Chapter 4

Forced vibrations of systems with one degree of freedom.

41.Introduction

Forced vibrations (oscillations) occur when the system is subjected to along its vibrations to periodic external forces. Often these forces are called external excitations. The resulting movement is called the response of the system to external excitement. The excitation force can be harmonic, periodic or harmonic, non-periodic or random. In this course, we are only interested in harmonic excitations. The excitement harmonic can be given mathematically by

$$f(t) = f_0 \sin(w.t + \varphi), \quad f(t) = f_0 \cos(w.t + \varphi), \quad f(t) = f_0 e^{i(w.t+\varphi)}.$$

Φ The phase of the excitation (depends on the value of the force f at t = 0.

4. .2 Harmonic excitation of an undamped system.

- **Equations of motion :**

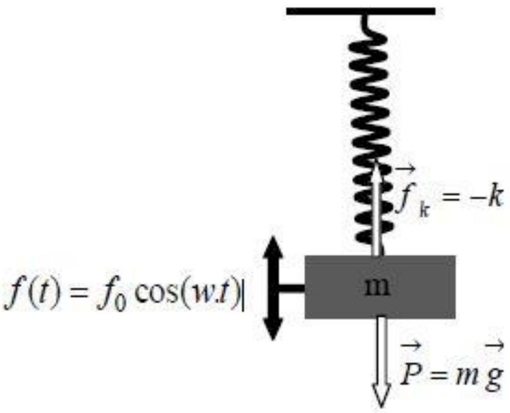


Figure 3.1 : Système masse ressort harmoniquement excité

From the application of Lagrange's theorem, which is recommended in this course?

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)-\frac{\partial L}{\partial q}=f_q$$

$$\text{Où}\begin{cases} L=T-U & \textit{le Langrangien du système} \\ q\equiv x & \textit{la coordonée généralisée} \\ f_q=f_0\sin(w.t+\varphi_0) & \textit{la force congégiée} \end{cases}$$

$$L=T-U=\frac{1}{2}m\dot{x}^2+\frac{1}{2}kx$$

- $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)=m\ddot{x}$
- $\frac{\partial L}{\partial q}=kx$
- $f_q=f_0\cos(w.t)$

Par remplacement,

$$\ddot{x}+\frac{k}{m}x=\frac{f_0}{m}\cos(w.t)$$

- **Solution of the differential equation.**

The equation of motion is a second order differential equation with second member. The solution to this equation is the sum of two solutions

- General solution x_g of the homogeneous equation
- Particular solution x_p of the non-homogeneous equation

The solution x g of the homogeneous equation

$$\ddot{x}+\frac{k}{m}x=0 \Rightarrow \ddot{x}+ \omega_0{}^2 x=0$$

Is given by: $x_g = A_1 \cos(\omega_0 t) + A_2 \sin(\omega_0 t)$

Since n w is not a solution of the characteristic equation, the particular solution is given by: $x_p = C_1 \cos(\omega t) + C_2 \sin(\omega t)$

To find the constants C_1 and C_2, we must replace this solution in the equation differential. We have,

$$x_p = C_1 \cos(\omega t) + C_2 \sin(\omega t)$$

$$\dot{x}_p = -C_1\omega \sin(\omega t) + C_2\omega \cos(\omega t)$$

$$\ddot{x}_p = -C_1\omega^2 \cos(\omega t) - C_2\omega^2 \sin(\omega t)$$

By replacement we find,

$$-C_1\omega^2 \cos(\omega t) - C_2\omega^2 \sin(\omega t) + C_1 \omega_n \cos(\omega t) + C_2 \omega_n \sin(\omega t) = \frac{f_0}{m} \cos(\omega t)$$

By comparing the two sides of the equation, we find:

$$C_1=\frac{\frac{f_0}{m}}{\omega_0^2-\omega^2} \quad \text{and} \quad C_2=0$$

We put C_1 =X and divide the numerator and denominator by k we find,

$$X=\frac{\frac{f_0}{m}}{\frac{m}{k}(\omega_0^2-\omega^2)} \quad \Rightarrow \quad X=\frac{\delta_{st}}{1-(\frac{\omega}{\omega_0})^2}$$

$\delta_{st} = \frac{f_0}{k}$:Represents the deflection of the mass under the application of the force f_0 and is called the static deflection of the mass.

Finally, the particular solution is written:

$$\mathbf{x_p} = \frac{\boldsymbol{\delta_{st}}}{\mathbf{1} - (\frac{\boldsymbol{\omega}}{\boldsymbol{\omega_0}})^2} \mathbf{cos}(\boldsymbol{\omega}.\boldsymbol{t})$$

We sum the solutions and p x, the solution of the differential equation of motion (1) is written as follows:

$$x = A_1 \cos(w_n t) + A_2 \sin(w_n t) + \frac{\delta_{st}}{\left(1 - \left(\frac{w}{w_n}\right)^2\right)} \cos(w.t)$$

(3.15)

Les constantes A_1 et A_2 seront trouvées par les conditions initiales.

$$t = 0 \begin{cases} x = x_0 \\ \dot{x} = \dot{x}_0 \end{cases}, \qquad \begin{cases} A_1 = x_0 - \dfrac{\delta_{st}}{\left(1 - \left(\frac{w}{w_n}\right)^2\right)} \\ A_2 = \dfrac{\dot{x}_0}{w_n} \end{cases}$$

$$x = \left(x_0 - \frac{\delta_{st}}{\left(1 - \left(\frac{w}{w_n}\right)^2\right)} \right) \cos(w_n t) + \frac{\dot{x}_0}{w_n} \sin(w_n t) + \frac{\delta_{st}}{\left(1 - \left(\frac{w}{w_n}\right)^2\right)} \cos(w.t)$$

4.5. Amplification factor:

The amplification factor is given by the ratio $\frac{A}{\delta_{st}}$

Where: $A = \frac{\delta_{st}}{1-(\frac{\omega}{\omega_0})^2}$

So : $\frac{A}{\delta_{st}} = \frac{1}{1-(\frac{\omega}{\omega_0})^2}$

We distinguish three cases depending on the value of $\frac{A}{\delta_{st}}$

- $\frac{A}{\delta_{st}} > 0 \Rightarrow \frac{\omega}{\omega_0} < 1 \Rightarrow x(t) et\ f(t)$ are in phase
- $\frac{A}{\delta_{st}} < 0 \Rightarrow \frac{\omega}{\omega_0} > 1 \Rightarrow x(t) et\ f(t)$ in phase opposition
- $\frac{A}{\delta_{st}} \rightarrow \Rightarrow \frac{\omega}{\omega_0} = 1 \Rightarrow$ in resonance

4.2. Harmonic excitation of a damped system.

4.2.1. Différentiel équation of motion

Pour un mouvement de translation, on écrit

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + \frac{\partial D}{\partial \dot{q}} = f(x)$$

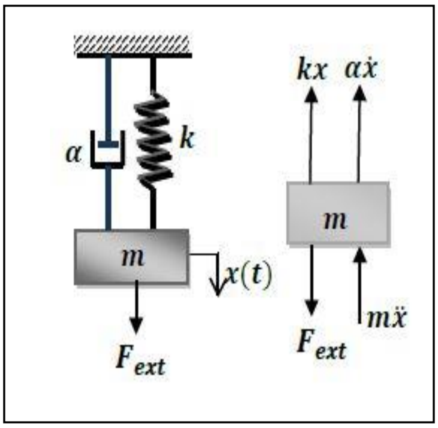
❖ **Example: mass-spring-damper system**

Let us return to the case of the elastic pendulum (vertical for example).

The study of the damped oscillator is done in the same way as previously but adding an external force

In one dimension, the Lagrange equation is written:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} + \frac{\partial D}{\partial \dot{x}} = f(x)$$



Consider a sinusoïdal force applied to mass m $F_{ext} = F_0 \cos \omega t$.

L: The Lagrangian of the system is given by : $\mathbf{L} = \boldsymbol{T} - \mathbf{U}$

q: The generalized coordinate, in this case $\boldsymbol{q} \equiv \boldsymbol{x}$

<u>The kinetic energy of the system</u>**:** $\boldsymbol{T} = \frac{1}{2}\ \mathbf{m}\dot{\boldsymbol{x}}^2$

<u>The potential energy of the system</u> :$\boldsymbol{U} = \frac{1}{2}\mathbf{k}\boldsymbol{x}^2$

<u>The dissipation energy is.:</u> $\boldsymbol{D} = \frac{1}{2}\boldsymbol{\alpha}\dot{\boldsymbol{q}}^2$

<u>The Lagrange function:</u> $\mathbf{L} = \boldsymbol{T} - \mathbf{U} = \frac{1}{2}\ \mathbf{m}\dot{\boldsymbol{x}}^2 - \frac{1}{2}\mathbf{k}\boldsymbol{x}^2$

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = k\ddot{x} \\ \frac{\partial L}{\partial x} = kx \\ \frac{\partial D}{\partial \dot{x}} = \alpha\dot{x} \end{cases} \quad \Rightarrow \quad \boldsymbol{m\ddot{x} + \alpha\dot{x} + kx = F_0\ \cos\omega t}$$

We then divide by m and we find : $\ddot{\boldsymbol{x}} + \frac{\boldsymbol{\alpha}}{\boldsymbol{m}}\dot{\boldsymbol{x}} + \frac{\boldsymbol{k}}{\boldsymbol{m}}\boldsymbol{x} = \frac{\boldsymbol{f_0}}{\boldsymbol{m}}\ \mathbf{cos}\,\boldsymbol{\omega t}$

Often the differential equation is written in reduced form:

$$\ddot{\boldsymbol{x}} + 2\boldsymbol{\delta}\dot{\boldsymbol{x}} + \boldsymbol{\omega}_0{}^2\boldsymbol{x} = \frac{\boldsymbol{f_0}}{\boldsymbol{m}}\ \mathbf{cos}\,\boldsymbol{\omega t}$$

Tels que : $\begin{cases} \delta = \frac{\boldsymbol{\alpha}}{\boldsymbol{m}}\ [1/s] \ \textbf{Facteur d'amortissement}. \\ \varepsilon = \frac{\delta}{\omega_0} \text{(Sans unité)} : \ \textbf{Rapport d'amortissement}. \end{cases}$

We therefore obtain a second order linear differential équation with constant coefficients with second member.

4.2 .2. Solution of the differential equation of motion

The general solution of this differential equation is the sum of two terms

• A solution of the equation without a second member: homogeneous solution $\boldsymbol{x}_{\mathrm{g}}(\boldsymbol{t})$.

• A solution of the equation with second member: particular solution $\boldsymbol{x_P}(\boldsymbol{t})$**.**

The total solution of the équation of motion Will therefore bé: $\boldsymbol{x}(\boldsymbol{t}) = \boldsymbol{x}_{\mathrm{g}}(\boldsymbol{t}) + \boldsymbol{x_P}(\boldsymbol{t})$

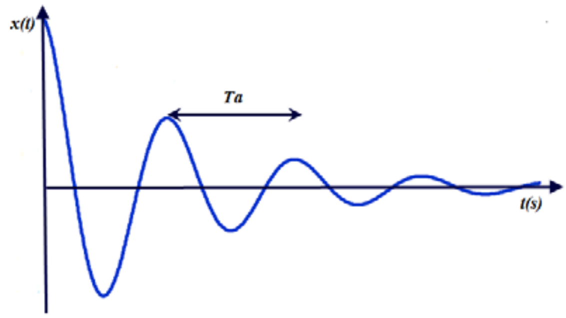
a) Homogeneous solution :

The homogeneous solution corresponds to the solution of the differential equation without a second member: $\ddot{x} + 2\delta\dot{x} + \omega_0{}^2 x = 0$

It appears that the solution of the homogeneous differential equation is quite simply the solution found for the damped harmonic oscillator in free regime in the case of weakly damped oscillations:

$$x_g = A_1 \cos(\omega_0 t) + A_2 \sin(\omega_0 t)$$

The general solution of the equation without a second member corresponds to a **transient** regime (which only lasts a certain time).



b) Special solution:

When the component $x_g(t)$ becomes truly negligible, all that remains is the particular solution, which is the solution imposed by the excitation function. We say that we are in a forced or permanent regime.

The exciting force forces the mechanical system to follow a temporal evolution equivalent to its own. So if F_{ext} is a sinusoïdal pulsation function ω; then the particular solution $x_P(t)$ will be a sinusoidal function with the same pulsation ω.

The oscillations of the mass are not necessarily in phase with the exciting force and present a noted phase shift φ. The particular solution corresponding to the steady state is written as: $x_g = X \cos(\omega_0 t + \varphi)$

To find the constants **X** et **φ**, we drift x_p and we replace in the equation.

$$\dot{x}_p = -Xw\sin(w.t+\varphi) \quad et \quad \ddot{x}_p = -Xw^2\cos(w.t+\varphi)$$

$$X\left[\left(k-mw^2\right)\cos(w.t+\varphi) - cw\sin(w.t+\varphi)\right] = f_0\cos(w.t)$$

On utilise : $\begin{cases}\cos(w.t+\varphi) = \cos(wt)\cos\varphi - \sin\varphi\sin wt \\ \sin(w.t+\varphi) = \cos\varphi\sin(w.t) + \sin\varphi\cos(w.t)\end{cases}$ on trouve,

$$\left(k-mw^2\right)\cos\varphi - cw\sin\varphi = \frac{f_0}{X}$$

$$\left(k-mw^2\right)\sin\varphi + cw\cos\varphi = 0$$

$$X = \frac{f_0}{\sqrt{\left(k-mw^2\right)^2 + c^2w^2}}$$

$$\varphi = \tan^{-1}\left(-\frac{cw}{k-mw^2}\right)$$

Nous avons, $w_n = \sqrt{\frac{k}{m}}$ et $\varepsilon = \frac{C}{C_c} = \frac{C}{2\sqrt{mk}}$ donc, on peut écrire : $\frac{c^2w^2}{k^2} = \left(2\varepsilon\frac{w}{w_n}\right)^2$

La réponse du système à la force d'excitation peut écrite sous la forme ;

$$X = \frac{\frac{f_0}{k}}{\sqrt{\left(1-\left(\frac{w}{w_n}\right)^2\right)^2 + 4\varepsilon^2\left(\frac{w}{w_n}\right)^2}}$$

$$\varphi = \tan^{-1}\left(-\frac{2\varepsilon\left(\frac{w}{w_n}\right)}{1-\left(\frac{w}{w_n}\right)^2}\right)$$

4.2.3 Amplification factor:

The amplification factor is given by the ratio $\frac{A}{\delta_{st}}$

Where: $\boldsymbol{A} = \frac{\delta_{st}}{1-\left(\frac{\omega}{\omega_0}\right)^2} \Leftrightarrow \frac{A}{\delta_{st}} = \frac{1}{1-\left(\frac{\omega}{\omega_0}\right)^2}$

We distinguish three cases depending on the value of $\frac{A}{\delta_{st}}$

- $\frac{A}{\delta_{st}} > 0 \Rightarrow \frac{\omega}{\omega_0} < 1 \Rightarrow x(t) \text{ and } f(t) \text{ are in phase}$
- $\frac{A}{\delta_{st}} < 0 \Rightarrow \frac{\omega}{\omega_0} > 1 \Rightarrow x(t) \text{ and } f(t) \text{ are in phase oppositions}$
- $\frac{A}{\delta_{st}} \to \infty \Rightarrow \frac{\omega}{\omega_0} = \mathbf{1} \Rightarrow \text{in resonance}$

4.2.4.Bandwidth:

In the case of a sinusoidal excitation of variable pulsation $\boldsymbol{\omega}$ and in the case where $\boldsymbol{\delta} < \frac{\omega_0}{\sqrt{2}}$, we define the pulsation bandwidth of the oscillator by the interval

$$\Delta\boldsymbol{\omega} = \boldsymbol{\omega_2} - \boldsymbol{\omega_1} \; (\omega_2 > \omega_1)$$

Where the pulsations $\boldsymbol{\omega_1}$ and $\boldsymbol{\omega_2}$correspond to the amplitudes A (ω_1) and A (ω_2) such that $\boldsymbol{A(\omega_1)} = \boldsymbol{A(\omega_2)} = \frac{A_{(\omega_r)}}{\sqrt{2}}$

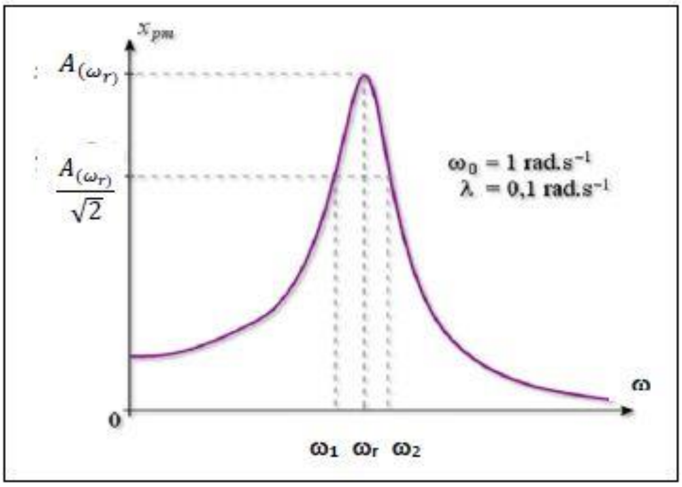


Fig.21. Bandwidth

Then the bandwidth is given by: $\boldsymbol{B} = \Delta\boldsymbol{\omega} = \boldsymbol{\omega_2} - \boldsymbol{\omega_1} = \mathbf{2}\boldsymbol{\delta}$

4.2. 5.The quality factor

The quality factor is defined by the ratio of the specific pulsation to the bandwidth

$$\boldsymbol{Q} = \frac{\boldsymbol{\omega_0}}{\mathbf{B}} = \frac{\boldsymbol{\omega_0}}{\mathbf{2\delta}}$$