Chapter II

Free vibrations of systems with one degree of freedom.

II. 1.Introduction

Any system vibrates far from external excitations (the movement is due to an initial disturbance); the vibrations that follow are called free. If the system requires a single coordinate for its study, the system therefore has one degree of freedom. If the system does not lose energy, the vibrations are said to be undamped, otherwise they are said to be damped.

To study vibrational systems, you must follow the following steps:

- Establish the differential equation that represents the movement.
- Solving the differential equation.
- Deduce the physical parameters: Amplitude, Beat, Frequency.

II.2 Free undamped vibrations

A system which oscillates in the absence of any excitation force ($f(t)=0$), and the forces of Friction and friction ($f\,(q)^{\cdot} = 0$) is called, undamped free oscillator (Harmonic).

II.2.1 Equation of motion

To determine the differential equation representing motion, we use only two methods: Newton's method and Lagrange's method.

a- Newton's method

We take the system composed of a mass **m** suspended from a spring of stiffness **k** and initial length $\mathbf{l_0}$.

We apply the laws of PFD we obtain

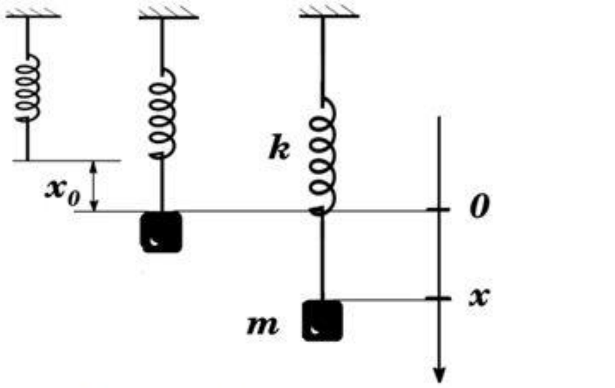


Figure 2.1 : Système masse-ressort à un degré

$$\begin{cases} -kx_0 + mg = 0 \\ et \\ -k(x_0 + x) + mg = m\ddot{x} \end{cases} \Rightarrow \quad \ddot{x} + \frac{k}{m}x = 0$$

b- Lagrange method:

The differential equation of an undamped free motion is of the form: ($\ddot{q}$+ ω^2q=0). This equation is obtained by the Lagrange formalism of conservative systems:

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial \mathrm{L}}{\partial \dot{\mathrm{q}}}\right) - \frac{\partial L}{\partial \mathrm{q}} = 0$$

For the mass-spring system example with one degree of freedom, the Lagrange equation is written as follows:

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial \mathrm{L}}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

L: The Lagrangian of the system is given by $\boldsymbol{L = T - U}$

q: The generalized coordinate, in this case q $\equiv x$

<u>The kinetic energy of the system:</u> it is the kinetic energy of the mass **m**: $T = \frac{1}{2}\mathrm{m}\dot{x}^2$

<u>The potential energy of the system</u>: this is the energy stored in the spring: $\mathrm{U} = \frac{1}{2}\mathrm{k}x^2$

The Lagrange function: $\boldsymbol{L} = \boldsymbol{T} - \boldsymbol{U} = \frac{1}{2}\,\mathrm{m}\dot{x}^2 - \frac{1}{2}\mathrm{k}x^2$

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{dt}}\left(\dfrac{\partial \mathrm{L}}{\partial \dot{x}}\right) = \mathrm{m}\,\ddot{x} \\ \left(\dfrac{\partial \mathrm{L}}{\partial \mathrm{x}}\right) = -kx \end{cases}$$

By replacing in the Lagrange equation we will have $m\,\ddot{x} + kx = 0$

We then divide by **m** and we find $\ddot{x} + \frac{k}{m}x = 0$

II.2.2 Resolution of the equation of motion:

The differential equation of an undamped free motion ($\ddot{q}+\omega^2 q=0$) is a second order equation without a second member, its solution is a sinusoidal function of time, of the form: $\boldsymbol{x} = \boldsymbol{A}\,\mathbf{sin}(\boldsymbol{\omega_0} + \varphi)$

A: Amplitude of oscillations.

$\boldsymbol{\phi}$: Initial phase.

ω: the pulsation of the system, it depends on the constituent elements (mass, spring, wires, etc.)

A and $\boldsymbol{\phi}$: are calculated from the initial conditions:

$$t = 0 \begin{cases} q = q_0 \\ \dot{q} = \dot{q}_0 \end{cases} \quad \Rightarrow \quad \begin{cases} A\sin\varphi = q_0 \\ -A\omega_0\cos\varphi = \dot{q}_0 \end{cases}$$

II.2.3.The own pulsation (the natural beat)

We call proper pulsation ω_0 because it only depends on the proper quantities of the oscillator (*pour* the spring mass system)

$$\omega_0 = \sqrt{\frac{k}{m}}$$

II.3. The total energy of a harmonic oscillator

For the harmonic oscillator $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$

$$x = A\sin(\omega_0 t + \varphi) \quad \Rightarrow \quad \dot{x} = A\omega_0 \cos(\omega_0 t + \varphi)$$

$$\begin{cases} E = \frac{1}{2}mA^2 x{\omega_0}^2 \cos^2(\omega_0 t + \varphi) + \frac{1}{2}kA^2 \sin^2(\omega_0 t + \varphi) \\ Alor \\ E = \frac{1}{2}mA^2 x{\omega_0}^2(\cos^2(\omega_0 t + \varphi) + \sin^2(\omega_0 t + \varphi)) \end{cases} \Rightarrow \quad E = \frac{1}{2}mA^2$$

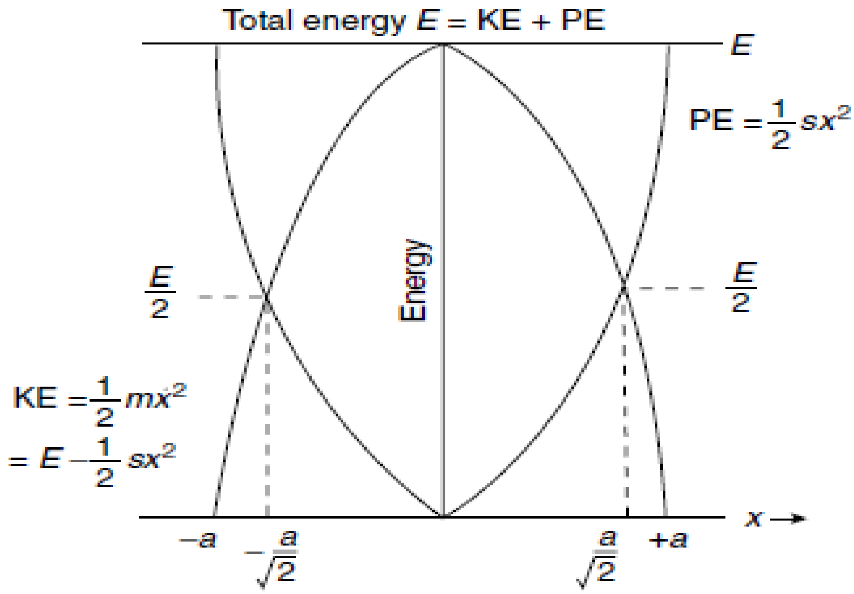


Figure 2.6: Variation of kinetic energy and potential energy

in harmonic vibrations

II. 4. Undamped free rotation vibrations

Given a system oscillating around an axis, the resulting movement is called rotational vibratory movement. We take the example of two masses m_1 and m_2 connected by a metal rod of negligible mass and length r. The rod is mounted on a circular metal bar which is in turn fixed by the other end to a fixed frame (fig.2.4). The metal bar has length **l**, moment of inertia $\mathbf{I_0}$ and shear coefficient **G**. If the system is displaced from its equilibrium position by a given angle θ, a torque $\boldsymbol{M_t = \frac{GI_0}{l}\theta}$ results in the bar Then, the bar acts as a torsion spring with a torsional stiffness $\boldsymbol{k_t = \frac{M_t}{\theta} = \frac{GI_0}{l}}$

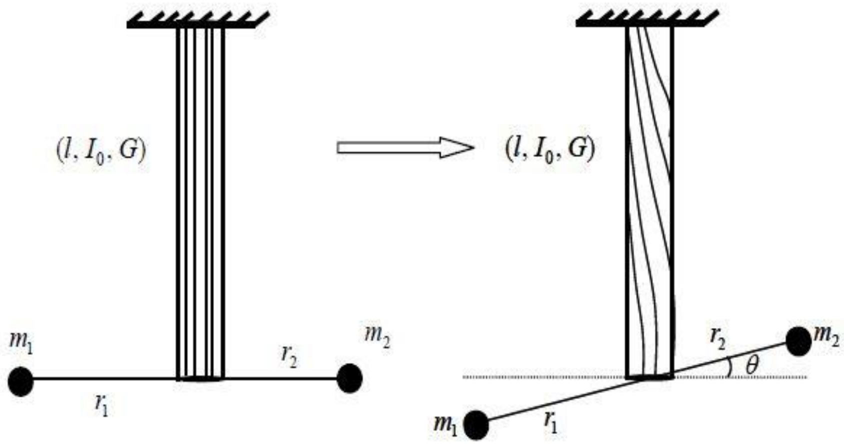


Figure 2. 4 Mouvement vibratoire rotationnel.

The system has one degree of freedom; the only coordinate necessary to study it is the angle θ. We use the Lagrange equation to deduce the differential equation representing the movement,

The generalized coordinate q in this case is the angle θ. SO

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial \mathrm{L}}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$\boldsymbol{L = T - U}$ *and* $\mathrm{U} = \frac{1}{2}\mathrm{k_t}\theta^2$

$$T = \frac{1}{2}\,m_1 v_1{}^2 + \frac{1}{2}\,m_2 v_2{}^2\,; \qquad v_1 = r_1\,\dot{\theta} \text{ and } v_2 = r_2\,\dot{\theta}$$

$$\Rightarrow T = \frac{1}{2}\,(m_1 r_1{}^2 + m_2 r_2{}^2)\dot{\theta}^2 \qquad \Rightarrow T = \frac{1}{2}\,I\,\dot{\theta}^2$$

$I = (m_1 r_1{}^2 + m_2 r_2{}^2)$ is the moment of inertia of the system.

At the end we write the Lagrangian of the system as follows:

$$L = \frac{1}{2}\,I\,\dot{\theta}^2 - \frac{1}{2}k_t\theta^2$$

Applying Lagrange's theorem: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = I\ddot{\theta}$ and $\frac{\partial L}{\partial \theta} = -\,k_t\theta$

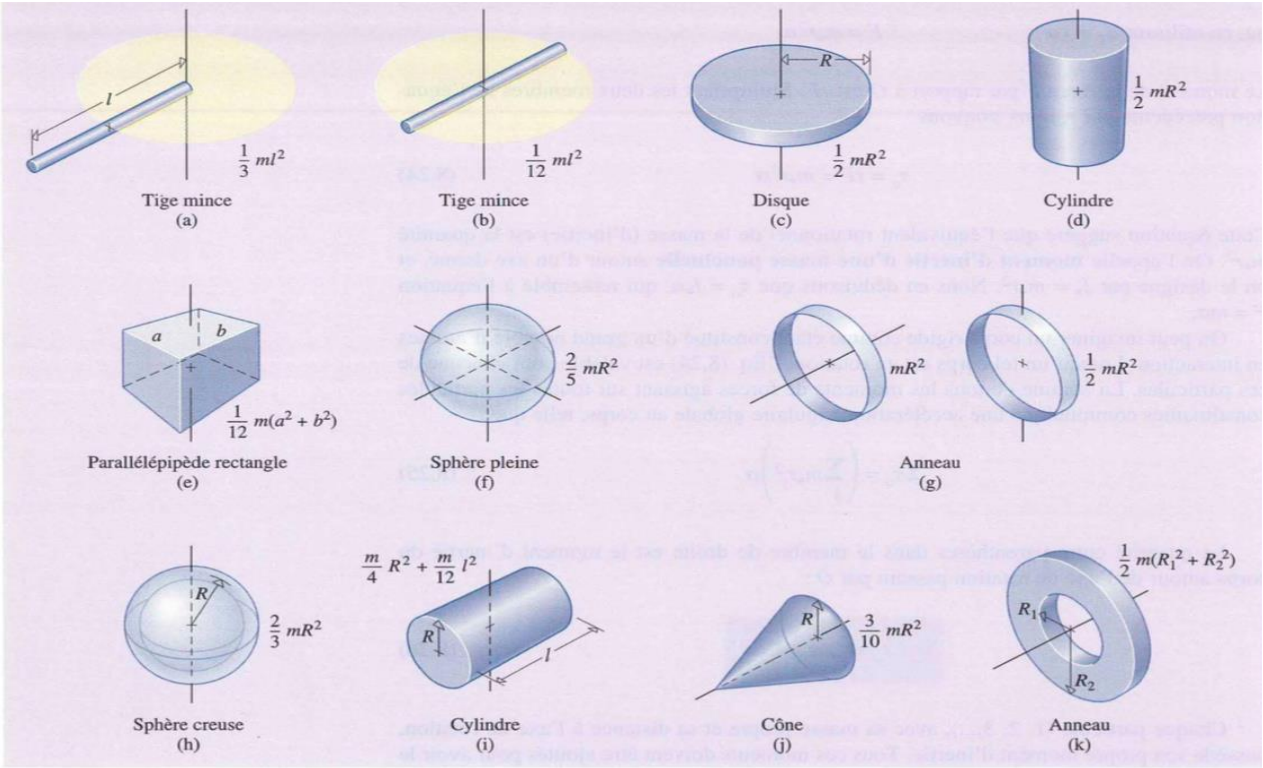
Finally the differential equation which describes the movement can be written as follows:

$$\boldsymbol{\ddot{\theta} + \frac{k_t}{I}\,\theta = 0 \quad \Leftrightarrow \quad \ddot{\theta} + \omega^2\theta = 0}$$

- $\boldsymbol{\omega_n = \sqrt{\frac{k_t}{I}}}$: The natural beat of the vibratory movement in rad .s^{-1}.
- $\boldsymbol{T_n = \frac{2\pi}{\omega_n}}$: The natural period of vibrational movement.
- $\boldsymbol{f_n = \frac{1}{T_n}}$: the natural frequency of the vibratory movement in s^{-1}.

Noticed :

The inertia of a body depends on its dimensions, its mass and its axis of rotation, the figure presents the inertia of different bodies



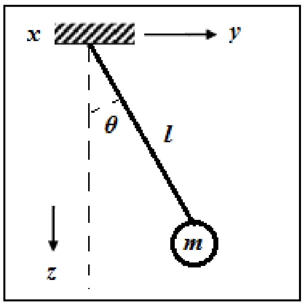
II.5 Huygens' theorem

The moment of inertia varies according to the axis of rotation, if **I_0** is the inertia of a body of mass **m** when the axis of rotation passes through the center of the mass of the body and **ΔI** is the moment of inertia if the axis of rotation is **Δ** with a distance d between the center of mass and the axis of rotation, Huygens' theorem gives the inertia by the following formula:

$$\Delta I = I_0 + md^2$$

Exemple

Consider the following system



1-What is the kinetic energy and potential energy of the system.

2-What is the Lagrangian of the system.

3-Find the differential equation of motion.

4-At initial conditions x (0) = 0 and $\dot{x}$ (0) = 1, find the amplitude of the movement and the phase shift.

Answer

The system is free (no external force applied) and conservative (undamped). The Lagrange equation is given by the relation **L = T−U**

The kinetic energy of the system

$$T = \frac{1}{2} m\, v^2 \quad as \;\; v = l\dot{\theta} \;\; \Rightarrow T = \frac{1}{2} ml^2\dot{\theta}^2$$

The potential energy of the system $U = mgl(1 - \cos\theta)$

$$\begin{cases} L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta) \\ \text{Alor} \\ L = \frac{1}{2}ml^2\dot{\theta}^2 + ml^2\dot{\theta}^2 - mgl + mgl\cos\theta \end{cases} \Rightarrow \begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \theta}\right) = ml^2\ddot{\theta} \\ \left(\frac{\partial L}{\partial \theta}\right) = -\,mgl\sin\theta \end{cases}$$

We consider that the pendulum oscillates (vibrates) with small angles, which allowed us to write $\sin\theta \approx \theta$

$$\frac{\partial L}{\partial \theta} = -mgl\theta \;\; \Rightarrow \;\; ml^2\ddot{\theta} + mgl\theta = 0$$

We replace the equations in the Lagrange equation we find

$$ml^2\ddot{\theta} + mgl\theta = 0$$

We then divide by ml^2 and we find

$$\ddot{\theta} \;\; + \frac{g}{l}\,\theta = 0$$

We take for example the case of the spring mass system and put the equation in the form

$$\ddot{x} + \omega_0{}^2 x = 0 \quad avec \quad \omega_0{}^2 = \frac{k}{m} \quad \Rightarrow \quad x = A_1\cos(\omega_0 t) + A_2\sin(\omega_0 t)$$

Solving the equation is of the form cos$\omega_n t$

$$x = A_1 \cos(\omega_0 t) + \mathrm{A}_2 \sin(\omega_0 \mathrm{t})$$

Where the proper pulsation is:

$$\boldsymbol{\omega}_0 = \sqrt{\frac{\boldsymbol{k}}{\boldsymbol{m}}}$$