Chapter 03: Real Functions of a Real Variable

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Plan

Références

بالعربية:

- بابا حامد ، بن حبيب، التحليل 1 تذكير بالدروس و تمارين محلولة عدد 300 ترجمة الحفيظ مقران، ديوان المطبوعات الجامعية (**الفصلين الثالث و الرابع**) .

In English:

- Murray R. Spiegel, Schaum's outline of theory and problems of advanced calculus, Mcgraw-Hill (1968), **(Chapters 2 and 4)** .
- Robert C. Wrede, Murray R. Spiegel, Schaum's Outlines: Advanced Calculus, (2011), **(Chapter 3 and 4)**.
- Terence Tao, Analysis 1 (3rd edition), Springer (2016).

En français:

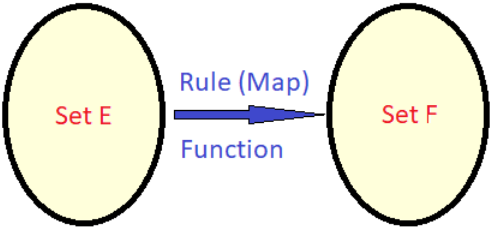
- BOUHARIS Epouse, OUDJDI DAMERDJI Amel, Cours et exercices corrigés d'Analyse 1, Première année Licence MI Mathématiques et Informatique, U.S.T.O 2020-2021 **(Chapitres 4 et 5)**.
- Benzine BENZINE, Analyse réelle cours et exercices corriges, première année maths et informatique (2016), **(Chapitres 3 et 4)**.

Definition:

- A real function of a real variable is a mapping f from a set $E \subset \mathbb{R}$ to a set $F \subset \mathbb{R}$, written as:

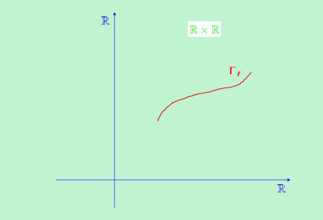
$$f : E \to F, \quad x \mapsto f(x).$$

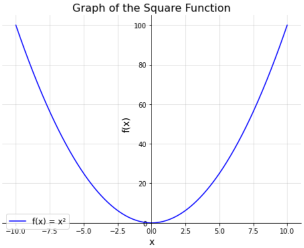
- Here, x is called the real variable, and $f(x)$ is called the image of x under f.
- The domain of definition of f is the set of values $x \in E$ for which $f(x) \in F$, denoted D_f.
- The set of all functions from E to F is denoted as $F(E, F)$.

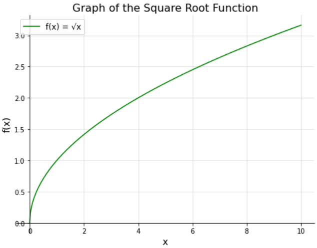


The graph of f is the subset Γ_f of the Cartesian product $\mathbb{R} \times \mathbb{R}$ defined as:

$$\Gamma_f = \{(x, f(x)) \mid x \in E\}.$$







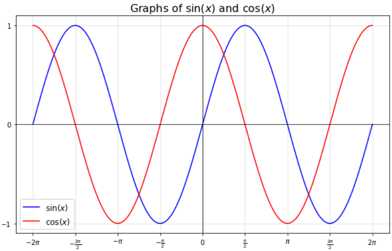
Definition: (Parity of a Function)
Let f be a function from $\mathbb{R}$ to $\mathbb{R}$.
- f is called even if $\forall x \in D_f,\ f(-x) = f(x)$, meaning the graph of f is symmetric with respect to the y-axis.
- f is called odd if $\forall x \in D_f,\ f(-x) = -f(x)$, meaning the graph of f is symmetric with respect to the origin.

Definition: (Periodicity of a Function) A function f is said to be periodic if there exists a strictly positive real number T such that:

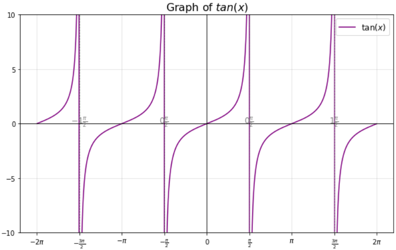
$$\forall x \in D_f,\ f(x + T) = f(x).$$

Examples:

- For $f(x) = \sin x$ or $f(x) = \cos x$, the period is $T = 2\pi$.



- For $f(x) = \tan x$, the period is $T = \pi$.



- For $f(x) = x - \lfloor x \rfloor$, the period is $T = 1$.

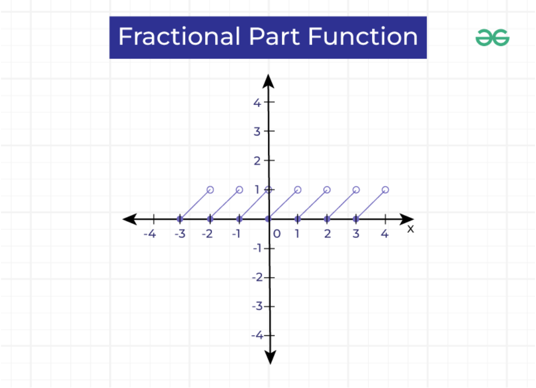


Figure: Source: https://www.geeksforgeeks.org/fractional-part-function/

- For $f(x) = \cos\left(\frac{5x}{2}\right)$, the period is $T = \frac{4\pi}{5}$.

Remarks:

1. If f is even or odd, it suffices to study it over half its domain.
2. There exist functions that are neither even nor odd.
3. If f is periodic with period T, it suffices to study it over a single period.

1. A function $f(x)$ is said to be **bounded above** in an interval (or set) if there exists a constant M such that:

$$f(x) \leq M \quad \text{for all } x \text{ in the interval.}$$

Here, M is called an **upper bound** for $f(x)$.
2. Similarly, $f(x)$ is **bounded below** if there exists a constant m such that:

$$f(x) \geq m \quad \text{for all } x \text{ in the interval.}$$

In this case, m is called a **lower bound**.
3. If both conditions are satisfied—i.e., there exist constants m and M such that:

$$m \leq f(x) \leq M \quad \text{for all } x \text{ in the interval,}$$

then $f(x)$ is called a **bounded function**. To denote that $f(x)$ is bounded, we can write:

$$|f(x)| \leq P, \quad P > 0.$$

Examples

1) For $f(x) = \cos(x)$ in the interval $-\infty < x < \infty$:

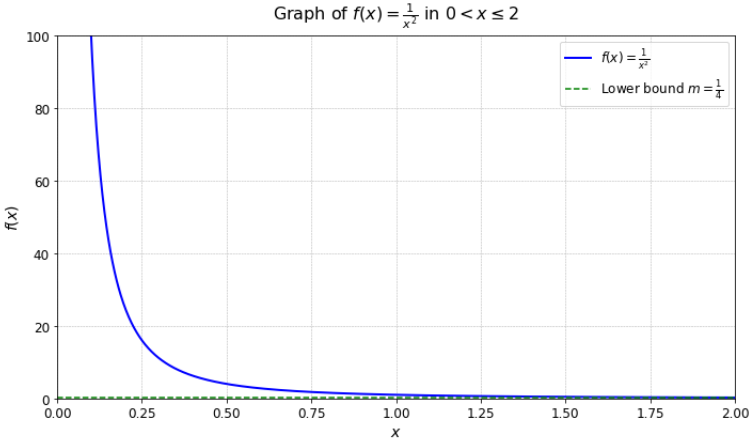
- The function is bounded since $-1 \leq f(x) \leq 1$ for all x.
- $M = 1$ is an **upper bound**, and $m = -1$ is a **lower bound**.

2) For $f(x) = x^3$ in the interval $-2 \leq x \leq 2$:

- The function is bounded because $-8 \leq f(x) \leq 8$ in this interval.
- $M = 8$ is an **upper bound**, and $m = -8$ is a **lower bound**.

3) For $f(x) = \frac{1}{x^2}$ in the interval $0 < x \leq 2$:

- The function is **not bounded above**, as $f(x) \to \infty$ as $x \to 0^+$.
- However, it is bounded below with $f(x) \geq \frac{1}{4}$.


Graph of $f(x) = \frac{1}{x^2}$ in $0 < x \leq 2$
$f(x) = \frac{1}{x^2}$
Lower bound $m = \frac{1}{4}$
$f(x)$
x
100
80
60
40
20
0
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00

Definition of Monotonic Functions

Monotonically Increasing:

- A function $f(x)$ is monotonically increasing on an interval if, for any x_1, x_2 such that $x_1 < x_2$, we have:

$$f(x_1) \leq f(x_2).$$

- If $f(x_1) < f(x_2)$ strictly for all $x_1 < x_2$, then $f(x)$ is **strictly increasing**.

Monotonically Decreasing:

- A function $f(x)$ is monotonically decreasing on an interval if, for any x_1, x_2 such that $x_1 < x_2$, we have:

$$f(x_1) \geq f(x_2).$$

- If $f(x_1) > f(x_2)$ strictly for all $x_1 < x_2$, then $f(x)$ is **strictly decreasing**.

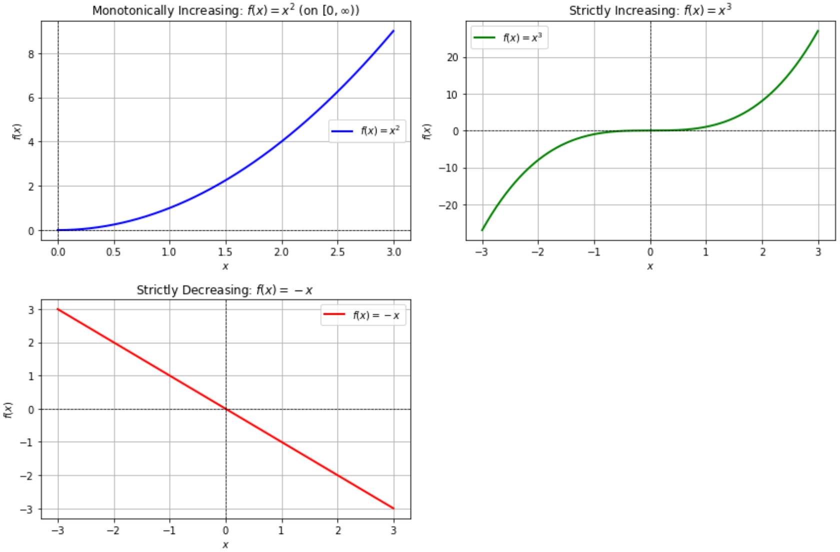
Examples:

Monotonically Increasing:

- The function $f(x) = x^2$ is monotonically increasing on $[0, \infty)$.
- The function $f(x) = x^3$ is strictly increasing on $\mathbb{R}$.

Monotonically Decreasing:

- The function $f(x) = -x$ is strictly decreasing on $\mathbb{R}$.


Monotonically Increasing: $f(x) = x^2$ (on $[0, \infty)$)
$f(x) = x^2$
0.0 0.5 1.0 1.5 2.0 2.5 3.0
0 2 4 6 8
x
$f(x)$
Strictly Increasing: $f(x) = x^3$
$f(x) = x^3$
−3 −2 −1 0 1 2 3
−20 −10 0 10 20
x
$f(x)$
Strictly Decreasing: $f(x) = -x$
$f(x) = -x$
−3 −2 −1 0 1 2 3
−3 −2 −1 0 1 2 3
x
$f(x)$

Introduction to Maxima and Minima

- The study of maxima and minima, or extreme values of functions, was a key motivation for the development of calculus in the 17th century.
- Extreme values are important in both mathematical theory and real-world applications.
- They are classified into:
  - **Relative extrema** (local maxima and minima).
  - **Absolute extrema** (global maxima and minima).

Relative Extrema

Definition:

- A function $f(x)$ has a **relative maximum** at c if there exists an interval (a, b) containing c such that:

$$f(x) < f(c) \quad \text{for all } x \neq c \text{ in } (a, b).$$

- Similarly, $f(x)$ has a **relative minimum** at c if:

$$f(x) > f(c) \quad \text{for all } x \neq c \text{ in } (a, b).$$

Key Points:

- Relative extrema are the "high points" (maximum) or "low points" (minimum) within a local neighborhood.
- Functions may have multiple relative extrema or none at all.

Absolute Extrema

Definition:

- A function $f(x)$ has an **absolute maximum** at c if:

$$f(x) \leq f(c) \quad \text{for all } x \text{ in the domain of } f.$$

- A function $f(x)$ has an **absolute minimum** at c if:

$$f(x) \geq f(c) \quad \text{for all } x \text{ in the domain of } f.$$

Key Points:

- Absolute extrema consider the entire domain of the function.
- Strictly increasing or decreasing functions may have absolute extrema at the endpoints of a closed interval.
- Absolute extrema are not always unique, as in the case of constant functions.

Examples and Observations

Examples:

- Relative extrema:
  - The highest point on a specific hill is a **relative maximum**.
  - The lowest point in a valley is a **relative minimum**.
- Absolute extrema:
  - The tallest hill overall is the **absolute maximum**.
  - The deepest valley is the **absolute minimum**.

Special Cases:

- Strictly increasing or decreasing functions have no relative extrema but may have absolute extrema on closed intervals.
- Constant functions have infinite absolute extrema, as all points are maxima and minima.

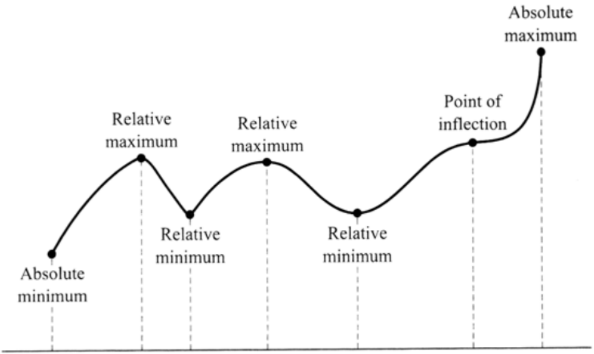


Figure 3.3

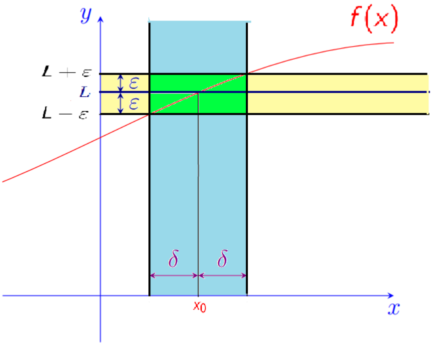
Figure: Source:Schaum's Outlines: Advanced Calculus By Robert C. Wrede, Murray R. Spiegel · 2011

Definition: Let f be a real-valued function defined on a set $E \subset \mathbb{R}$. We say that $f(x)$ tends toward a limit L as x approaches x_0 if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$\forall x \in E,\ 0 < |x - x_0| < \delta \implies |f(x) - L| < \epsilon.$$

This is written as:

$$\lim_{x \to x_0} f(x) = L.$$



Theorems on limits:

1. If $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = M$, then:

$$\lim_{x\to x_0} [f(x) + g(x)] = L + M.$$

2. If $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = M$, then:

$$\lim_{x\to x_0} [f(x) \cdot g(x)] = L \cdot M.$$

3. If $\lim_{x\to x_0} f(x) = L$, $\lim_{x\to x_0} g(x) = M$, and $M \neq 0$, then:

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

Remark: If $f(x)$ is continuous at x_0, then:

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Definition: A function $f(x)$ is said to have a limit L as x approaches infinity if, for every $\epsilon > 0$, there exists $N > 0$ such that:

$$\forall x > N, \ |f(x) - L| < \epsilon.$$

This is written as:

$$\lim_{x \to \infty} f(x) = L.$$

Theorem If f admits a limit at the point x_0, then this limit is unique.

Proof

Suppose, for contradiction, that f admits two different limits, l_1 and l_2 ($l_1 \neq l_2$). When x tends to x_0, we have:

$$\lim_{x \to x_0} f(x) = l_1 \iff$$

$$\forall \epsilon > 0, \exists \alpha_1 > 0, \forall x \in I, \ |x - x_0| < \alpha_1 \implies |f(x) - l_1| < \frac{\epsilon}{2}$$

$$\lim_{x \to x_0} f(x) = l_2 \iff$$

$$\forall \epsilon > 0, \exists \alpha_2 > 0, \forall x \in I, \ |x - x_0| < \alpha_2 \implies |f(x) - l_2| < \frac{\epsilon}{2}$$

Let $\epsilon > 0$, then:

$$|l_1 - l_2| = |(l_1 - f(x)) + (f(x) - l_2)|$$

For $\alpha = \min(\alpha_1, \alpha_2)$, we have:

$$|l_1 - l_2| \leq |f(x) - l_1| + |f(x) - l_2| < \epsilon$$

Thus, for all $\epsilon > 0$ (no matter how small), it follows that $l_1 = l_2$.

Definition:

- f has a **left-hand limit** l_l as $x \to x_0^-$:

$$\lim_{x \to x_0^-} f(x) = l_l \iff$$

$$\forall \epsilon > 0, \exists \alpha > 0, \forall x \in I,\ x_0 - \alpha < x < x_0 \implies |f(x) - l_l| < \epsilon$$

- f has a **right-hand limit** l_r as $x \to x_0^+$:

$$\lim_{x \to x_0^+} f(x) = l_r \iff$$

$$\forall \epsilon > 0, \exists \alpha > 0, \forall x \in I,\ x_0 < x < x_0 + \alpha \implies |f(x) - l_r| < \epsilon$$

Remarks

1. If f has a limit l as $x \to x_0$, then:

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = l$$

2. If f has a left-hand limit l_l and a right-hand limit l_r at x_0, and $l_l = l_r$, then:

$$\lim_{x \to x_0} f(x) = l_l = l_r$$

3. If $l_l \neq l_r$, then f does not have a limit as $x \to x_0$.

1.

$$\lim_{x \to x_0} f(x) = +\infty \iff$$

$$\forall A > 0, \exists \alpha > 0, \forall x \in I, \ |x - x_0| < \alpha \implies f(x) > A$$

2.

$$\lim_{x \to x_0} f(x) = -\infty \iff$$

$$\forall A < 0, \exists \alpha > 0, \forall x \in I, \ |x - x_0| < \alpha \implies f(x) < A$$

1. $$\lim_{x\to+\infty} f(x) = l \iff \forall\epsilon > 0, \exists\alpha > 0, \forall x > \alpha \implies |f(x)-l| < \epsilon$$

2. $$\lim_{x\to-\infty} f(x) = l \iff \forall\epsilon > 0, \exists\alpha < 0, \forall x < \alpha \implies |f(x)-l| < \epsilon$$

Definition: A function $f(x)$ is said to be continuous at a point $x_0 \in D_f$ if:

$$\lim_{x \to x_0} f(x) = f(x_0).$$

If $f(x)$ is continuous at every point in D_f, it is said to be continuous on D_f.

A function $f(x)$ is said to be **continuous** at a point $x = x_0$ if the following conditions hold:

1. The limit $\lim_{x \to x_0} f(x)$ exists.
2. The value $f(x_0)$ exists (i.e., f is defined at x_0).
3. The limit matches the function's value:

$$\lim_{x \to x_0} f(x) = f(x_0).$$

These conditions ensure the function behaves smoothly at x_0, with no jumps, gaps, or breaks.

Alternative Definition

A function $f(x)$ is continuous at x_0 if:

$\forall\epsilon > 0, \exists\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$.

This means that we can make $f(x)$ arbitrarily close to $f(x_0)$ by taking x sufficiently close to x_0.

Intuitive Understanding of Continuity

- If $f(x)$ is continuous at x_0, the graph of $f(x)$ near x_0 can be drawn without lifting the pencil from the paper.
- If there is a gap or jump in the graph at x_0, the function is **not continuous** there.

In simple terms, $f(x)$ is continuous at x_0 if:

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Example 1: A Discontinuous Function

Let:

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 2, \\ 0 & \text{if } x = 2. \end{cases}$$

- For $x \neq 2$, $f(x) = x^2$, so $\lim_{x\to 2} f(x) = 4$.
- At $x = 2$, $f(2) = 0$.
- Since $\lim_{x\to 2} f(x) \neq f(2)$, $f(x)$ is **not continuous** at $x = 2$.

Example 2: A Continuous Function

Let $f(x) = x^2$ for all x:

- The limit $\lim_{x \to 2} f(x) = 4$.
- The value of the function at $x = 2$ is $f(2) = 4$.
- Since $\lim_{x \to 2} f(x) = f(2)$, $f(x)$ is **continuous** at $x = 2$.

Points of Discontinuity

Discontinuities are points where $f(x)$ fails to be continuous. These can occur if:

- $f(x)$ is undefined at the point (a gap),
- $f(x)$ jumps to a different value,
- The left-hand and right-hand limits do not match.

For example: - In Example 1, $f(x)$ is discontinuous at $x = 2$ because the limit and function value differ.

Property:

The sum, product, and quotient (where the denominator is nonzero) of continuous functions are continuous.

Examples:

1. The function $f(x) = x^2 + 3x + 2$ is continuous on $\mathbb{R}$.
2. The function $g(x) = \frac{1}{x}$ is continuous on $\mathbb{R} \setminus \{0\}$.

Theorem 1:
If $f(x)$ and $g(x)$ are continuous at $x = x_0$, then the following functions are also continuous at $x = x_0$:
$f(x) + g(x)$, $f(x) - g(x)$, $f(x) \cdot g(x)$, $\frac{f(x)}{g(x)}$
provided $g(x_0) \neq 0$.

Theorem 2:
The following functions are continuous over every finite interval:

- Polynomials.
- $\sin x$ and $\cos x$.
- Exponential functions a^x, where $a > 0$.

Theorem 3: If $f(x)$ is continuous at x_0, and $g(y)$ is continuous at $y_0 = f(x_0)$, then the composite function $g(f(x))$ is continuous at x_0.

Theorem 4: If $f(x)$ is continuous over a closed interval, then it is bounded on that interval.
Theorem 5: If $f(x)$ is continuous at x_0 and $f(x_0) > 0$ (or $f(x_0) < 0$), then there exists an interval around x_0 where $f(x) > 0$ (or $f(x) < 0$).
Theorem 6: If $f(x)$ is continuous and strictly monotonic (strictly increasing or decreasing) on an interval, then its inverse $f^{-1}(x)$ exists, is single-valued, and is also continuous.
Theorem 7: (Intermediate Value Theorem) If $f(x)$ is continuous on a closed interval $[a, b]$ and $f(a) = A$, $f(b) = B$, then for any C between A and B, there exists at least one $c \in [a, b]$ such that:

$$f(c) = C.$$

Theorem 8: (Zero Value Theorem) If $f(x)$ is continuous on $[a, b]$ and $f(a) \cdot f(b) < 0$, then there exists at least one $c \in (a, b)$ such that:

$$f(c) = 0.$$

Theorem 9: (Extreme Value Theorem) If $f(x)$ is continuous on a closed interval $[a, b]$, then it attains both its maximum value M and minimum value m at some points in the interval:

$$M = \max_{x \in [a,b]} f(x), \quad m = \min_{x \in [a,b]} f(x).$$

Theorem 10: If $f(x)$ is continuous on $[a, b]$, then its least upper bound (l.u.b.) and greatest lower bound (g.l.b.) are attained within the interval.

Piecewise Continuity

A function $f(x)$ is piecewise continuous on (a, b) if the interval can be divided into a finite number of subintervals where $f(x)$ is continuous, and $f(x)$ has finite left-hand and right-hand limits at any points of discontinuity.

Uniform Continuity

Definition: A function $f(x)$ is uniformly continuous on an interval if, for any $\epsilon > 0$, there exists a $\delta > 0$ such that:

$$|f(x_1) - f(x_2)| < \epsilon \quad \text{whenever} \quad |x_1 - x_2| < \delta,$$

where x_1, x_2 are any two points in the interval.

Theorem: If $f(x)$ is continuous on a closed interval, it is also uniformly continuous on that interval.

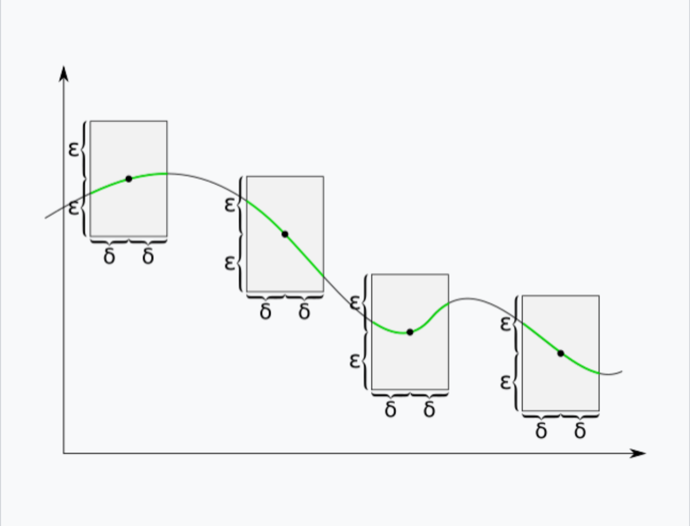


Figure: Source:https://en.wikipedia.org

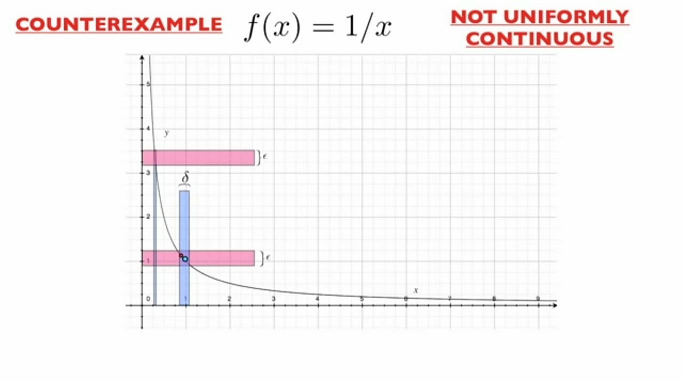


Figure:
Source:https://math.stackexchange.com/questions/2283008/uniform-continuity-of-function

Definition: Let $f(x)$ be a real-valued function defined on an open interval containing x_0. The derivative of f at x_0, denoted $f'(x_0)$, is defined as:

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exists. If $f'(x_0)$ exists, $f(x)$ is said to be differentiable at x_0.

Higher-Order Derivatives: If $f'(x)$ is differentiable, the derivative of $f'(x)$ is called the second derivative of $f(x)$, denoted $f''(x)$. Similarly, higher-order derivatives are defined recursively.

Examples of Differentiation

1. For $f(x) = x^2$, the derivative is:
$$f'(x) = 2x.$$
2. For $g(x) = \sin x$, the derivative is:
$$g'(x) = \cos x.$$
3. For $h(x) = \ln x$, the derivative is:
$$h'(x) = \frac{1}{x}.$$
4. For $k(x) = e^x$, the derivative is:
$$k'(x) = e^x.$$

Definition of a Derivative

Let $f(x)$ be defined at a point x_0 in (a, b). The derivative of $f(x)$ at $x = x_0$ is defined as:

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

if this limit exists.

Note: The derivative can also be defined in other equivalent ways.
Key Property: If $f(x)$ is differentiable at $x = x_0$, it must be continuous there.
Caution: Continuity does not necessarily imply differentiability.

Right and Left-Hand Derivatives

Right-Hand Derivative:

$$f'_{+}(x_0) = \lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h},$$

where h approaches 0 from positive values.

Left-Hand Derivative:

$$f'_{-}(x_0) = \lim_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h},$$

where h approaches 0 from negative values.

Condition for Differentiability:

$$f(x) \text{ is differentiable at } x_0 \iff f'_{+}(x_0) = f'_{-}(x_0).$$

A function $f(x)$ is said to be **differentiable** in an interval if it has a derivative at every point in the interval.

Closed Interval Case: For $f(x)$ defined in $[a, b]$, $f(x)$ is differentiable if:

- $f'(x_0)$ exists for all x_0 where $a < x_0 < b$.
- The right-hand derivative exists at $x = a$ and the left-hand derivative exists at $x = b$.

Continuous Derivatives: If $f(x)$ has a continuous derivative, it is said to be **continuously differentiable**.

Sectional Differentiability

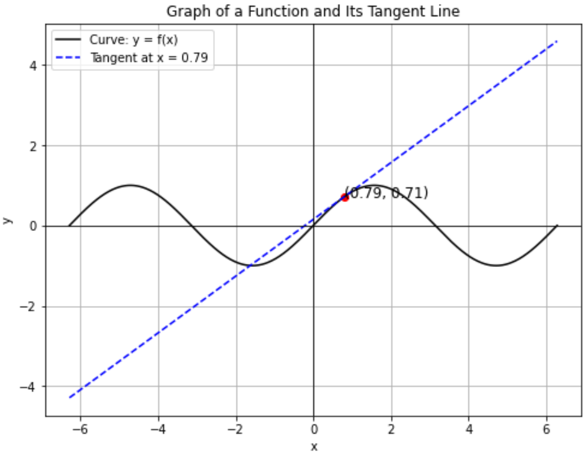
A function $f(x)$ is **sectionally differentiable** in $[a, b]$ if its derivative $f'(x)$ is piecewise continuous in the interval.

Example: Sectional continuity is commonly observed in functions defined piecewise over different subintervals of $[a, b]$.

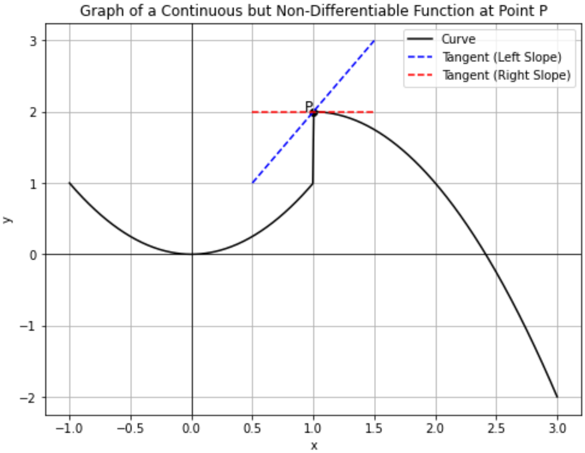
The equation of the tangent line to the curve $y = f(x)$ at $x = x_0$ is:

$$y - f(x_0) = f'(x_0)(x - x_0).$$

This equation represents the linear approximation of $f(x)$ near x_0.



A function can be continuous but not differentiable at a point. This occurs at sharp corners or cusps in the graph. At such points, multiple tangent lines may exist. For instance, point P in the figure has two tangent lines with slopes $f_1'(x_0)$ and $f_2'(x_0)$.



Differentials

Let $\Delta x = dx$ be a small increment in x. The corresponding change in $y = f(x)$ is:

$$\Delta y = f(x + \Delta x) - f(x).$$

For differentiable functions, this change can be approximated as:

$$\Delta y = f'(x)\Delta x + \varepsilon \Delta x,$$

where $\varepsilon \to 0$ as $\Delta x \to 0$. The term $dy = f'(x)dx$ is the differential of y, representing the principal part of Δy.

Rules of Differentiation

For differentiable functions $f(x)$ and $g(x)$, the following rules apply:

- **Sum Rule:** $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$,
- **Product Rule:** $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$,
- **Quotient Rule:** $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$,
- **Chain Rule:** If $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

Thanks