Chapter 4

Work and energy

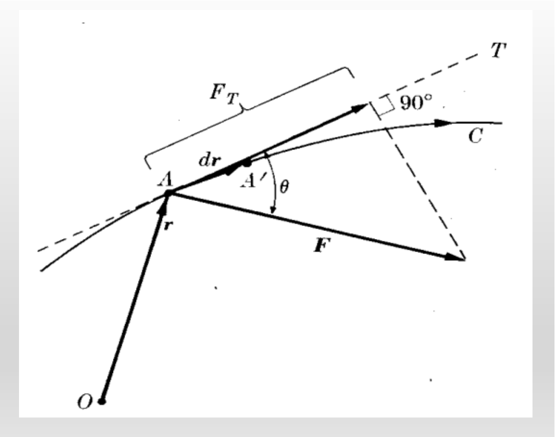
4.1 Introduction

Newton's laws (dynamics) allowed to predict the evolution of the system over time by knowing the forces applied to the particle as well as the initial position and speed. However, in practice it is not always possible to know all the forces affecting the system. In addition, we sometimes obtain a large number of equations that are not easy to solve. The notions of work and energy offer the possibility to study a complex system on easy way.

4.2 Work

4. 2. 1. Elementary work

Suppose a particle is moved from one point A to another point B very close to A under the effect of a force f. The displacement of the particle is $d\vec{r}$



The work done by this force is given by

$$dw = \vec{f} d\vec{r}$$

Work is a scalar quantity; its sign depends on the angle between the force and the displacement of the particle.

$$dw = \|\vec{f}\|\|d\vec{r}\|\cos(\theta)$$

- $0 \leq \theta < 90^{\circ} \rightarrow dw > 0$, the work is driven.
- $90^{\circ} < \theta < 180^{\circ} \rightarrow dw < 0$, the work is resistant.
- $\theta = 90^{\circ} \rightarrow dw = 0$, the force perpendicular to the path does not work.

❖ **Unit of work**

Force times displacement gives an energy. In the SI the unit of force is Newton (N) and the displacement unit is the meter (m). The product of N and m gives the Joule (J). So,

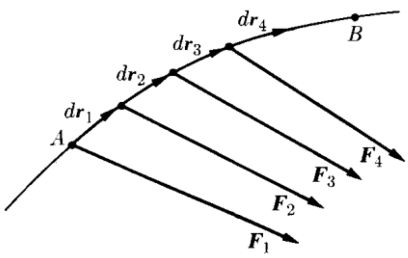
$$[w] = Joul$$

In physics, work is the energy transferred to or from an object via the application of force along a displacement.

4. 2 .2. Work along a path

The work done by a force along a path (from an initial point to the final point) is the sum of all infinitesimal works done by this force. We may represent this work by the integration,

$$w_{ab} = \int_{\vec{r}_a}^{\vec{r}_b} \vec{f} d\vec{r}$$



- If the force is constant along the path,

$$w_{ab} = \vec{f} \int_{\vec{r}_a}^{\vec{r}_b} d\vec{r} = \vec{f} \cdot (\vec{r}_b - \vec{r}_a)$$

$$w_{ab} = \vec{f} \cdot \overrightarrow{AB} = \|\vec{f}\|\|\overrightarrow{AB}\|\cos(\theta)$$

- $\|\vec{f}\|\cos(\theta): represent\ the\ tangent\ component\ of\ the\ force$

In words, we may express this by saying that

Work is equal to the displacement times the tangent component of the force.

- In the special case where $\theta = 90 \rightarrow \cos(\theta) = 0$

$$w_{ab} = \|\vec{f}\|\|\overrightarrow{AB}\|$$

- We arrive to the simplest (common) definition of the work,

Work is equal to the product of displacement and force.

4 .3 Power

In practical applications, especially in connection with machines and engineering, it is important to know the rate at which work is done. The instantaneous power is defined by

$$P = \lim_{\Delta t \to 0} \frac{dw}{dt} = \lim_{\Delta t \to 0} \frac{\vec{f} d\vec{r}}{dt}$$

<u>Power</u> is defined as the work done during a very small time interval dt,

$$P = \lim_{\Delta t \to 0} \frac{\vec{f} d\vec{r}}{dt}$$

We have,

$$\frac{d\vec{r}}{dt} = \vec{v}.$$

$$P = \vec{f}\vec{v}$$

Thus, power can be defined as the product of force velocity.

- **Unit of power**

The unity of power is the watt ($\underline{watt = Nms^{-1} = Js^{-1}}$)

The idea of power is crucial to engineering because, in machine design, the rate of operation is more significant to an engineer than the entire quantity of work the machine can accomplish.

4. 4. Kinetic Energy

From the definition of work and the definition of force (second newton's law).

$$\begin{cases} dw = \vec{f} d\vec{r} \\ \vec{f} = m\dfrac{d\vec{v}}{dt} \end{cases} \rightarrow \quad dw = m\frac{d\vec{v}}{dt} d\vec{r}$$

$$dw = md\vec{v}\frac{d\vec{r}}{dt} \quad \rightarrow \quad dw = m\vec{v} \cdot d\vec{v}$$

$$dw = \frac{1}{2}md(\vec{v} \cdot \vec{v}) \quad \rightarrow dw = \frac{1}{2}md(v^2) \quad \rightarrow \quad dw = d(\frac{1}{2}mv^2)$$

We know that $\frac{1}{2}mv^2$ refers to the kinetic energy E_k

$$dw = dE_k$$

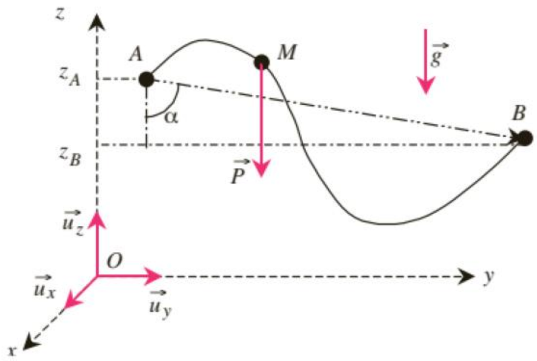
Therefore

$$w_{ab} = \int_a^b dE_k \;\rightarrow\; w_{ab} = E_k^b - E_k^a$$

The work done by or on a particle is equal to the change in its kinetic energy. This result is valid in general, no matter what the nature of the force.

4. 5 Potential Energy

Consider a body moves only under the effect of its weight along a path from the point A to the point B.



In Cartesian coordinates $d\vec{r} = dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k}$

$$\vec{f} = f_x\vec{\imath} + f_y\vec{\jmath} + f_z\vec{k},$$

$$\vec{f} = \vec{p} = -mg\vec{k}$$

$$w_{ab} = \int_{\vec{r}_a}^{\vec{r}_b} (-mg\vec{k}) \cdot (dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k})$$

$$w_{ab} = -mg\vec{k}\int_{\vec{r}_a}^{\vec{r}_b} dz\vec{k} = -mg(z_b - z_a)$$

- *Work of constant force (force that does not depend on the path) depends only on a quantity, which is function of the position.*

This example is one of important class of forces, which are called ***conservative forces***.

We may write, the last equation, as

$$w_{ab} = -mg(z_b - z_a) = mgz_a - mgz_b$$

In other words, we may say that the work of a conservative force depends only on a quantity that characterizes the position. This quantity is called potential energy E_p.

$$w_{conservative\ force} = -(E_p^B - E_p^A) = -\Delta E_p$$

Potential energy is a function of the coordinates such that the difference between its value at the initial and final positions is equal to the work done on the particle to move it from the initial to the final position.

4. 6. Conservation of Energy of a particle (Mechanical energy)

From the definition of work,

- For the general case,

$$w_{ab} = E_k^b - E_k^a$$

- For the conservative force

$$w_{ab} = E_p^a - E_p^b$$

When the force acting on a particle is conservative, we may combine and write,

$$E_k^b - E_k^a = E_p^a - E_p^b$$

$$E_k^b + E_p^b = E_k^a + E_p^a$$

The sum of kinetic energy and potential energy is called the mechanical energy of the particle. Then,

$$(E_k + E_p)_A = (E_k + E_p)_B$$

$$E_A^M = E_B^M$$

- When the forces are conservative, the mechanical energy E^M of the particle remains constant.

4. 5. None conservative Force

At first sight, we find some forces in nature that are not conservative. One example is friction. Sliding friction always opposes the displacement. Its work will depend on the path followed, even on a closed path the work is not zero, so that the total energy it does not conserved. Similarly, fluid friction opposes the velocity, and depends on velocity but not on position. A particle may thus be subject to conservative and to non-conservative forces at the same time.

If the particle is the subject conservative forces ($\vec{F}$), and non-conservative forces ($\vec{F}_f$), the total work will be given as,

$$w_{ab} = \int_{\vec{r}_a}^{\vec{r}_b} \vec{f} d\vec{r} + \int_{\vec{r}_a}^{\vec{r}_b} \vec{f}_f d\vec{r} = E_k^b - E_k^a$$

$$w_{ab} = w_{ab}^{f} + w_{ab}^{f_f} = E_k^b - E_k^a$$

We have, $w_{ab}^{f} = (E_p^a - E_p^b)$,

$$w_{ab} = (E_p^a - E_p^b) + w_{ab}^{f_f} = E_k^b - E_k^a$$

$$(E_p^a - E_p^b) + w_{ab}^{f_f} = E_k^b - E_k^a$$

$$w_{ab}^{f_f} = \left(E_k^b - E_k^a\right) - \left(E_p^a - E_p^b\right)$$

$$w_{ab}^{f_f} = (E_k^b + E_p^b) - (E_k^a + E_p^a)$$

$$w_{ab}^{f_f} = E_b^M - E_a^M$$

The variation of the total energy represents the work done by the non-conservative forces. This energy is dissipated in the surrounding as heat.