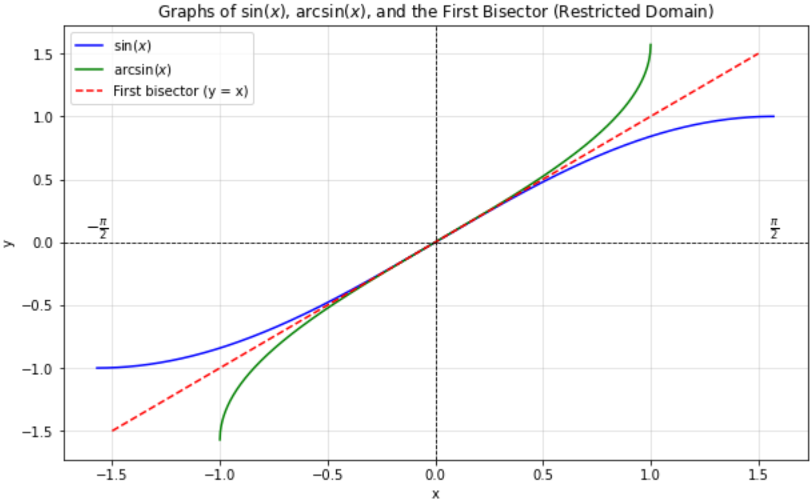
$$f\left(\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right) = [-1, 1]$$

and

$$f^{-1} : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \quad y \mapsto f^{-1}(y) = \arcsin y.$$

Thus, we get:

$$\arcsin y = x \iff \sin x = y, \quad -1 \leq y \leq 1, \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}.$$



Graphs of sin(x), arcsin(x), and the First Bisector (Restricted Domain)
sin(x)
arcsin(x)
First bisector (y = x)
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
y
−π/2
π/2
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
x

Function arccos

The function:

$$f : [0, \pi] \to [-1, 1], \quad x \mapsto f(x) = \cos x$$

is continuous and strictly decreasing on $[0, \pi]$. Therefore, f is bijective, and its inverse function exists. It is continuous and strictly decreasing. We have:

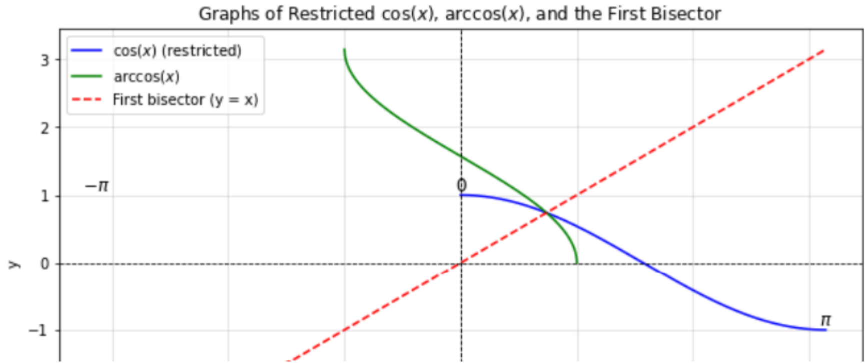
$$f([0, \pi]) = [-1, 1]$$

and

$$f^{-1} : [-1, 1] \to [0, \pi], \quad y \mapsto f^{-1}(y) = \arccos y.$$

Thus, we get:

$$\arccos y = x \iff \cos x = y, \quad -1 \le y \le 1, \quad 0 \le x \le \pi.$$


Graphs of Restricted cos(x), arccos(x), and the First Bisector
cos(x) (restricted)
arccos(x)
First bisector (y = x)
3
2
1
0
−1
y
−π
0
π

Function arctan

The function:

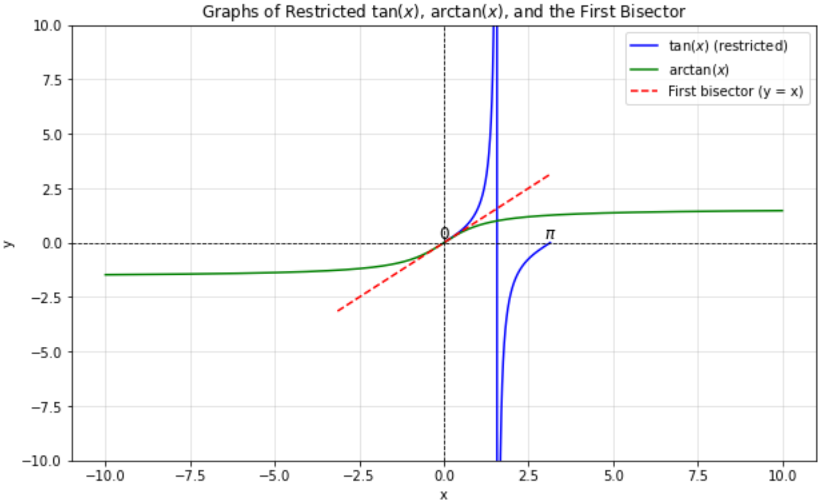
$$f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}, \quad x \mapsto f(x) = \tan x = \frac{\sin x}{\cos x}$$

is continuous and strictly increasing on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Therefore, f is bijective, and its inverse function exists. It is continuous and strictly increasing. We have:

$$f\left(\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right) = \mathbb{R}$$

and

$$f^{-1} : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad y \mapsto f^{-1}(y) = \arctan y.$$



Graphs of Restricted tan(x), arctan(x), and the First Bisector
tan(x) (restricted)
arctan(x)
First bisector (y = x)
10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0
y
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0
x
0
π

Function arccot

The function:

$$f : (0, \pi) \to (-\infty, +\infty), \quad x \mapsto f(x) = \cot x = \frac{\cos x}{\sin x}$$

is continuous and strictly decreasing on $(0, \pi)$. Therefore, f is bijective, and its inverse function exists. It is continuous and strictly decreasing. We have:

$$f((0, \pi)) = (-\infty, +\infty)$$

and

$$f^{-1}:(-\infty,+\infty)\to(0,\pi),\quad y\mapsto f^{-1}(y)=arccot\ y.$$

Thus, we get:

$$arccot\ y=x \iff \cot x=y,\quad 0<x<\pi.$$

Properties of Inverse Trigonometric Functions

Properties:

1. For all $x \in [-1, 1]$:

$$\arcsin x + \arccos x = \frac{\pi}{2}.$$

2. If $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$:

$$\sin t = x \iff \arcsin x = t.$$

Otherwise:

$$\sin t = x \iff t = \arcsin x + 2k\pi \text{ or } t = (\pi - \arcsin x) + 2k\pi, \quad k \in \mathbb{Z}.$$

3. If $t \in [0, \pi]$:

$$\cos t = x \iff \arccos x = t.$$

Otherwise:

$$\cos t = x \iff t = \arccos x + 2k\pi \text{ or } t = -\arccos x + 2k\pi, \quad k \in \mathbb{Z}.$$

Hyperbolic Functions

Definition: The hyperbolic cosine function, cosh, and hyperbolic sine function, sinh, are defined as:

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}.$$

Properties:

1. $\cosh^2 x - \sinh^2 x = 1.$
2. $\frac{d}{dx}\cosh x = \sinh x, \quad \frac{d}{dx}\sinh x = \cosh x.$
3. $\cosh(-x) = \cosh x, \quad \sinh(-x) = -\sinh x.$

Properties: $\cosh x$ is even, continuous, and strictly increasing on $[0, +\infty)$. Its inverse function $\cosh^{-1}$ exists, is continuous, and strictly increasing. We have:

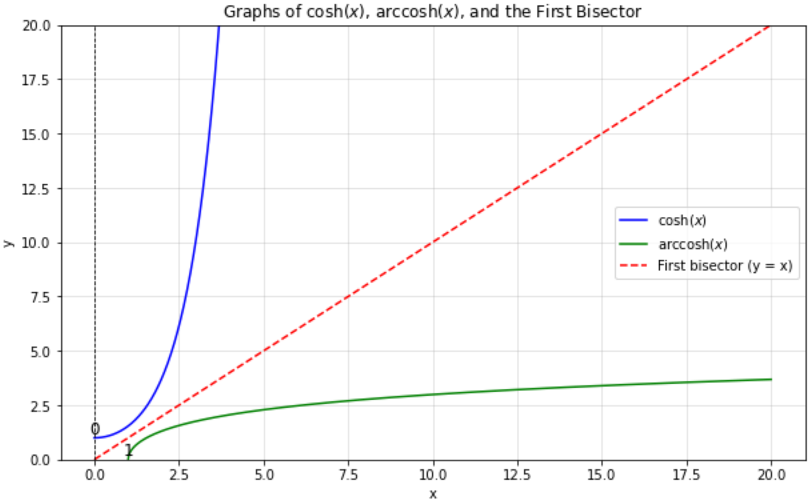
$$f([0, +\infty)) = [1, +\infty)$$

and

$$f^{-1} : [1, +\infty) \to [0, +\infty), \quad y \mapsto f^{-1}(y) = \operatorname{arccosh}(y).$$

Thus, we get:

$$\operatorname{arccosh}(y) = x \iff \cosh x = y, \quad x \geq 0.$$


Graphs of cosh(x), arccosh(x), and the First Bisector
20.0
17.5
15.0
12.5
10.0
7.5
5.0
2.5
0.0
y
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
x
0
1
cosh(x)
arccosh(x)
First bisector (y = x)

Hyperbolic Sine Function

Definition: Hyperbolic Sine Function sinh:

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \sinh x = \frac{e^x - e^{-x}}{2}$$

Properties: $\sinh x$ is odd, continuous, and strictly increasing on $\mathbb{R}$. Its inverse function $\sinh^{-1}$ exists, is continuous, and strictly increasing. We have:

$$f(\mathbb{R}) = \mathbb{R}$$

and

$$f^{-1} : \mathbb{R} \to \mathbb{R}, \quad y \mapsto f^{-1}(y) = \text{arcsinh}(y).$$

Thus, we get:

$$\text{arcsinh}(y) = x \iff \sinh x = y, \quad x \in \mathbb{R}.$$

Definition: Hyperbolic Tangent Function tanh:

$$f : \mathbb{R} \to (-1, 1), \quad x \mapsto \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Properties: $\tanh x$ is odd, continuous, and strictly increasing on $\mathbb{R}$. Its inverse function $\tanh^{-1}$ exists, is continuous, and strictly increasing. We have:

$$f(\mathbb{R}) = (-1, 1)$$

and

$$f^{-1} : (-1, 1) \to \mathbb{R}, \quad y \mapsto f^{-1}(y) = \operatorname{arctanh}(y).$$

Thus, we get:

$$\operatorname{arctanh}(y) = x \iff \tanh x = y, \quad |y| < 1.$$

Hyperbolic Cotangent Function

Definition: Hyperbolic Cotangent Function coth:

$$f : (0, +\infty) \to (1, +\infty), \quad x \mapsto \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Properties: $\coth x$ is odd, continuous, and strictly decreasing on $(0, +\infty)$. Its inverse function $\coth^{-1}$ exists, is continuous, and strictly decreasing. We have:

$$f((0, +\infty)) = (1, +\infty)$$

and

$$f^{-1} : (1, +\infty) \to (0, +\infty), \quad y \mapsto f^{-1}(y) = \text{argcoth}(y).$$

Thus, we get:

$$\text{argcoth}(y) = x \iff \coth x = y, \quad x > 0.$$

Properties

1. $\cosh x + \sinh x = e^x$:
2. $\cosh x - \sinh x = e^{-x}$:
3. $\cosh^2 x - \sinh^2 x = 1$:
4. $1 - \tanh^2 x = \operatorname{sech}^2 x$:
5. $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$:
6. $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$:

Expression in Logarithmic Form

The inverse functions of hyperbolic functions can be expressed using the natural logarithm as follows:

$$arctanhx = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \quad x \in]-1,1[$$

$$arccothx = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right), \quad x \in]-1,-1[\cup[1,+1[$$

$$arcsinhx = \ln\left(x+\sqrt{1+x^2}\right), \quad x \in \mathbb{R}$$

$$arccoshx = \ln\left(x+\sqrt{x^2-1}\right), \quad x \geq 1$$

Proof:

1.

- Let $x \in]-1,1[$, and set $arctanh x = y$.

$$\tanh y = x \implies \frac{e^{y} - e^{-y}}{e^{y} + e^{-y}} = x$$

$$1 - e^{-2y} = x(1 + e^{-2y}) \implies e^{-2y}(1 + x) = 1 - x$$

$$e^{2y} = \frac{1+x}{1-x} \implies 2y = \ln\left(\frac{1+x}{1-x}\right) \implies y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

2. **For** $x \in]-1,-1[\cup[1,+1[$:
   - Assume $arccothx = y$.

$$\coth y = x \implies \frac{e^{y}+e^{-y}}{e^{y}-e^{-y}} = x$$

$$\frac{1+e^{-2y}}{1-e^{-2y}} = x \implies 1+e^{-2y} = x(1-e^{-2y})$$

$$e^{-2y}(1+x) = x-1 \implies e^{2y} = \frac{1+x}{x-1}$$

$$2y = \ln\left(\frac{1+x}{x-1}\right) \implies y = arccothx = \frac{1}{2}\ln\left(\frac{1+x}{x-1}\right)$$

3. For $x \in \mathbb{R}$:

- Assume $arcsinhx = y$.

$$\sinh y = x$$

- Using the identities:

$$e^y = \sinh y + \cosh y \quad \text{and} \quad \cosh y = \sqrt{\sinh^2 y + 1}$$

- We have:

$$e^y = x + \sqrt{x^2 + 1} \implies y = arcsinhx = \ln\left(x + \sqrt{x^2 + 1}\right)$$

4. **For $x \geq 1$:**

- Assume $arccoshx = y$.

$$\cosh y = x$$

- Using the identities:

$$e^y = \cosh y + \sinh y \quad \text{and} \quad \sinh y = \sqrt{\cosh^2 y - 1}$$

- We have:

$$e^y = x + \sqrt{x^2 - 1} \implies y = arccoshx = \ln\left(x + \sqrt{x^2 - 1}\right)$$

Thanks