Planar Kenetics of a Rigid Body

Force and Acceleration

Work and Energy

Impulse and Momentum

Moment of Inertia

Definition

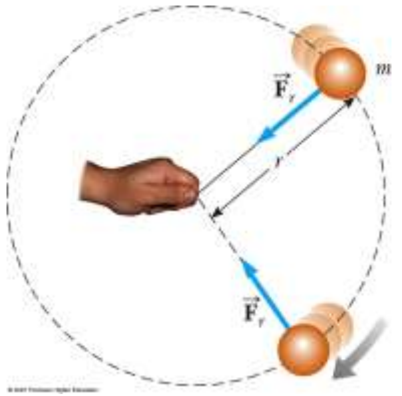
Measure of resistance of an object to changes in its rotational motion.
Equivalent to mass in linear motion.

For a single particle, the definition of moment of inertia is

m is the mass of the single particle

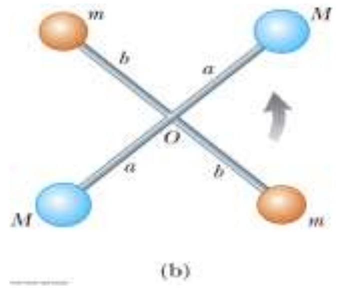
r is the rotational radius

$$I = mr^2$$



For a composite particle, the definition of moment of inertia is

$$I = \sum m_i r_i^{\ 2} = m_1 r_1^{\ 2} + m_2 r_2^{\ 2} + m_3 r_3^{\ 2} + m_4 r_4^{\ 2} + \ldots$$



m_i is the mass of the i_{th} single particle

r_i is the rotational radius of i_{th} particle

SI units of moment of inertia are kg·m^2

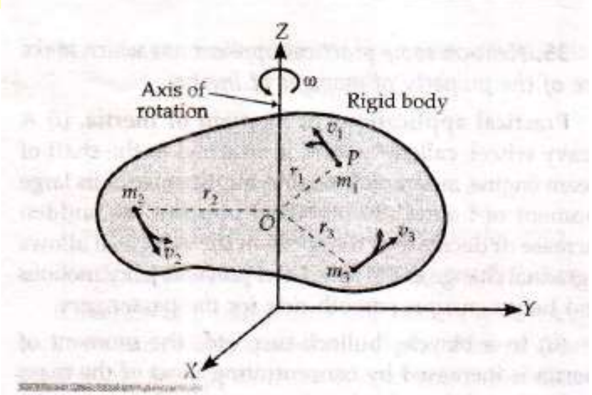


Fig. 7.36 M.I. and rotational K.E. of a rigid body.

Example 1

Three mass point m1,m2 and m3 are located at the vertices of an equilateral triangle a. What is the moment inertia of the system about an axis along the altitude of the triangle passing through m1.

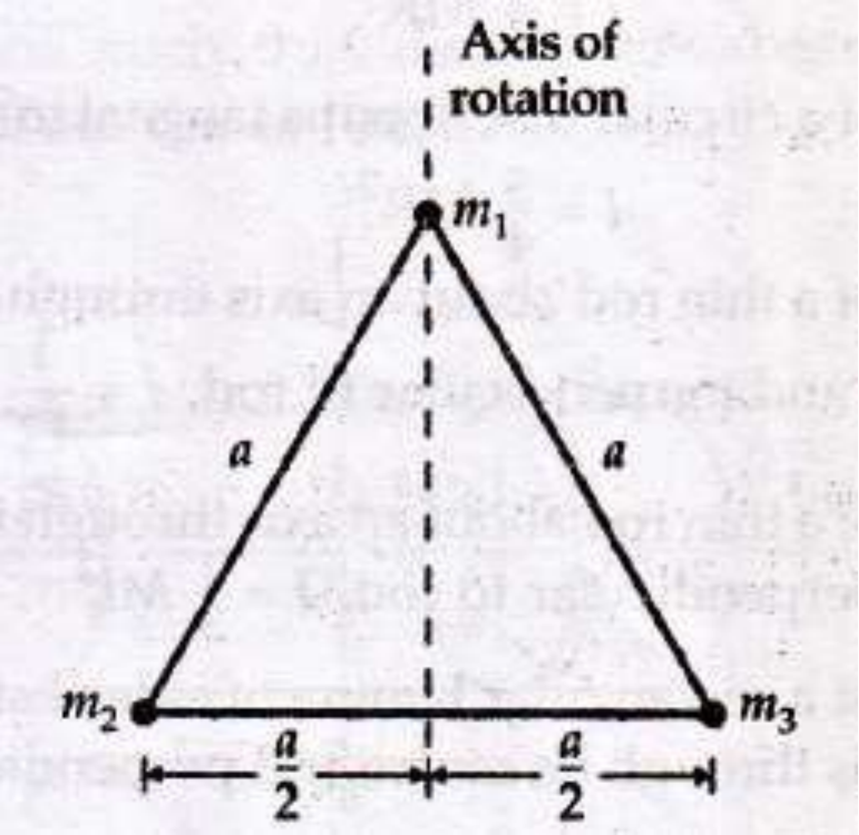
Solution. As shown in Fig. 7.54, the axis of rotation passes through m_1. The distances of m_1, m_2 and m_3 from the axis of rotation are 0, $a/2$ and $a/2$ respectively.

$\therefore$ M.I. of the system about the altitude through m_1 is

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2$$

$$= m_1 (0)^2 + m_2 \left(\frac{a}{2}\right)^2 + m_3 \left(\frac{a}{2}\right)^2$$

or $$I = \frac{a^2}{4}(m_2 + m_3).$$



Calculation of Moments of Inertia

Moments of inertia for large objects can be computed, if we assume the object consists of small volume elements with mass, Δm_i.

The moment of inertia for the large rigid object is $I = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i = \int r^2 dm$

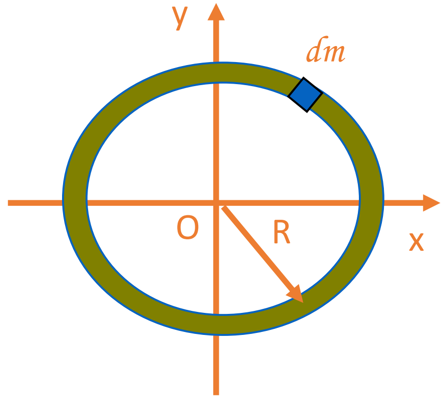
It is sometimes easier to compute moments of inertia in terms of volume of the elements rather than their mass

$$\rho = \frac{dm}{dV} \qquad dm = \rho dV$$

The moments of inertia becomes $$I = \int \rho r^2 dV$$

Example 2:

Find the moment of inertia of a uniform hoop of mass ***M*** and radius ***R*** about an axis perpendicular to the plane of the hoop and passing through its center.



The moment of inertia is $$I = \int r^2 dm = R^2 \int dm = MR^2$$

Moment of Inertia of a Uniform Rigid Rod

The shaded area has a mass

$dm = \lambda\, dx$

Then the moment of inertia is

$$I_y = \int r^2 dm = \int_{-L/2}^{L/2} x^2 \frac{M}{L}\, dx$$

$$I = \frac{1}{12} ML^2$$

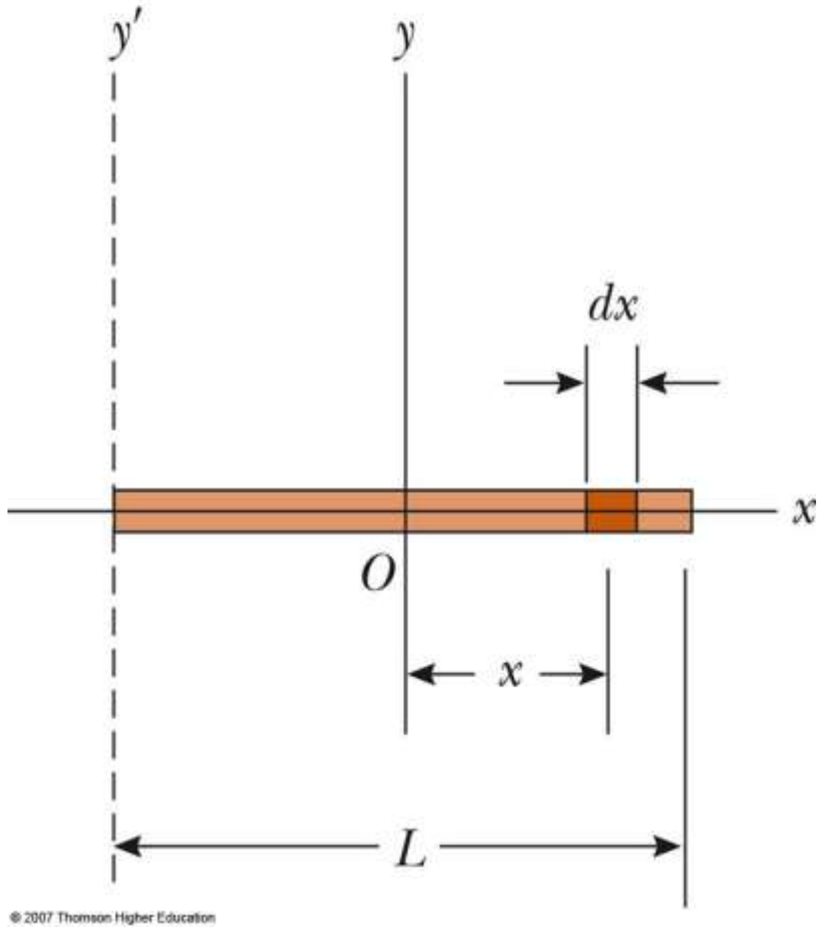
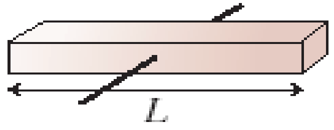
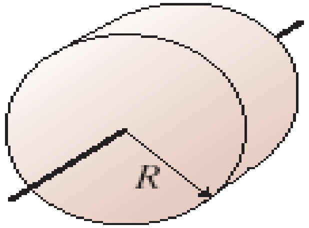
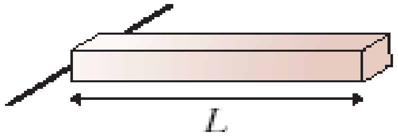
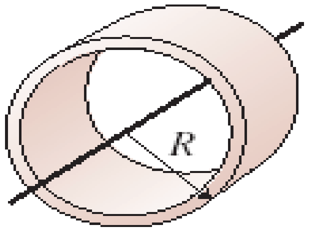
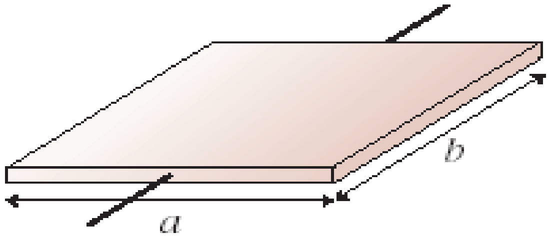
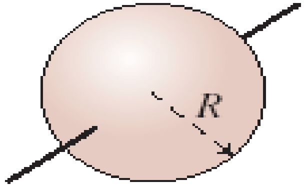
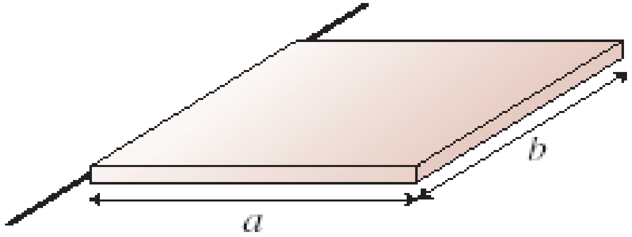
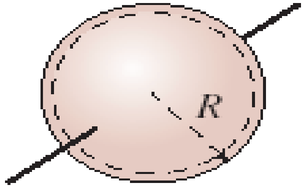


TABLE 12.2 Moments of inertia of objects with uniform density

Object and axis	Picture	I	Object and axis	Picture	I
Thin rod, about center		$\frac{1}{12}ML^2$	Cylinder or disk, about center		$\frac{1}{2}MR^2$
Thin rod, about end		$\frac{1}{3}ML^2$	Cylindrical hoop, about center		MR^2
Plane or slab, about center		$\frac{1}{12}Ma^2$	Solid sphere, about diameter		$\frac{2}{5}MR^2$
Plane or slab, about edge		$\frac{1}{3}Ma^2$	Spherical shell, about diameter		$\frac{2}{3}MR^2$

Center of Mass

The centre of mass (***CM***) is the point where the mass-weighted position vectors (moments) relative to the point sum to zero ; the ***CM*** is the mean location of a distribution of mass in space.

Take a system of ***n*** particles, each with mass ***mi*** located at positions *ri* , the position vector of the ***CM*** is defined by:

$$\sum_{i=1}^{n} m_i(\underline{\mathbf{r}}_\mathbf{i} - \underline{\mathbf{r}}_\mathbf{cm}) = 0$$

Solve for $\underline{\mathbf{r}}_\mathbf{cm}$:

$$\underline{\mathbf{r}}_\mathbf{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i\,\underline{\mathbf{r}}_\mathbf{i}$$

where $M = \sum_{i=1}^{n} m_i$

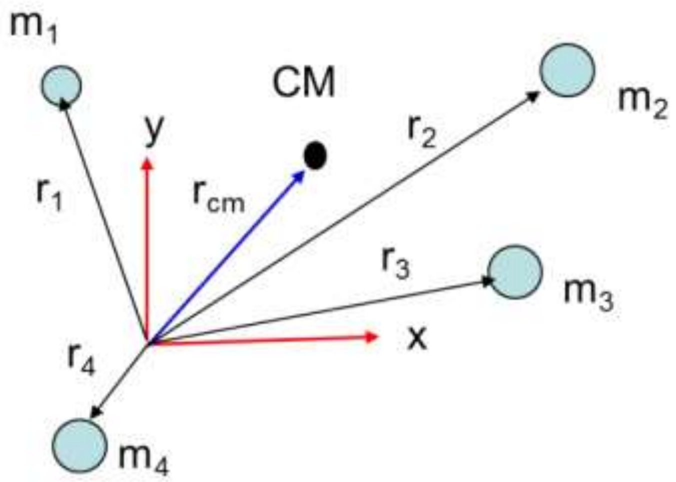
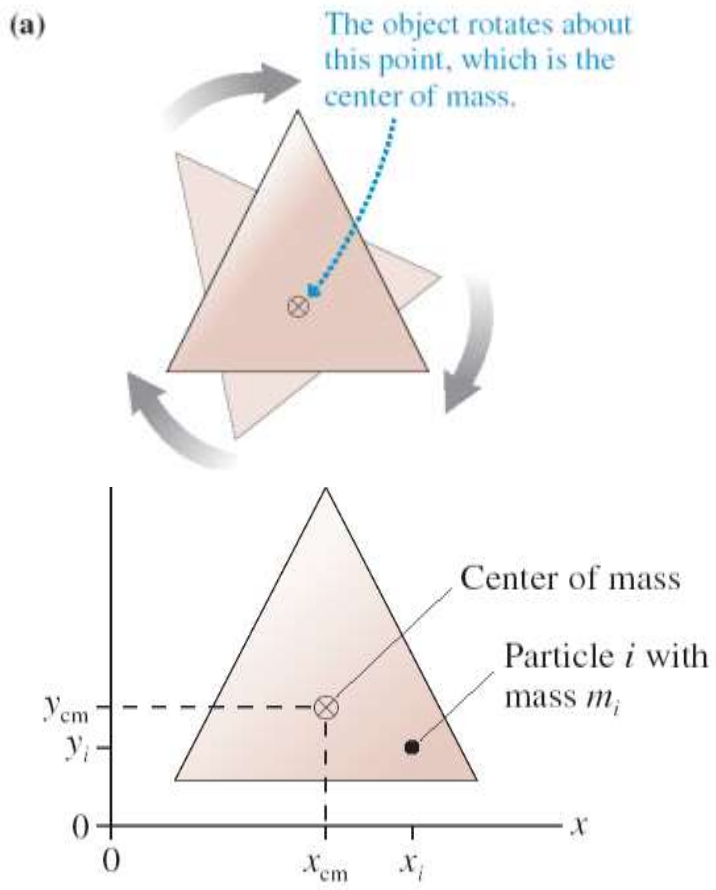


FIGURE 12.5 Rotation about the center of mass.

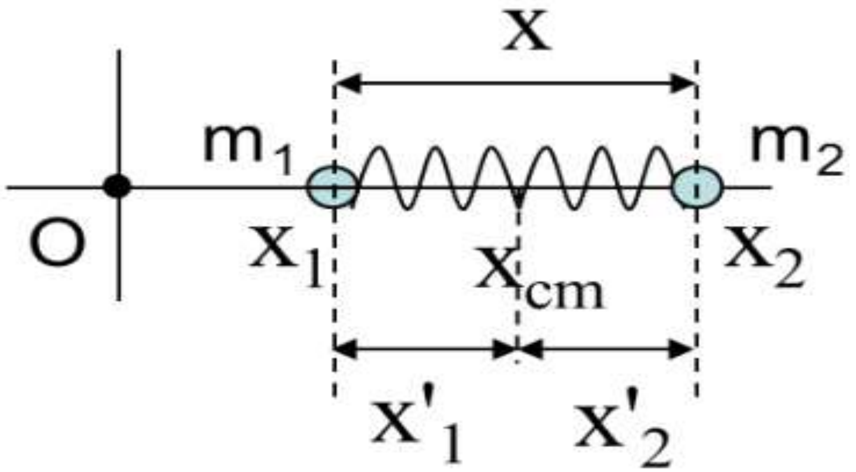


Example : SHM of two connected masses in 1D

SHM between two masses m_1 and m_2 connected by a spring

- $x = x_2 - x_1$; Natural length L
- $F_{int} = -k\,(x - L) = \mu\,\ddot{x}$

  ($\mu = \frac{m_1 m_2}{m_1 + m_2}$ = reduced mass)
- $\ddot{x} + \frac{k}{\mu}(x - L) = 0$



Solution: $x = x_0 \cos(\omega t + \phi) + L$

where $\omega = \sqrt{\frac{k}{\mu}}$

With respect to the CM:

- $x_{CM} = \frac{m_1 x_1 + m_2 x_2}{M}$ where $M = m_1 + m_2$
- $x_1' = x_1 - x_{cm} = \frac{Mx_1 - m_1 x_1 - m_2 x_2}{M} = -\frac{m_2 x}{M}$
- $x_2' = x_2 - x_{cm} = \frac{Mx_2 - m_1 x_1 - m_2 x_2}{M} = \frac{m_1 x}{M}$

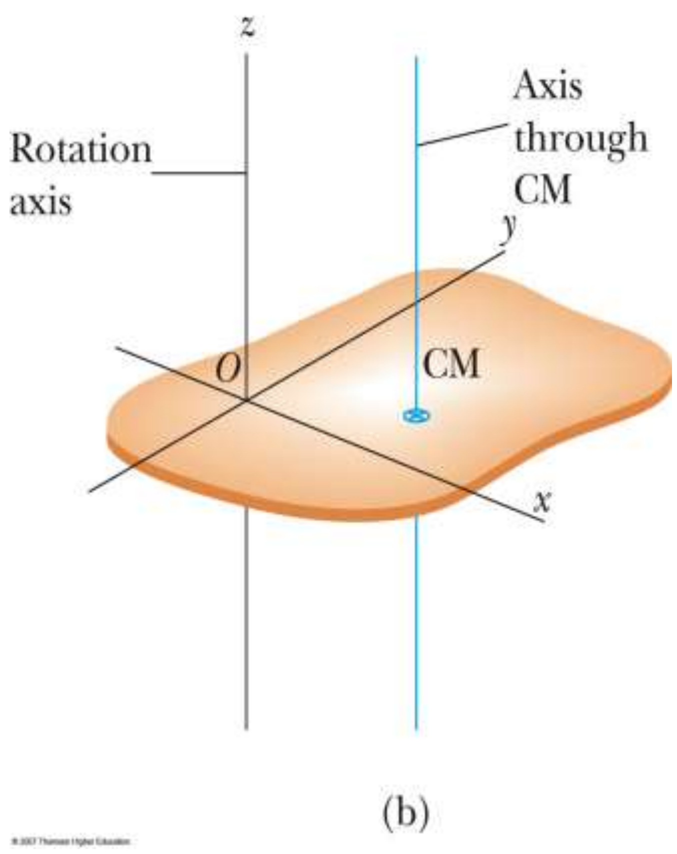
Eg. take $m_1 = m_2 = m \;\rightarrow\; \omega = \sqrt{\frac{2k}{m}}$; $x_1' = -\frac{1}{2}x,\; x_2' = \frac{1}{2}x$

Parallel-Axis Theorem

For an arbitrary axis, the parallel-axis theorem often simplifies calculations

The theorem states

$$I = I_{CM} + MD^2$$



I is about any axis parallel to the axis through the center of mass of the object

I_{CM} is about the axis through the center of mass

D is the distance from the center of mass axis to the arbitrary axis

Moment of Inertia of a Uniform Rigid Rod

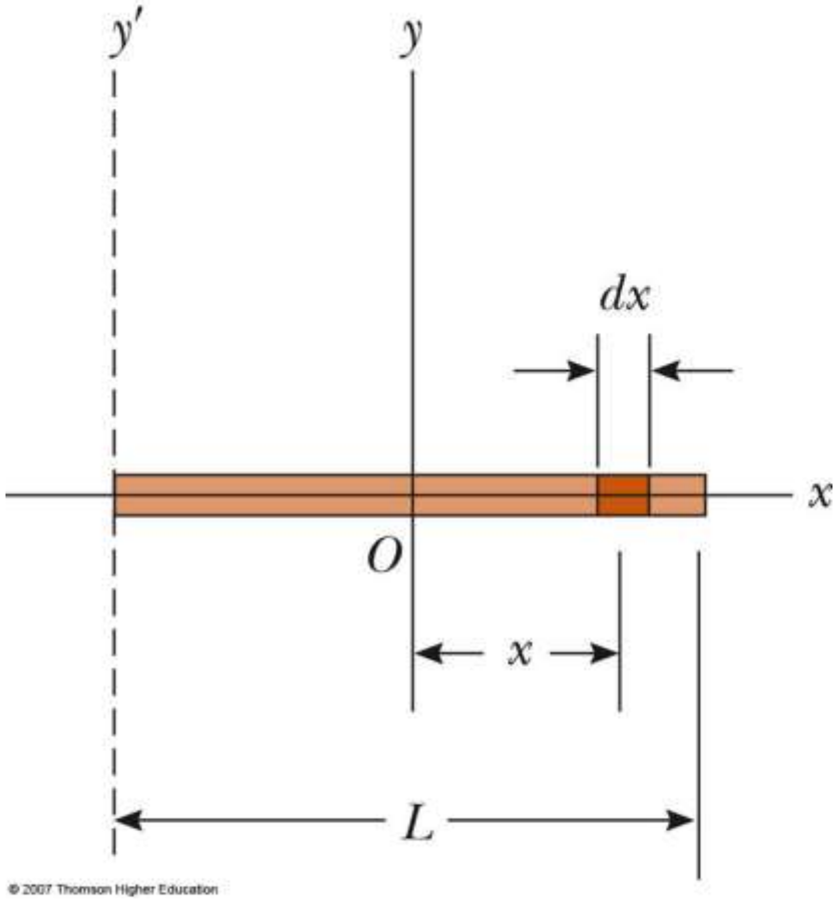
The moment of inertia about ***y*** is

$$I_{CM} = \int r^2 dm = \int_{-L/2}^{L/2} \frac{x^2 M}{L} dx = \frac{M}{L}\left[\frac{1}{3}x^3\right]_{-L/2}^{L/2}$$

$$= \frac{M}{3L}\left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right] = \frac{M}{3L}\left(\frac{L^3}{4}\right) = \frac{ML^2}{12}$$

The moment of inertia about ***y'*** is

$$I_{y'} = I_{CM} + MD^2 = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}ML^2$$



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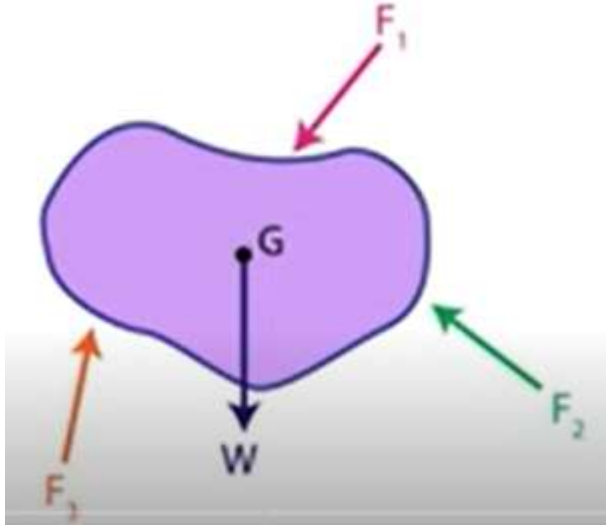
Planar Kinetic equations of motion: Equation of translational motion

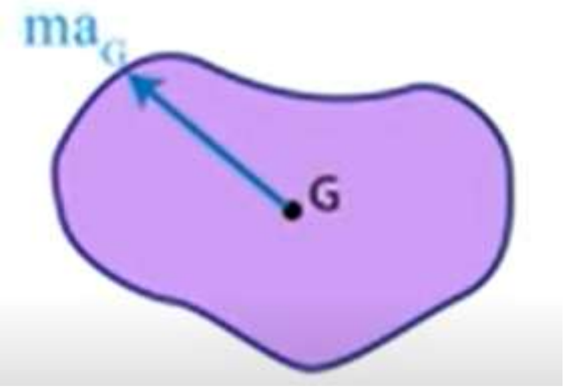
A-Rectilinear motion

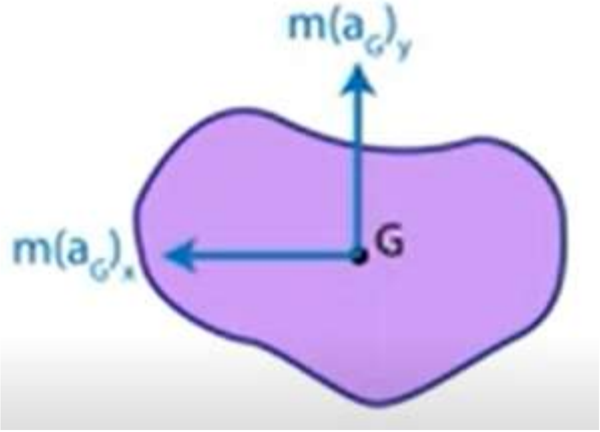
If a body undergoes translational motion, the equation of motion is $\Sigma \boldsymbol{F} = m\,\boldsymbol{a}_G$

The sum of all the external forces acting on the body is equal to the body′s mass times the acceleration of its mass center.

The scalar form as: $\Sigma F_x = m(a_G)_x$ and $\Sigma F_y = m(a_G)_y$ $\Sigma M_G = 0$







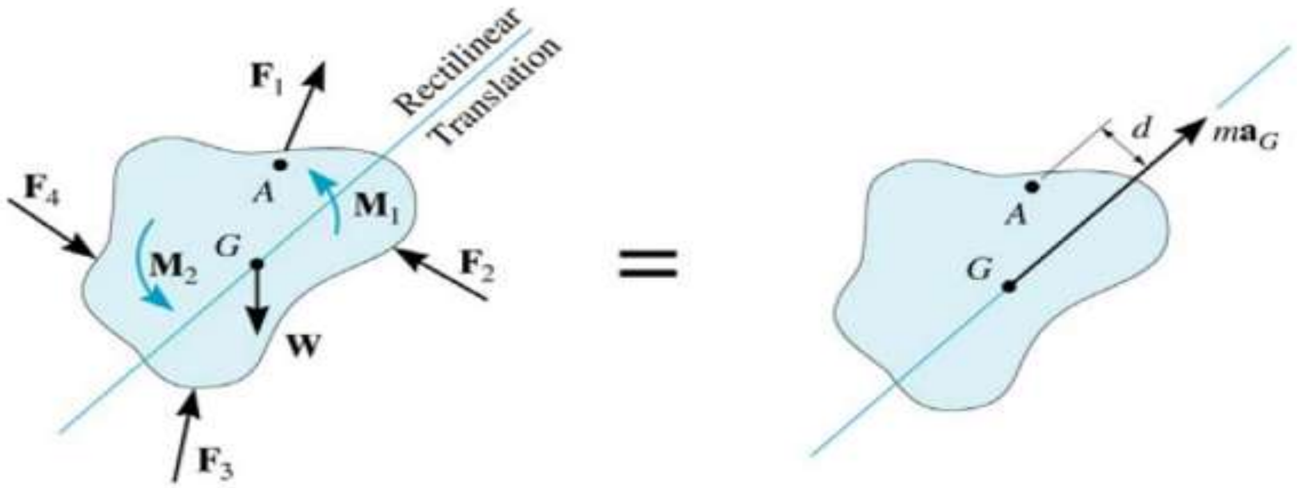
Equation of motion Translation

For ***a planar rectilinear translation***, all the particles of the body have the same acceleration so $a_G = a$ and $\alpha = 0$. The equation of motion are as follows.

$$\Sigma F_x = m(a_G)_x$$

$$\Sigma F_y = m(a_G)_y$$

$$\Sigma M_G = 0$$



The moment equation can be applied about other points instead of the mass center. In this case.

$$\Sigma M_A = r_G \times m a_G = (m\, a_G)\, d\,.$$

B-Curvilinear motion

Similarly, for ***a planar curvilinear translation***, all the particles of the body have the same accelerati $\Sigma \boldsymbol{F} = m\,\boldsymbol{a}_G$ equation of motion are as follows, for n-t coordinates.

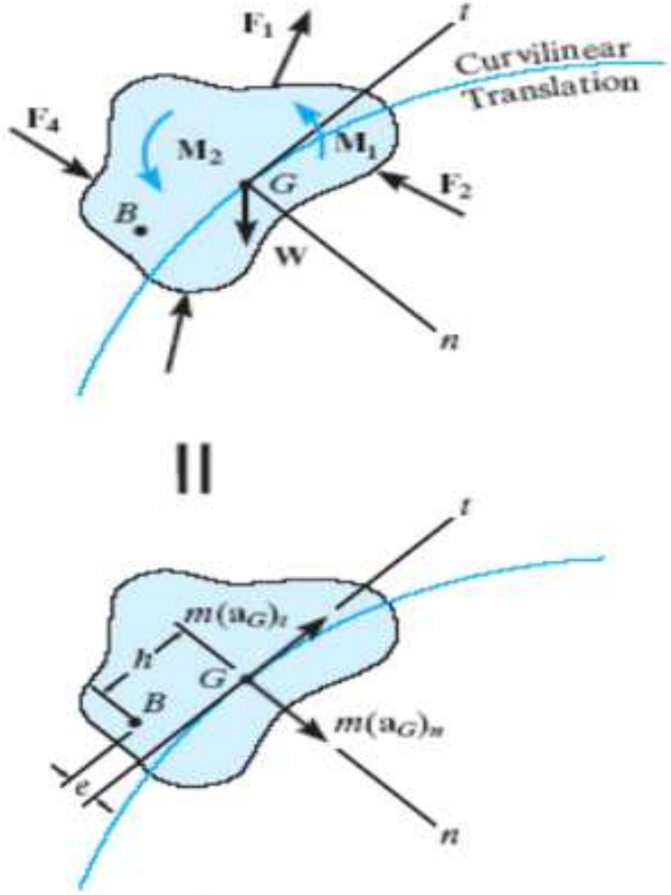
$$\Sigma F_n = m(a_G)_n \qquad \Sigma F_n = m\,(a_G)_n = m\,r_G\,\omega^2$$

$$\Sigma F_t = m\,(a_G)_t = m\,r_G\,\alpha$$

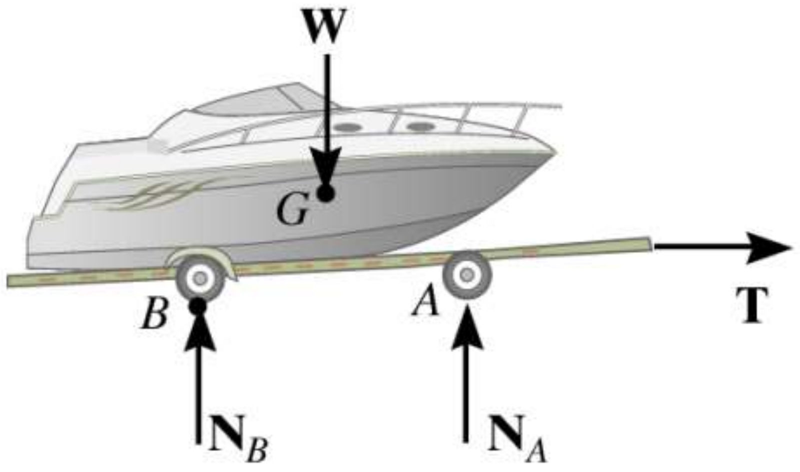
$$\Sigma F_t = m(a_G)_t$$

$$\Sigma M_G = 0 \quad \text{or}$$

$$\Sigma M_B = \boldsymbol{r}_G \times m\boldsymbol{a}_G = e[m(a_G)_t] - h[m(a_G)_n]$$



The boat and trailer undergo rectilinear motion. In order to find the reactions at the trailer wheels and the acceleration of the boat at its center of mass, we need to draw the FBD for the boat and trailer.



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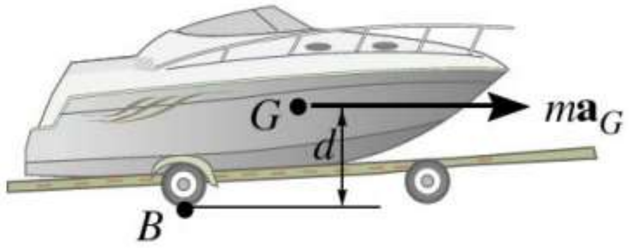
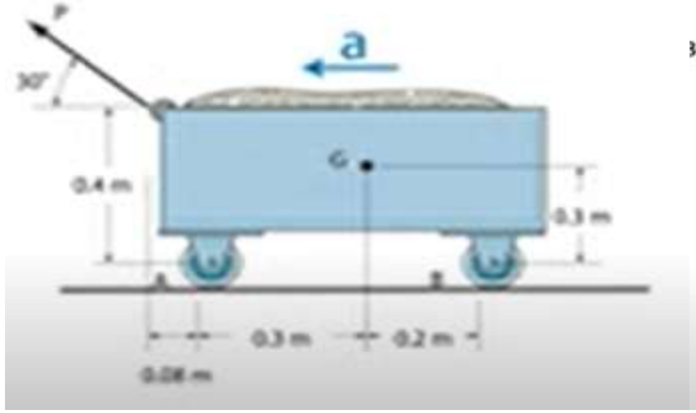


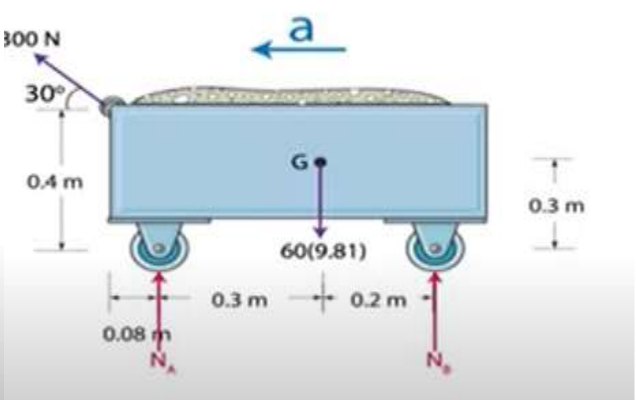
Free body diagram

Kenetic diagram

Example 3

A force of ***P= 300 N*** is applied to the ***60 kg*** cart. Determine **the reactions** at both the wheels at ***A*** and the wheels at ***B***. Also, what is the acceleration of the cart ? The mass center of the cart is at G.





$$\overset{+}{\leftarrow} \sum F_x = m(a_G)_x$$

$$300 \cos 30° = 60a$$

$$a = 4.33 \text{ m/s}^2$$

$$+\uparrow \sum F_y = m(a_G)_y$$

$$N_A + N_B + 300 \sin 30° - 60(9.81) = 60(0)$$

$$\circlearrowleft^{+} \sum M_G = 0$$

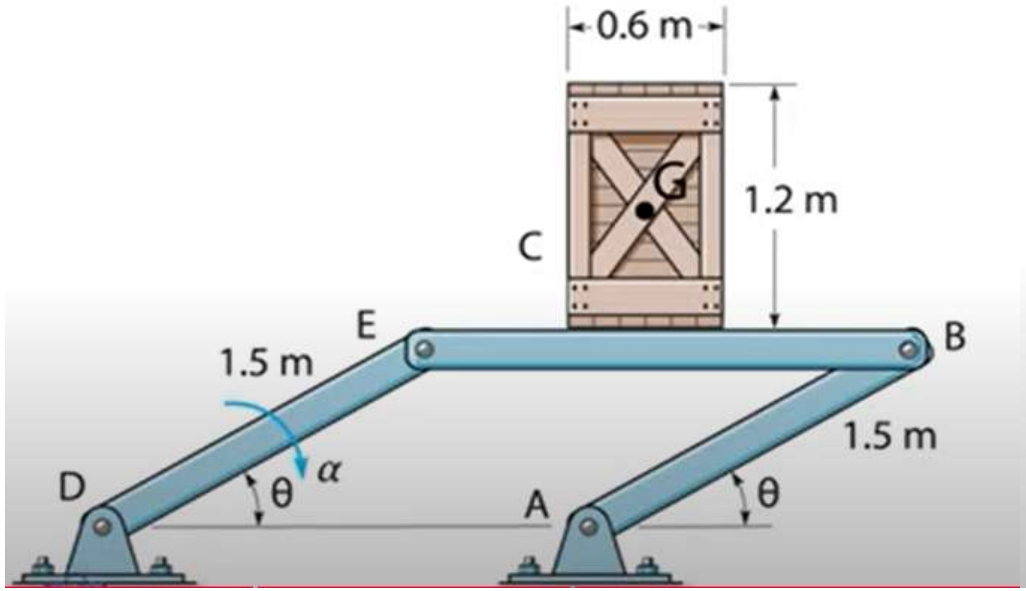
$$-N_A(0.3) + N_B(0.2) - 300 \sin 30° \,(0.38) + 300 \cos 30° \,(0.1) = 0$$

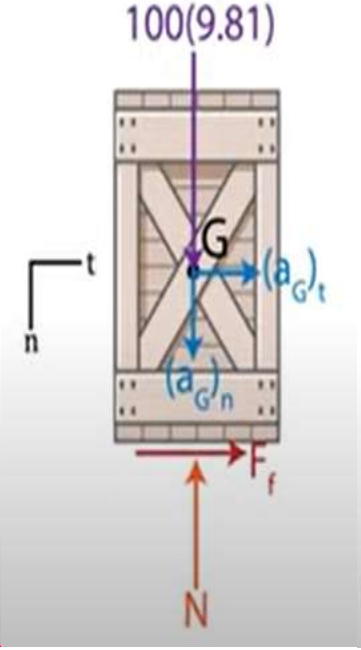
$$N_A = 113.4 \text{ N}$$

$$N_B = 325.2 \text{ N}$$

Example 4

The **100 kg** uniform crate ***C*** rests on the elevator floor where the coefficient of statics friction is ***μ_s= 0.4***. **Determine** the largest initial angular acceleration α, starting from rest at ***θ=90°***, without causing the crate to slip. No tipping occurs.





$$(a_G)_n = 0$$

$$+\downarrow \sum F_n = m(a_G)_n$$

$$100(9.81) - N = 100(0)$$

$$N = 981\ \text{N}$$

$$\omega = 0\ \text{rad/s}$$

$$(a_G)_n = \omega^2 r$$

$$(a_G)_n = 0$$

$$(a_G)_t = \alpha(1.5)$$

$$\xrightarrow{+} \sum F_t = m(a_G)_t$$

$$(0.4)(981) = 100(1.5\alpha)$$

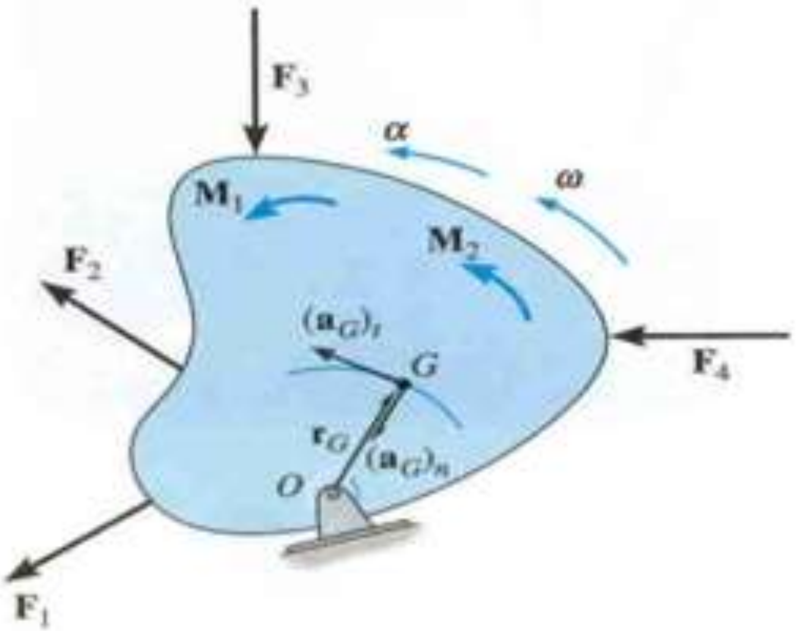
$$(a_G)_t = \alpha r$$

~~$\alpha = 2.616\ \text{rad/s}^2$~~

Equation of motion for pure rotation

When a rigid body rotates about a fixed axis perpendicular to the plane of the body at point a O, the body,s center of *gravity G* moves in a circular path of radius *r_G*. Thus, the acceleration of point G can be represented by *a tangential component* $(a_G)_t = r_G \alpha$ and *a nomal component* $(a_G)_n = r_G \omega^2$. The *scalar equations* of motion can be states as:

$$\sum F_n = m(a_G)_n = m r_G \omega^2$$
$$\sum F_t = m(a_G)_t = m r_G \alpha$$
$$\sum M_G = I_G \alpha$$



Equation rotational motion

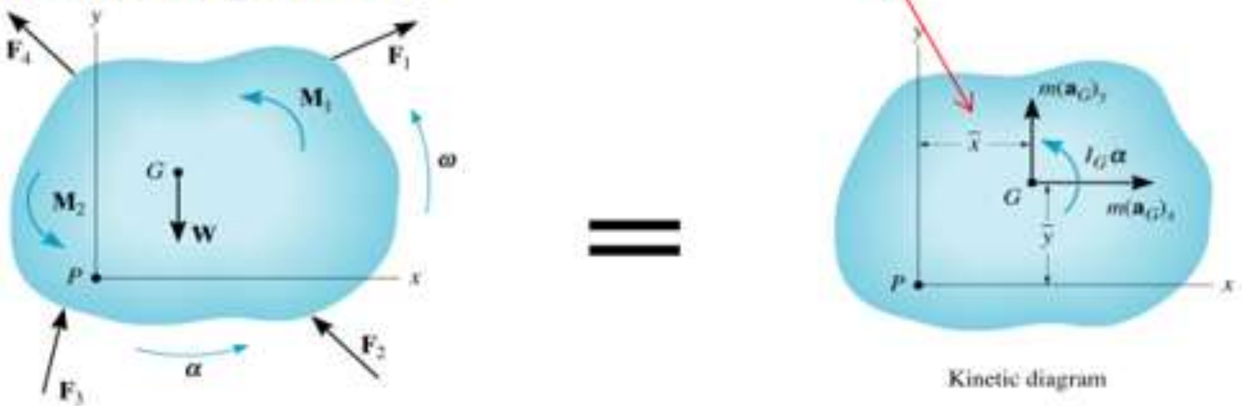
We need to determine the effects caused by the moments of the external force system. The moment about point P can be written as

$$\Sigma (\boldsymbol{r}_i \times \boldsymbol{F}_i) + \Sigma \boldsymbol{M}_i = \boldsymbol{r}_G \times m\boldsymbol{a}_G + I_G\boldsymbol{\alpha}$$

$$\Sigma M_p = \Sigma(M_k)_p$$

Wont have this equation if it's particle

where ΣM_p is the resultant moment about P due to all the external forces. The term $\Sigma(\mathbf{M}_k)_\mathbf{p}$ is called the kinetic moment about point P.



Example 5

The uniform 24 kg plate is release from rest shown. **Determine** its initial angular acceleration and the horizontal acceleration and the horizontal and vertical reactions at the pin ***A***.