Analysis I: Solutions of Tutorial Exercise Sheet 2

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This document is supplemented for the second chapter lecture notes (Analyses 1).

Exercise 01:

Let us take $n \in \mathbb{N}$, then we have:

(a) For $n \in \mathbb{N}$ we have $-\frac{1}{2}, \frac{1}{5}, \frac{3}{8}, \frac{5}{11}, \frac{7}{14}$.

(b) For $n \in \mathbb{N}^*$ we have $2, 0, \frac{2}{9}, 0, \frac{2}{25}$.

(c) For $n \in \mathbb{N}^*$ we have $\frac{1}{2}, -\frac{1}{8}, \frac{1}{48}, -\frac{1}{384}, \frac{1}{3840}$.

(d) For $n \in \mathbb{N}^*$ we have $\frac{1}{2}, \frac{1}{2} + \frac{1}{4}, \frac{1}{2} + \frac{1}{4} + \frac{1}{8}, \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}, \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$

Exercise 02:

(a) For values of $u_1 = 0.22222\ldots$, $u_5 = 0.56000\ldots$, $u_{10} = 0.64444\ldots$, $u_{100} = 0.73827\ldots$, $u_{1000} = 0.74881\ldots$, $u_{10000} = 0.74988\ldots$ and $u_{100000} = 0.74998\ldots$. A reasonable guess is that the limit is $3/4$. It's important to note that this limit becomes evident only for sufficiently large values of n.

(b)To verify this limit by using the definition: We need to demonstrate that for any given positive value ε, there exists a corresponding number N (dependent on ε) such that $|u_n - \frac{3}{4}| < \varepsilon$ for all $n > N$. By manipulating the expression, $\left|\frac{3n-1}{4n+5} - \frac{3}{4}\right| < \varepsilon$, we find that $\left| - \frac{19}{4(4n+5)}\right| < \varepsilon$, which leads to $\frac{19}{4(4n+5)} < \varepsilon \iff \frac{4(4n+5)}{19} > \frac{1}{\varepsilon}$.

We can simplify and derive conditions such as $4n+5 > \frac{19}{4\varepsilon} \iff n > \frac{1}{4}\left(\frac{19}{4\varepsilon} - 5\right)$. Choosing $N = \left[\frac{1}{4}\left(\frac{19}{4\varepsilon} - 5\right)\right] + 1$ accordingly ensures that the limit as n approaches infinity is indeed $3/4$.

Exercise 03:

(1) $\lim_{n\to+\infty} \frac{3n-1}{2n+3} = \frac{3}{2} \iff \forall \varepsilon > 0, \exists ? n_\varepsilon \in \mathbb{N}, \forall n \in \mathbb{N}; n \geq n_\varepsilon \Rightarrow \left|\frac{3n-1}{2n+3} - \frac{3}{2}\right| < \varepsilon$

We use $\left|\frac{3n-1}{2n+3} - \frac{3}{2}\right| < \varepsilon$, then we have $\left|\frac{2(3n-1)-3(2n+3)}{2(2n+3)}\right| = \left|\frac{6n-2-6n-9}{(4n+6)}\right| = \frac{11}{(4n+6)} < \varepsilon \iff \frac{11}{4\varepsilon} - \frac{3}{2} < n$.

For this, it is sufficient to take $n_\varepsilon = \lceil \frac{11}{4\varepsilon} - \frac{3}{2} \rceil + 1$.

(2) $\lim_{n\to+\infty} \frac{(-1)^n}{2n} = 0 \iff \forall \varepsilon > 0, \exists ? n_\varepsilon \in \mathbb{N}, \forall n \in \mathbb{N}; n \geq n_\varepsilon \Rightarrow \left|\frac{1}{2^n}\right| < \varepsilon$. We have $\frac{1}{2^n} < \varepsilon$ leads to $-\frac{\ln \varepsilon}{\ln 2} < n$.

For this, it is sufficient to take $n_\varepsilon = \lceil -\ln(\varepsilon)/\ln(2) \rceil + 1$.

(3) $\lim_{n\to+\infty} \frac{2\ln(1+n)}{\ln(n)} = 2 \iff \forall \varepsilon > 0, \exists ? n_\varepsilon \in \mathbb{N}, \forall n \in \mathbb{N}; n \geq n_\varepsilon \Rightarrow \left|\frac{2\ln(1+n)}{\ln(n)} - 2\right| < \varepsilon$.

So we take, $\left|\frac{2\ln(1+n)}{\ln(n)} - 2\right| = \left|\frac{2\ln(1+n) - 2\ln(n)}{\ln(n)}\right| = 2\left|\frac{\ln(\frac{1+n}{n})}{\ln(n)}\right| = \frac{2\ln\left(\frac{1}{n}+1\right)}{\ln n}$

Then we can use: $\forall n \in \mathbb{N}^* : \frac{1}{n} \leq 1$ so that we have $\frac{1}{n} + 1 \leq 2$, which leads to $\frac{2\ln(\frac{1}{n}+1)}{\ln n} \leq \frac{2\ln 2}{\ln n}$. Thus, we can choose $\left|\frac{2\ln(1+n)}{\ln(n)} - 2\right| < \frac{2\ln(\frac{1}{n}+1)}{\ln n} < \varepsilon$, which leads to $n > e^{\frac{2\ln 2}{\ln n}}$. For this, it is sufficient to take $n_\varepsilon = \lceil e^{2\ln(2)/\varepsilon} \rceil + 1$.

(4) $\lim_{n\to+\infty} 3^n = +\infty \iff (\forall A>0, \exists? n_A \in \mathbb{N}, \forall n \in \mathbb{N}; n \geq n_A \Rightarrow 3^n > A)$. We have $3^n > A \iff n > \frac{\ln A}{\ln 3}$. For this, take $n_A = \lceil |\ln(A)/\ln(3)| \rceil + 1$.

(5) $\lim_{n\to+\infty} \frac{-5n^2-2}{4n} = -\infty \iff \forall B<0, \exists? n_B \in \mathbb{N}, \forall n \in \mathbb{N}; n \geq n_B \Rightarrow \frac{-5n^2-2}{4n} < B$.

We have $\frac{-5n^2-2}{4n} < B \iff \frac{5n^2+2}{4n} > -B$. It is obvious that; for all $n \in \mathbb{N}^*$ we have $5n^2+2 > 5n^2$, which leads to $\frac{5n^2+2}{4n} > \frac{5n}{4}$. Thus, for $\frac{5n^2+2}{4n} > -B$ it is sufficient to take $\frac{5n}{4} > -B \iff n > \frac{-4B}{5}$. For this, take $n_B = \lceil -4B/5 \rceil + 1$.

(6) $\lim_{n\to+\infty} \ln(\ln(n)) = +\infty \iff (\forall A>0, \exists? n_A \in \mathbb{N}, \forall n \in \mathbb{N}; n \geq n_A \Rightarrow \ln(\ln(n)) > A)$

For this, take $n_A = \lceil e^{e^A} \rceil + 1$.

Exercise 04:

(v_n) increasing $\iff \forall n > 0;\ v_{n+1} \geq v_n \iff v_{n+1} - v_n \geq 0$. So we have

$$\begin{aligned} v_{n+1} - v_n &= \frac{u_1+u_2+\ldots+u_{n+1}}{n+1} - \frac{u_1+u_2+\ldots+u_n}{n} \\ &= \frac{(nu_1+nu_2+\ldots+nu_n)+nu_{n+1}}{n(n+1)} - \frac{(nu_1+nu_2+\ldots+nu_n)+u_1+u_2+\cdots+u_n}{n(n+1)} \\ &= \frac{-u_1-u_2-\ldots-u_n+nu_{n+1}}{n(n+1)} \\ &= \frac{(u_{n+1}-u_1)+(u_{n+1}-u_2)+\ldots+(u_{n+1}-u_n)}{n(n+1)} \end{aligned}$$

Since the sequence (u_n) is increasing, for all integers $k,\ k=1,2,\ldots,n,\ u_k \leq u_{n+1},$ and thus $v_{n+1} - v_n \geq 0$. Therefore, the sequence (v_n) is increasing.

Exercise 05:

1. Let's assume by contradiction that $(u_n)_{n\in\mathbb{N}}$ converges to two different limits l_1 and l_2 such that $l_1 \neq l_2$. Then we have:

   $(\lim_{n\to+\infty} u_n = l_1) \Rightarrow \big(\forall \varepsilon > 0, \exists n_{\varepsilon_1} \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_{\varepsilon_1} \Rightarrow |u_n - l_1| < \frac{\varepsilon}{2}\big)$

   $(\lim_{n\to+\infty} u_n = l_2) \Rightarrow \big(\forall \varepsilon > 0, \exists n_{\varepsilon_2} \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_{\varepsilon_2} \Rightarrow |u_n - l_2| < \frac{\varepsilon}{2}\big)$

   Now, let $n_{\varepsilon_0} = \max(n_{\varepsilon_1}, n_{\varepsilon_2})$, then for all $n \geq n_{\varepsilon_0}$, we have:

   $$|l_2 - l_1| = |(u_n - l_1) + (l_2 - u_n)| \leq |(u_n - l_1)| + |(u_n - l_2)| < \varepsilon$$

   This leads to $|l_2 - l_1| < \varepsilon$. Regardless of how small the positive number ε, this statement holds true. So, ε must be zero , which contradicts the assumption $l_1 \neq l_2$. Therefore, $l_1 = l_2$, which is absurd.

2.
   - The sequence $(u_n)_{n\in\mathbb{N}}$ is bounded $\iff |u_n| < P;\ \forall n \in \mathbb{N}$, where $P \geq 0$.
   - $\lim_{n\to+\infty} u_n = l \implies \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_\varepsilon \Rightarrow |u_n - l_2| < \varepsilon$.

     Let us take:

     $$|u_n| = |u_n - l + l| \leq |u_n - l| + |l| < \varepsilon + |l|;\ \ \forall n > n_\varepsilon$$

     So it is sufficient to choose $P = \varepsilon + |l|$.

3. $(\lim_{n\to+\infty} u_n = A) \Rightarrow \big(\forall \varepsilon > 0, \exists n_{\varepsilon_1} \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_{\varepsilon_1} \Rightarrow |u_n - A| < \frac{\varepsilon}{2}\big)$

   $(\lim_{n\to+\infty} v_n = B) \Rightarrow \big(\forall \varepsilon > 0, \exists n_{\varepsilon_2} \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_{\varepsilon_2} \Rightarrow |u_n - B| < \frac{\varepsilon}{2}\big)$

   We have:

   $$|(u_n + v_n) - (A+B)| \leq |u_n - A| + |v_n - B| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall n > n_\varepsilon$$

where $n_\varepsilon = \max(n_{\varepsilon_1}, n_{\varepsilon_2})$.

we get:

$$\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_\varepsilon \Rightarrow |(u_n + v_n) - (A + B)| < \varepsilon$$

which is the definition of: $\lim\limits_{n\to+\infty}(u_n + v_n) = A + B$.

4. We have:

$$|u_n \cdot v_n - A \cdot B| = |u_n(v_n - B) + B(u_n - A)| \leq |u_n||v_n - B| + |B||u_n - A| \leq P|v_n - B| + (|B| + 1)|u_n - A| \quad (1)$$

where we use the fact that $|u_n| < P$ because the sequence $(v_n)_{n\in\mathbb{N}}$ is convergent.

- $(\lim_{n\to+\infty} u_n = A) \Rightarrow \left(\forall \varepsilon > 0, \exists n_{\varepsilon_1} \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_{\varepsilon_1} \Rightarrow |u_n - A| < \frac{\varepsilon}{2P}\right)$
- $(\lim_{n\to+\infty} v_n = B) \Rightarrow \left(\forall \varepsilon > 0, \exists n_{\varepsilon_2} \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_{\varepsilon_2} \Rightarrow |u_n - B| < \frac{\varepsilon}{2(|B|+1)}\right)$

So we find: $\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_\varepsilon \Rightarrow |u_n \cdot v_n - A \cdot B| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, where $n_\varepsilon = \max(n_{\varepsilon_1}, n_{\varepsilon_2})$. We get the definition of: $\lim\limits_{n\to+\infty}(u_n \cdot v_n) = A \cdot B$.

5. (a) We need to demonstrate that: $\forall \varepsilon > 0, \exists n_\varepsilon$ such that $\forall n \in \mathbb{N},\ \ n > n_\varepsilon \implies \left|\frac{1}{v_n} - \frac{1}{B}\right| < \varepsilon$.

First, we have

$$\left|\frac{1}{v_n} - \frac{1}{B}\right| = \frac{|v_n - B|}{|v_n||B|}$$

We use:

$$|B| = |B + v_n - v_n| < |B - v_n| + |v_n| \quad (2)$$

From the definition of limit, we have:

$$\left(\lim_{n\to+\infty} v_n = B\right) \Rightarrow (\forall \delta > 0, \exists n_\delta \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_\delta \Rightarrow |v_n - B| < \delta) \quad (3)$$

so, we can choose: $\delta = \frac{|B|}{2}$. Then from (2) and (3) we find: $|v_n| > \frac{|B|}{2}$, and we get $|v_n||B| > \frac{|B|^2}{2} \iff \frac{1}{|v_n||B|} < \frac{2}{|B|^2}$, so

$$\frac{|v_n - B|}{|v_n||B|} < \frac{2|v_n - B|}{|B|^2} \quad (4)$$

also we have the choice to take: $\delta = \frac{\varepsilon|B|^2}{2}$ in the definition (3), which leads to:

$$\left|\frac{1}{v_n} - \frac{1}{B}\right| = \frac{|v_n - B|}{|v_n||B|} < \varepsilon \quad (5)$$

Thus, the proof is concluded.

(b) We have

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} u_n \cdot \frac{1}{v_n} = \lim_{n\to\infty} u_n \cdot \lim_{n\to\infty} \frac{1}{v_n} = A \cdot \frac{1}{B} = \frac{A}{B}$$

Exercise 06:

(a) $\lim\limits_{n\to+\infty} \frac{3n^2-5n}{5n^2+2n-6} = \lim\limits_{n\to+\infty} \frac{3-\frac{5}{n}}{5+\frac{2}{n}-\frac{6}{n^2}} = \frac{3+0}{5+0+0} = \frac{3}{5}$.

(b) $\lim\limits_{n\to+\infty}(\sqrt{n+1} - \sqrt{n}) = \lim\limits_{n\to+\infty}(\sqrt{n+1} - \sqrt{n})\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \lim\limits_{n\to+\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$.

(c) $\lim\limits_{n\to+\infty} \frac{1+2\cdot10^n}{5+3\cdot10^n} = \lim\limits_{n\to+\infty} \frac{(1+2\cdot10^n)\cdot10^{-n}}{(5+3\cdot10^n)\cdot10^{-n}} = \lim\limits_{n\to+\infty} \frac{1\cdot10^{-n}+2}{5\cdot10^{-n}+3} = \frac{2}{3}$

(d) We have:

$$-1 \leq \cos(2n^3 - 5) \leq 1 \tag{6}$$

So, we can write

$$\frac{-1}{3n^3 + 2n^2 + 1} \leq \frac{\cos(2n^3 - 5)}{3n^3 + 2n^2 + 1} \leq \frac{1}{3n^3 + 2n^2 + 1} \tag{7}$$

Thus, we find

$$-\lim_{n\to+\infty} \frac{1}{3n^3 + 2n^2 + 1} \leq \frac{\cos(2n^3 - 5)}{3n^3 + 2n^2 + 1} \leq \lim_{n\to+\infty} \frac{1}{3n^3 + 2n^2 + 1} \tag{8}$$

In fact, we have

$$\lim_{n\to+\infty} \frac{1}{3n^3 + 2n^2 + 1} = 0 \tag{9}$$

Then, we get

$$\lim_{n\to+\infty} \frac{\cos(2n^3 - 5)}{3n^3 + 2n^2 + 1} = 0 \tag{10}$$

(e) $\lim\limits_{n\to+\infty} \frac{e^{2n}-e^n+1}{2e^n+3} = \lim\limits_{n\to+\infty} \frac{(e^{2n}-e^n+1)e^{-2n}}{(2e^n+3)e^{-2n}} = \lim\limits_{n\to+\infty} \frac{1-e^{-n}+e^{-2n}}{2e^{-n}+3e^{-2n}} = +\infty.$

(f)

Exercise 07:

1. (a) For $U_n = \sum_{k=1}^{n} \frac{n}{n^5+k}$; we have For all $k = 1, \ldots, n$ the following inequalities:

$$n^5 + 1 \leq n^5 + k \leq n^5 + n \iff \frac{n}{n^5 + n} \leq \frac{n}{n^5 + k} \leq \frac{n}{n^5 + 1}$$

Then, we can write

$$\sum_{k=1}^{n} \frac{n}{n^5 + n} \leq \sum_{k=1}^{n} \frac{n}{n^5 + k} \leq \sum_{k=1}^{n} \frac{n}{n^5 + 1}$$

which means that

$$\frac{n^2}{n^5 + n} \leq U_n \leq \frac{n^2}{n^5 + 1}$$

As

$$\lim_{n\to\infty} \frac{n^2}{n^5 + n} = \lim_{n\to\infty} \frac{n^2}{n^5 + 1} = 0,$$

we have

$$\lim_{n\to\infty} U_n = 0.$$

(b) For $U_n = \sum_{k=1}^{n} \frac{1}{\sqrt{n^3+k}}$

So, we have for all $k = 1, \ldots, n$:

$$n^3 + 1 \leq n^3 + k \leq n^3 + n \iff \sqrt{n^3 + 1} \leq \sqrt{n^3 + k} \leq \sqrt{n^3 + n}$$

which leads to

$$\frac{1}{\sqrt{n^3 + n}} \leq \frac{1}{\sqrt{n^3 + k}} \leq \frac{1}{\sqrt{n^3 + 1}}$$

Thus, we find

$$\sum_{k=1}^{n} \frac{1}{\sqrt{n^3 + n}} \leq \sum_{k=1}^{n} \frac{1}{\sqrt{n^3 + k}} \leq \sum_{k=1}^{n} \frac{1}{\sqrt{n^3 + 1}}$$

which gives

$$\frac{n}{\sqrt{n^3 + n}} \leq U_n \leq \frac{n}{\sqrt{n^3 + 1}}$$

As

$$\lim_{n\to+\infty} \frac{n}{\sqrt{n^3 + n}} = \lim_{n\to+\infty} \frac{n}{\sqrt{n^3 + 1}} = 0.$$

Then

$$\lim_{n\to+\infty} U_n = 0$$

2. Let $U_n = \sum_{k=1}^{\infty} \frac{1}{2+|\cos k|\sqrt{k}}$.

For all $k = 1, \dots, n$, we have

$$|\cos k| \leq 1$$

$\Leftrightarrow 2 + |\cos k|\sqrt{k} \leq 2 + \sqrt{k} \leq 2 + \sqrt{n}$.

$\Rightarrow 2 + |\cos k|\sqrt{k} \leq 2 + \sqrt{n} \Leftrightarrow \frac{1}{2+\sqrt{n}} \leq \frac{1}{2+|\cos k|\sqrt{k}}$.

$\Rightarrow \sum_{k=1}^{n} \frac{1}{2+\sqrt{n}} \leq \sum_{k=1}^{n} \frac{1}{2+|\cos k|\sqrt{k}} \Leftrightarrow n\left(\frac{1}{2+\sqrt{n}}\right) \leq U_n, \forall n \in \mathbb{N}$.

As $n \to +\infty$, $\sum_{k=1}^{n} \frac{1}{2+\sqrt{n}} = +\infty$, therefore, $\lim_{n\to+\infty} U_n = +\infty$.

Exercise 8

1. By induction:
For $n = 0$: we have $0 \leq U_0 < 2$.
Assume $0 \leq U_n < 2$, then:

$$2 \leq U_n + 2 < 4 \implies \sqrt{2} \leq U_{n+1} < 2 \implies 0 \leq U_{n+1} < 2.$$

Thus,

$$0 \leq U_n < 2, \forall n \in \mathbb{N}.$$

2. Monotonicity of $(U_n)_{n\in\mathbb{N}}$:

$$U_{n+1} - U_n = \sqrt{U_n + 2} - U_n = \frac{U_n + 2 - U_n^2}{\sqrt{U_n + 2} + U_n}.$$

Since $0 \leq U_n < 2, \forall n \in \mathbb{N}$, we have:

$$(U_n + 1)(2 - U_n) > 0.$$

Hence, $(U_n)_{n\in\mathbb{N}}$ is increasing.
3. Define $V_n = 2 - U_n, \forall n \in \mathbb{N}$:

(a) From part 1, $U_n < 2, \forall n \in \mathbb{N}$, so:

$$0 < V_n, \forall n \in \mathbb{N}.$$

(b)

$$\frac{V_{n+1}}{V_n} = \frac{2 - U_{n+1}}{2 - U_n} = \frac{2 - \sqrt{U_n + 2}}{2 - U_n}.$$

Simplify:

$$\frac{V_{n+1}}{V_n} = \frac{(2 - U_n)}{(2 - U_n)(2 + \sqrt{U_n + 2})} = \frac{1}{2 + \sqrt{U_n + 2}}.$$

For all $n \in \mathbb{N}$:

$$0 \leq U_n \implies 2 + \sqrt{2} \leq 2 + \sqrt{U_n + 2} \implies \frac{1}{2 + \sqrt{U_n + 2}} \leq \frac{1}{2}.$$

(c) By induction:
For $n = 1$: $V_1 \leq 1$.
Assume $V_n \leq \left(\frac{1}{2}\right)^{n-1}$, then:

$$V_{n+1} \leq \frac{1}{2}V_n \leq \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^n.$$

Thus:

$$V_n \leq \left(\frac{1}{2}\right)^{n-1}, \forall n \in \mathbb{N}^*.$$

(d) Since:

$$0 < V_n \leq \left(\frac{1}{2}\right)^{n-1}, \forall n \in \mathbb{N}^*,$$

and:

$$\lim_{n \to +\infty} \left(\frac{1}{2}\right)^{n-1} = 0,$$

it follows that:

$$\lim_{n \to +\infty} V_n = 0.$$

Since $U_n = 2 - V_n$, we conclude:

$$\lim_{n \to +\infty} U_n = 2.$$

Exercise 9:

1. By induction:
For $n = 0$: $U_0 > 0$.
Assume $U_n > 0$, then:

$$U_n e^{-U_n} > 0 \implies U_{n+1} > 0, \forall n \in \mathbb{N}.$$

2. Monotonicity of $(U_n)_{n\in\mathbb{N}}$:

$$U_{n+1} - U_n = U_n(e^{-U_n} - 1).$$

Since $U_n > 0, \forall n \in \mathbb{N}$, we have:

$$e^{-U_n} - 1 < 0 \implies U_{n+1} - U_n < 0.$$

Thus, $(U_n)_{n\in\mathbb{N}}$ is decreasing.
3. Convergence of $(U_n)_{n\in\mathbb{N}}$:
As $(U_n)_{n\in\mathbb{N}}$ is bounded below and decreasing, it converges to its infimum.
Let:

$$\lim_{n \to +\infty} U_n = l = \lim_{n \to +\infty} U_{n+1}.$$

Then:

$$U_{n+1} = U_n e^{-U_n} \implies l = le^{-l} \implies l(e^{-l} - 1) = 0 \implies l = 0.$$

Thus:

$$\lim_{n \to +\infty} U_n = 0.$$

4. By induction:
For $n = 0$: $U_1 = e^{-U_0} = e^{-S_0}$.
Assume $U_{n+1} = e^{-S_n}$, and show $U_{n+2} = e^{-S_{n+1}}$:

$$U_{n+2} = U_{n+1}e^{-U_{n+1}} = e^{-S_n}e^{-U_{n+1}} = e^{-S_{n+1}}.$$

Thus:

$$U_{n+1} = e^{-S_n}, \forall n \in \mathbb{N}.$$

5. We have:

$$S_n = -\ln U_{n+1}, \quad \text{and} \quad \lim_{n \to +\infty} U_{n+1} = \lim_{n \to +\infty} U_n = 0.$$

Thus:

$$\lim_{n \to +\infty} S_n = +\infty.$$

Exercise 10

1. By induction:
For $n = 0$: $0 \leq U_0 \leq 2$.
Assume $0 \leq U_n \leq 2$, then:

$$U_n \geq 0 \implies \frac{7U_n + 4}{3U_n + 3} \geq 0,$$

and:

$$U_n \leq 2 \implies 7U_n + 4 \leq 6 + 6U_n \implies \frac{7U_n + 4}{3U_n + 3} \leq 2.$$

Thus:

$$0 \leq U_{n+1} \leq 2.$$

Consequently:

$$0 \leq U_n \leq 2, \forall n \in \mathbb{N}.$$

2. Monotonicity of $(U_n)_{n\in\mathbb{N}}$:

$$U_{n+1} - U_n = \frac{7U_n + 4}{3U_n + 3} - U_n = \frac{-3U_n^2 + 4U_n + 4}{3U_n + 3}.$$

This simplifies to:

$$U_{n+1} - U_n = \frac{(2 + 3U_n)(2 - U_n)}{3U_n + 3}.$$

Since $0 \leq U_n \leq 2, \forall n \in \mathbb{N}$, we have:

$$(2 + 3U_n)(2 - U_n) \geq 0.$$

Moreover, $3U_n + 3 > 0$, so $(U_n)_{n\in\mathbb{N}}$ is increasing.
3. Convergence of $(U_n)_{n\in\mathbb{N}}$:
Since $(U_n)_{n\in\mathbb{N}}$ is bounded and increasing, it converges to its supremum.
Let:

$$\lim_{n\to+\infty} U_n = l = \lim_{n\to+\infty} U_{n+1}.$$

Then:

$$U_{n+1} = \frac{7U_n + 4}{3U_n + 3} \implies l = \frac{7l + 4}{3l + 3}.$$

Simplify:

$$3l^2 - 4l - 4 = 0 \implies (2 + 3l)(l - 2) = 0.$$

Thus:

$$l = 2 \quad \text{or} \quad l = -\frac{2}{3}.$$

Since $U_n \geq 0$, we have $l \geq 0$, so:

$$\lim_{n\to+\infty} U_n = 2.$$

4. Extremes of $E = \{U_n \mid n \in \mathbb{N}\}$:
From above:

$$\sup E = \lim_{n\to+\infty} U_n = 2.$$

Since $(U_n)_{n\in\mathbb{N}}$ is increasing, U_0 is a lower bound for E:

$$U_0 \leq U_n, \forall n \in \mathbb{N}.$$

Moreover, $U_0 \in E$, so:

$$\min E = U_0 = \inf E.$$

Exercise 11

1.

$$W_{n+1} = V_{n+1} - U_{n+1} = \frac{U_{n+3}V_n}{4} - \frac{U_{n+2}V_n}{3} = \frac{1}{12}W_n.$$

Hence, $(W_n)_{n\in\mathbb{N}^*}$ is a geometric sequence with ratio $r = \frac{1}{12}$:

$$W_n = W_1 \left(\frac{1}{12}\right)^{n-1} = \frac{11}{12^{n-1}},$$

and thus:

$$\lim_{n\to+\infty} W_n = 0.$$

2. Monotonicity of $(U_n)_{n\in\mathbb{N}^*}$:

$$U_{n+1} - U_n = \frac{U_n + 2V_n}{3} - U_n = \frac{2}{3}(V_n - U_n) = \frac{2}{3}W_n.$$

Since $W_n > 0, \forall n \in \mathbb{N}^*$, it follows that:

$$U_{n+1} - U_n > 0,$$

hence $(U_n)_{n\in\mathbb{N}^*}$ is increasing.

3. Monotonicity of $(V_n)_{n\in\mathbb{N}^*}$:

$$V_{n+1} - V_n = \frac{U_n + 3V_n}{4} - V_n = -\frac{1}{4}(V_n - U_n) = -\frac{1}{4}W_n.$$

Since $W_n > 0, \forall n \in \mathbb{N}^*$, it follows that:

$$V_{n+1} - V_n < 0,$$

hence $(V_n)_{n\in\mathbb{N}^*}$ is decreasing.

4. Adjacency of $(U_n)_{n\in\mathbb{N}^*}$ and $(V_n)_{n\in\mathbb{N}^*}$:

$$\lim_{n\to+\infty}(V_n - U_n) = \lim_{n\to+\infty} W_n = 0.$$

Thus, $(U_n)_{n\in\mathbb{N}^*}$ and $(V_n)_{n\in\mathbb{N}^*}$ are adjacent sequences.

Exercise 12

1. Cauchy sequence $(U_n)_{n\in\mathbb{N}}$:

$$U_n = \sum_{k=0}^{n} \frac{\sin k}{2^k}, \quad \forall n \in \mathbb{N}^*.$$

The sequence $(U_n)_{n\in\mathbb{N}}$ is a Cauchy sequence if and only if:

$$\forall \epsilon > 0, \exists n_\epsilon \in \mathbb{N}; \forall p, q \in \mathbb{N} : \begin{cases} n_\epsilon \leq p \\ n_\epsilon \leq q \end{cases} \implies |U_p - U_q| < \epsilon.$$

Given $\epsilon > 0$, $p, q \in \mathbb{N}$, assume $p > q$:

$$|U_p - U_q| = \left| \sum_{k=1}^{p} \frac{\sin k}{2^k} - \sum_{k=1}^{q} \frac{\sin k}{2^k} \right| = \left| \sum_{k=q+1}^{p} \frac{\sin k}{2^k} \right|.$$

Since $|\sin k| \leq 1, \forall k$, we have:

$$|U_p - U_q| \leq \sum_{k=q+1}^{p} \frac{1}{2^k}.$$

The sum:

$$\sum_{k=q+1}^{p} \frac{1}{2^k} = \frac{1}{2^{q+1}} + \frac{1}{2^{q+2}} + \cdots + \frac{1}{2^p},$$

is the sum of $p - q$ terms of a geometric sequence with ratio $\frac{1}{2}$. Hence:

$$\sum_{k=q+1}^{p} \frac{1}{2^k} = \frac{1}{2^{q+1}} \left(\frac{1 - \frac{1}{2^{p-q}}}{1 - \frac{1}{2}} \right) = \frac{1}{2^q} \left(1 - \frac{1}{2^{p-q}} \right).$$

Thus:

$$\sum_{k=q+1}^{p} \frac{1}{2^k} \leq \frac{1}{2^q}.$$

Therefore:

$$|U_p - U_q| \leq \frac{1}{2^q}.$$

To ensure $|U_p - U_q| < \epsilon$, it suffices that:

$$\frac{1}{2^q} < \epsilon \implies q > \frac{-\ln \epsilon}{\ln 2}.$$

Thus, take $n_\epsilon = \left\lceil \frac{-\ln \epsilon}{\ln 2} \right\rceil + 1$.
2. Non-Cauchy sequence $(V_n)_{n \in \mathbb{N}}$:

$$V_n = \sum_{k=2}^{n} \frac{1}{\ln k}, \quad \forall n \geq 2.$$

The sequence $(V_n)_{n \in \mathbb{N}}$ is not a Cauchy sequence if:

$$\exists \epsilon > 0, \forall n \in \mathbb{N}; \exists p, q \in \mathbb{N} : \begin{cases} n \leq p \\ n \leq q \end{cases} \wedge |V_p - V_q| \geq \epsilon.$$

Let $q = n$ and $p = 2n$:

$$|V_p - V_q| = |V_{2n} - V_n| = \sum_{k=n+1}^{2n} \frac{1}{\ln k}.$$

Since $\ln k < k, \forall k > 0$, we have:

$$\sum_{k=n+1}^{2n} \frac{1}{\ln k} > \sum_{k=n+1}^{2n} \frac{1}{k}.$$

For k such that $2 \leq k \leq 2n$:

$$\frac{1}{k} \geq \frac{1}{2n},$$

so:

$$\sum_{k=n+1}^{2n} \frac{1}{k} \geq \sum_{k=n+1}^{2n} \frac{1}{2n}.$$

Finally:

$$\sum_{k=n+1}^{2n} \frac{1}{2n} = \frac{1}{2},$$

hence:

$$|V_p - V_q| > \frac{1}{2}.$$

Take $\epsilon = \frac{1}{2}$.