Chapter 5: system with several degrees of freedom.

5.1. Degree of freedom

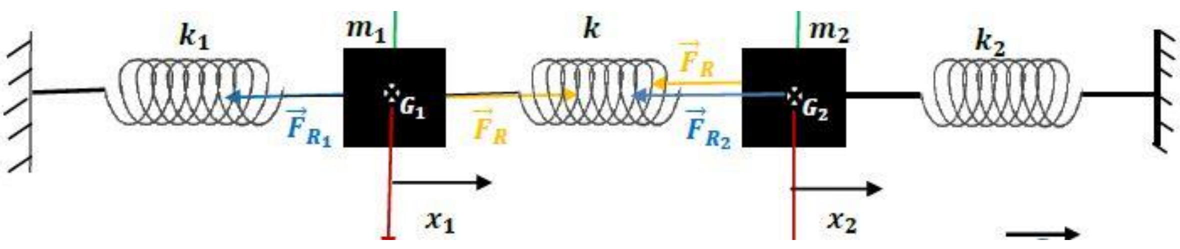
The number of degrees of freedom is defined by the number of independent variables necessary to describe the movement of a system.

Consider a system with n degrees of freedom, subject to forces which derive from a potential, viscosity friction forces and external forces. If the generalized coordinates are q_1, q_2.......q_n the Lagrange equations are written.

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \left(\frac{\partial L}{\partial q_1}\right) + \left(\frac{\partial D}{\partial \dot{q}_1}\right) = F_1 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \left(\frac{\partial L}{\partial q_2}\right) + \left(\frac{\partial D}{\partial \dot{q}_2}\right) = F_2 \\ \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n}\right) - \left(\frac{\partial L}{\partial q_n}\right) + \left(\frac{\partial D}{\partial \dot{q}_n}\right) = F_n \end{cases} \qquad (5.1)$$

5.2. Equations of motion of a system with two degrees of freedom

The two independent variables are x_1 and x_2; the coupling element is the spring K as shown in the following figure 5.1:



The kinetic energy of this system is given by

$$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$$

And the potential energy is given by

$$U=\frac{1}{2}kx_1^2+\frac{1}{2}kx_2^2+\frac{1}{2}k(x_1-x_2)^2$$

The Lagrangian is

$$L=T-U=\frac{1}{2}m\dot{x}_1^2+\frac{1}{2}m\dot{x}_2^2-\frac{1}{2}kx_1^2-\frac{1}{2}kx_2^2-\frac{1}{2}k(x_1-x_2)^2$$

The system of differential equations is written as follows:

$$\begin{cases}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right)-\left(\frac{\partial L}{\partial x_1}\right)=0\\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right)-\left(\frac{\partial L}{\partial x_2}\right)=0\end{cases}\Rightarrow\begin{cases}m_1\ddot{x}_1+(K_1+K)x_1-Kx_2=0\\ m_2\ddot{x}_2+(K_2+K)x_2-Kx_1=0\end{cases}\qquad(5.2)$$

We propose sinusoidal solutions to solve this system of linear differential equations, where the masses will oscillate at the same pulsation ω_p with different amplitudes and different phases.

5.3. Clean modes

The solution of the previous system is of the following form

$$x_1(t)=A\cos(\omega_p t+\varphi)$$

and

$$x_2(t)=B\cos(\omega_p t+\varphi)$$

SO

$$\dot{x}_1(t)=-A\omega\sin(\omega_p t+\varphi)$$

$$\dot{x}_2(t)=-B\omega\sin(\omega_p t+\varphi)$$

and

$$\ddot{x}_1(t)=A\omega^2\cos(\omega_p t+\varphi)$$

$$\ddot{x}_2(t)=B\omega^2\cos(\omega_p t+\varphi)$$

Let's replace it with

$$x_1(t)\ ,\ \ddot{x}_1(t),\quad x_2(t)\ \text{et}\ \ddot{x}_2(t),$$

The system becomes as follows:

$$\begin{cases} \left(-{\omega_p}^2+\dfrac{K_1+K}{m_1}\right)A-\dfrac{K}{m_1}B=0 \\ -\dfrac{K}{m_2}A+\left(-{\omega_p}^2+\dfrac{K_2+K}{m_2}\right)B=0 \end{cases}$$

$$\Leftrightarrow \begin{bmatrix} -{\omega_p}^2+\frac{K_1+K}{m_1} & -\frac{K}{m_1} \\ -\frac{K}{m_2} & -{\omega_p}^2+\frac{K_2+K}{m_2} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0 \quad (5.3)$$

The amplitude matrix must not cancel (because there is movement), therefore, the first matrix which must cancel.

$$\begin{vmatrix} -{\omega_p}^2+\dfrac{K_1+K}{m_1} & -\dfrac{K}{m_1} \\ -\dfrac{K}{m_2} & -{\omega_p}^2+\dfrac{K_2+K}{m_2} \end{vmatrix} = 0$$

$${\omega_p}^4-(\omega_1^2+\omega_2^2){\omega_p}^2+\omega_1^2\omega_2^2(1-k^2)=0$$

Avec

$$\omega_1^2=\frac{K_1}{m_1}\omega_2^2=\frac{K_2}{m_2} \qquad \text{et} \quad k=\frac{K^2}{(K_1+K)(K_2+K)}$$

k is the coupling coefficient.

The two proper pulsations are

$$\text{Où :}\left(\frac{k+k_1}{m_1}-\omega^2\right)\left(\frac{k+k_2}{m_2}-\omega^2\right)-\frac{k^2}{m_1m_2}=0$$

If we develop the relationship below we have:

$$\omega^4-\left({\omega_{0_1}}^2+{\omega_{0_2}}^2\right)\omega^2+{\omega_{0_1}}^2{\omega_{0_2}}^2-\frac{k^2}{m_1m_2}=0$$

$$\omega^4-\left(\frac{k+k_1}{m_1}+\frac{k+k_2}{m_2}\right)\omega^2+\left(\frac{k+k_1}{m_1}\right)\left(\frac{k+k_2}{m_2}\right)-\frac{k^2}{m_1m_2}=0 \qquad \text{(V-4)}$$

The determinant of the bisquare equation (V.3) is written:

$$\Delta=\left({\omega_{0_1}}^2+{\omega_{0_2}}^2\right)-4({\omega_{0_1}}^2{\omega_{0_2}}^2-\frac{k^2}{m_1m_2})$$

$$\Delta=\left({\omega_{0_1}}^2-{\omega_{0_2}}^2\right)+4\frac{k^2}{m_1m_2}$$

$$\Delta=\left(\frac{k+k_1}{m_1}-\frac{k+k_2}{m_2}\right)+4\frac{k^2}{m_1m_2}$$

We notice that the determinant is positive. The two bisquare solutions are real and positive, the quantity $({\omega_{0_1}}^2+{\omega_{0_2}}^2-\frac{k^2}{m_1m_2})$ is greater than zero, they are written

$$\omega^2{}_1=\frac{{\omega_{0_1}}^2+\omega^2{}_0+\sqrt{\Delta}}{2}$$

$$\omega^2{}_1=\frac{\frac{k+k_1}{m_1}+\frac{k+k_2}{m_2}+\sqrt{\Delta}}{2}$$

$$\omega^2{}_2=\frac{{\omega_{0_1}}^2+\omega^2{}_0-\sqrt{\Delta}^2}{2}$$

$$\omega^2{}_2=\frac{\frac{k+k_1}{m_1}+\frac{k+k_2}{m_2}+\sqrt{\Delta}}{2}$$

The two proper pulsations are

$$\begin{cases}(-m\omega_p^2 + 2K)A - KB = 0 \\ (-m\omega_p^2 + 2K)B - KA = 0\end{cases} \Leftrightarrow \begin{vmatrix}-m\omega_p^2 + 2K & -K \\ -K & -m\omega_p^2 + 2K\end{vmatrix} = 0$$

$$(-m\omega_p^2 + 2K)^2 - K^2 = 0 \Rightarrow \begin{cases}3K - m\omega_p^2 = 0 \\ K - m\omega_p^2 = 0\end{cases}$$

$$\begin{cases}\omega_{p1} = \sqrt{\dfrac{K}{m}} \\ \omega_{p2} = \sqrt{\dfrac{3K}{m}}\end{cases}$$

Alors les solutions générales sont :

$$\begin{cases}x_1(t) = A_1 \cos(\omega_{p1}t + \varphi_1) + A_2 \cos(\omega_{p2}t + \varphi_2) \\ x_2(t) = B_1 \cos(\omega_{p1}t + \varphi_1) + B_2 \cos(\omega_{p2}t + \varphi_2)\end{cases}$$

$$\begin{cases}x_1(t) = A_1 \cos\left(\sqrt{\dfrac{K}{m}}t + \varphi_1\right) + A_2 \cos\left(\sqrt{\dfrac{3K}{m}}t + \varphi_2\right) \\ x_2(t) = B_1 \cos\left(\sqrt{\dfrac{K}{m}}t + \varphi_1\right) + B_2 \cos\left(\sqrt{\dfrac{3K}{m}}t + \varphi_2\right)\end{cases}$$

Suppose that the two masses oscillate with the same beat ωp_1 then ωp_2

1^er^ cas ω=$\boldsymbol{\omega_{p1}}$

$$\begin{cases} x_1(t) = A_1 \cos\left(\sqrt{\dfrac{K}{m}}t + \varphi_1\right) \\ x_2(t) = B_1 \cos\left(\sqrt{\dfrac{K}{m}}t + \varphi_1\right) \end{cases}$$

Où $x_1(t)$ et $x_2(t)$ sont les solutions de des équations différentielle

$$\begin{bmatrix} -m\omega_{p1}^2 + 2K & -K \\ -K & -m\omega_{p1}^2 + 2K \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left(-m\omega_{p1}^2 + 2K\right)A_1 - KB_1 = 0$$

Et

$$-KA_1 + \left(-m\omega_{p1}^2 + 2K\right)B_1 = 0$$

Donc :$\frac{A_1}{B_1} = \frac{-K}{-m\omega_{p1}^2+2K}$ et$\frac{A_1}{B_1} = \frac{-m\omega_{p1}^2+2K}{-K}$

Alors :$\frac{A_1}{B_1} = 1 \Rightarrow A_1 = B_1$ (x_1 et x_2 sont en phase)

1^{er} cas $\omega=\omega_{p2}$

$$\begin{cases} x_1(t) = A_2 \cos\left(\sqrt{\frac{3K}{m}}t + \varphi_2\right) \\ x_2(t) = B_2 \cos\left(\sqrt{\frac{3K}{m}}t + \varphi_2\right) \end{cases}$$

$\Rightarrow$

$$\begin{bmatrix} -m\omega_{p2}^2 + 2K & -K \\ -K & -m\omega_{p2}^2 + 2K \end{bmatrix} \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow A_2 = -B_2$(x_1 et x_2Sonten opposition de phase)

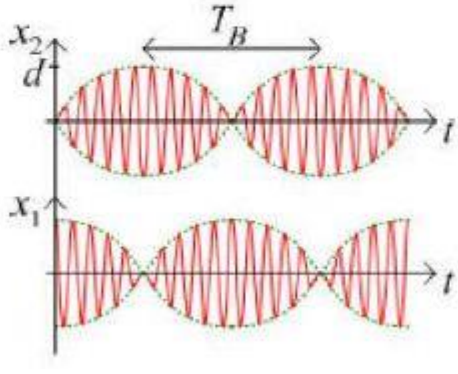
Alors :

$$x_1(t) = A_1 \cos\left(\sqrt{\frac{K}{m}}t + \varphi_1\right) + A_2 \cos\left(\sqrt{\frac{3K}{m}}t + \varphi_2\right)$$

$$x_2(t) = A_1 \cos\left(\sqrt{\frac{K}{m}}t + \varphi_1\right) - A_2 \cos\left(\sqrt{\frac{3K}{m}}t + \varphi_2\right)$$

On trouveA$_1$,A$_2$,$\varphi_1$et φ_2 à partir des conditions initiales.

Le phénomène étudié est le battement :



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5.2. Forced oscillations with two degrees of freedom

5.2.1. Forced oscillation without damping

We consider the system (two masses, three springs) in Figure

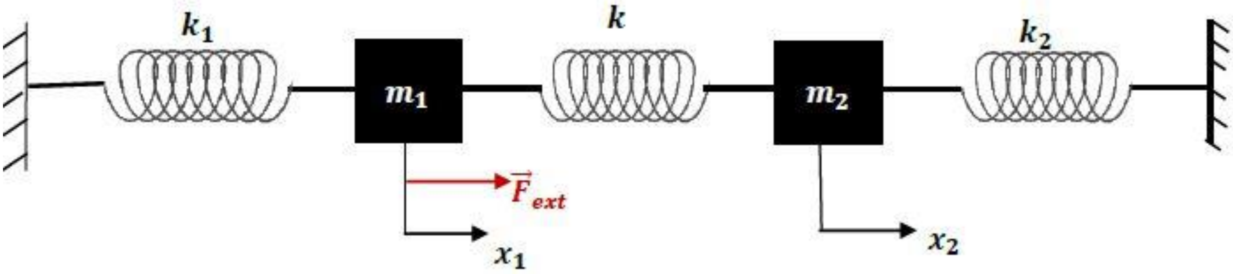

Figure 5.2. Mechanical oscillators coupled by elasticity are subjected to an external force

•The differential equations of motion are:

$$\begin{cases} m_1\ddot{x}_1 + (k_1 + k)x_1 - kx_2 = F_0 cos\Omega t \\ m_2\ddot{x}_2 + (k_2 + k)x_2 + kx_1 = 0 \end{cases}$$

We are interested in the permanent regime, that is to say the one which is governed by the particular solution of the system of differential equations. We are looking for sinusoidal solutions of pulsation ***Ω***. Using the expressions of the own pulsations of the decoupled oscillators defined by:

$$\omega_{01}^2 = \frac{k + k_1}{m_1} \quad and \quad \omega_{02}^2 = \frac{k + k_2}{m_2}$$

We obtain the following system of differential equations:

$$\begin{cases} \ddot{x}_1 + \omega_{01}^2 x_1 - \dfrac{k}{m_1} x_2 = \dfrac{F_0}{m_1} cos\Omega t \\ \ddot{x}_2 + \omega_{02}^2 x_2 - \dfrac{k}{m_2} x_1 = 0 \end{cases}$$

If we use the complex notion, we replace x_1 and x_2 the external force with their complex expressions.

$$\bar{A}_1 = A_1 e^{i\varphi_1} e^{i\Omega t} = \overline{A_1} e^{i\Omega t}$$

Avec : $\bar{A}_1 = A_1 e^{i\varphi_1}$

$$\bar{A}_2 = A_2 e^{i\varphi_2} e^{i\Omega t} = \overline{A_2} e^{i\Omega t}$$

Avec :

$$\bar{A}_2 = A_2 e^{i\varphi_2}$$

$$\bar{F}_0 = F_0 e^{i\Omega t}$$

We obtain the system:

$$\begin{cases} (\omega_{01}^2 - \Omega^2)\bar{A}_1 - \frac{k}{m_1}\bar{A}_2 = \frac{F_0}{m} \\ \frac{-k}{m_2}\bar{A}_1 + (\omega_{02}^2 - \Omega^2)\bar{A}_2 = 0 \end{cases}$$

For the system to admit solutions, the main determinant must be different from zero.

$$\Delta = \begin{vmatrix} \omega_{01}^2 - \Omega^2 & \frac{-k}{m_1} \\ \frac{-k}{m_2} & \omega_{02}^2 - \Omega^2 \end{vmatrix} \neq \mathbf{0}$$

The solutions of this system are:

$$\bar{A}_1 = \begin{vmatrix} \frac{F_0}{m_1} & \frac{-k}{m_1} \\ 0 & \omega_{02}^2 - \Omega^2 \end{vmatrix} = A_1 cos + iA_1 sin\boldsymbol{\varphi_1}$$

$$\bar{A}_2 = \begin{vmatrix} \omega_{01}^2 - \Omega^2 & \frac{F_0}{m_1} \\ -\frac{k}{m_2} & 0 \end{vmatrix} = A_2 cos\varphi_2 + iA_2 sin\varphi_2$$

• In the case of our example, the expressions found for A_1 *and* A_2 are purely real quantities.

$$\sin\varphi_1 = 0 \Rightarrow \quad \varphi_1 = 0 \ or\ \pi$$

$$\sin\varphi_2 = 0 \Rightarrow \quad \varphi_2 = 0 \ or\ \pi$$

• The movement of the mass considered is either in phase or in opposition to phase with the external force.

• If we take the two phase shifts equal to 0. The displacements take the following forms:

$$x_1(t) = \frac{F_0({\omega_{02}}^2 - \Omega^2)}{m_1\Delta} \cos\Omega t = A_1 \cos\Omega t$$

$$x_2(t) = \frac{-F_0 k}{m_1 m_2 \Delta} \cos\Omega t = A_2 \cos\Omega t$$

The main discriminant is zero for the values of the pulsation given by the square root of the expressions. The amplitudes $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ tend towards infinity. This is the resonance for an equal external pulsation a ω_{01}, the oscillator $\mathbf{m_1}$ is stationary and $\boldsymbol{A_1} = \frac{\boldsymbol{F_0}}{\boldsymbol{k}}$ This is the phenomenon of antiresonance. Oscillator 2 (right) vibrates sinusoidal at the own pulsation and exerts on oscillator 1 (left) via the coupling spring a force which is at each instant equal and opposite to the external force imposed by the oscillator I. We trace the variations of the amplitudes $\boldsymbol{A}_1$ and $\boldsymbol{A_2}$ as a function of the pulsation of the exciting force.

5.3. Oscillation force damped

Or the system opposite:

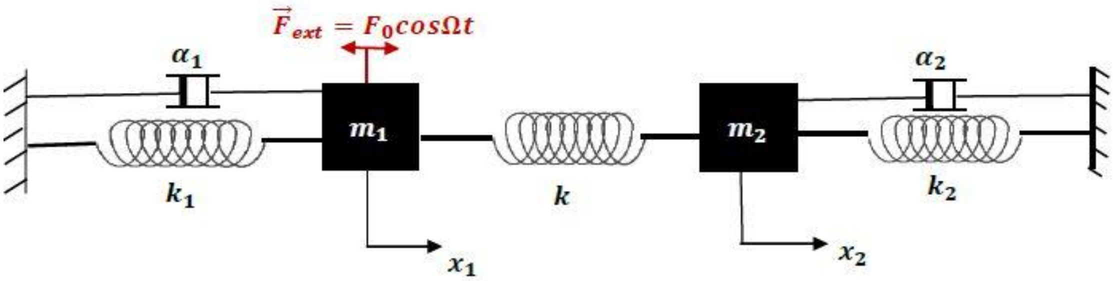

Figure.5.3. Mechanical oscillators by elasticity and subject to an external force and a shock absorber

❖ The differential equations of motion are:

$$\begin{cases} m_1\ddot{x}_1 + (k_1 + k)x_1 + \alpha_1\dot{x}_1 - kx_2 = F_0 cos\Omega t \\ m_2\ddot{x}_2 + (k_2 + k)x_2 + \alpha_2\dot{x}_2 - kx_1 = 0 \end{cases}$$