Chapter 4 : Complex numbers

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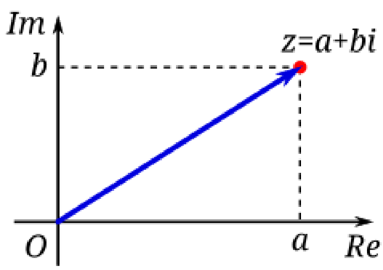
12 janvier 2025

The equation $3+x=2$ has no solutions on the set $\mathbb{N}$ of natural numbers, but if we restrict the domain to the set of integer number we get $x=-1$. In the same way if we look for solutions to $x^2+1=0$ we get no solution on the set of real numbers $\mathbb{R}$. In the set of the well known complex numbers $\mathbb{C}$ the solution is $x=i$.

1 Algebraic structure of complex numbers

If $z\in\mathbb{C}$ then we write $z=a+ib \mid a,b\in\mathbb{R}$. The complex number composed of two pats imaginary part $Im(z)=b$ and real part $Re(z)=a$. The number z It can be represent as follows :



2 Operations on complex numbers

Let z_1, z_2 and $z_3$be three complex numbers such that $z_1=a_1+ib_1$and $z_2=a_2+ib_2$.

1. Addition

$$z_3=z_1+z_2 \Rightarrow z_3=(a_1+a_2)+i(b_1+b_2)$$

2. Equality

$$z_1=z_2 \Longleftrightarrow a_1=a_2, b_1=b_2$$

3. Multiplication

$$\begin{aligned} z_3=z_1z_2 \Rightarrow z_3 &= (a_1+ib_1)(a_2+ib_2) \\ &= a_1a_2-b_1b_2+i(a_1b_2+a_2b_1) \end{aligned}$$

4. Complex conjugate

   The complex conjugate of $z=a+ib$ is deoted $\bar{z}=a-ib$. The product of a complex number and its conjugate gives the Modulus (or absolute value)

$$\begin{aligned} |z|=\sqrt{z\bar{z}} &= \sqrt{(a+ib)(a-ib)} \\ &= \sqrt{a^2+b^2} \end{aligned}$$

5. Division

$$z_3 = \frac{z_1}{z_2} = \frac{z_1\bar{z}_2}{|z_2|} = \frac{a_1a_2 + b_1b_2 + i\left(a_2b_1 - a_1b_2\right)}{a_2^2 + b_2^2}$$

Example :

— $\frac{1}{z} = \frac{1}{a+ib} = \frac{\bar{z}}{z\bar{z}} = \frac{a}{a^2+b^2} + \frac{b}{a^2+b^2}i$

— $z = 2+3i \Rightarrow \frac{1}{z} = \frac{2}{2^2+3^2} + \frac{3}{2^2+3^2}i$
$= \frac{2}{13} + \frac{3}{13}i$

3 Trigonometric presentation

The cmplex number $z = a + ib$ it can be represent in polar coordinates in which $a = \rho\cos(\theta)$ and $b = \rho\sin(\theta)$. Where θ is called argument ($\arg(z) = \theta$). Thus the trigonometric (or polar) formula is given as

$$z = \rho\left(\cos(\theta) + i\sin(\theta)\right)$$

where $\rho = \sqrt{a^2+b^2} = |z|$. From this equation we can extract De Moivre's formula.

$$z^n = \rho^n\left(\cos(n\theta) + i\sin(n\theta)\right)$$

The complex number $e^{i\theta}$also it can be written terms of trigonometric functions

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

This is called Euler's formula in which $Re(e^{i\theta}) = \cos(\theta)$ and $Im(e^{i\theta}) = \sin(\theta)$. Also we can get from this

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$
$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Example :$z = \frac{1}{1-2i}$

$$z = \frac{1+2i}{5} = \frac{1}{5} + \frac{2}{5}i \Longrightarrow |z| = \rho = \sqrt{z\bar{z}} = \sqrt{\left(\frac{1}{5}\right)^2 + \left(\frac{2}{5}\right)^2} = \frac{1}{\sqrt{5}}$$

$$\sin(\theta) = \frac{\frac{2}{5}}{\rho} = 2\frac{\sqrt{5}}{5}$$
$$\cos(\theta) = \frac{\frac{1}{5}}{\rho} = \frac{\sqrt{5}}{5}$$

$\Longrightarrow \theta \approx \frac{\pi}{\sqrt{8}}$

$$z = e^{i\frac{\pi}{\sqrt{8}}} = \frac{1}{\sqrt{5}}\left(\cos(\frac{\pi}{\sqrt{8}}) + i\sin(\frac{\pi}{\sqrt{8}})\right)$$