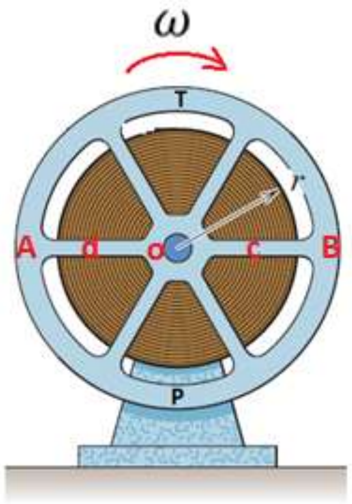
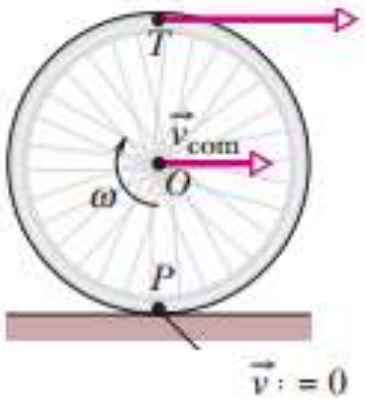
Kinematics of Rigid Bodies

Introduction

In Chapter 2 on particle kinematics, we developed the relationships governing the displacement, velocity, and acceleration of points as they moved along straight or curved paths. In rigid-body kinematics we use these same relationships but must also account for the rotational motion of the body. Thus rigid-body kinematics involves both linear and angular displacements, velocities, and accelerations, , and their variation with time.





A rigid body is a solid body in which deformation **is zero** or so small it can be neglected. The distance between any two given point on a rigid body remains constant in time regardless of external forces or moments exerted on it. A rigid body is usually considered as a continuous distribution of mass.

Characteristics of rigid body motion

All lines on a rigid body have the same angular velocity and the same angular acceleration. Rigid motion can be decomposed into the translation of an arbitrary point followed by a rotation about the point.

Motion

A body is said to be in motion if it **changes its position** with respect to a reference point A moving body can have three types of motions i.e., translational, rotational or general plane motion. The classification of motion is shown in figure 1.

Translation motion.

This type of motion occurs when a line in the body remains parallel to its original orientation throughout the motion. When the paths of motion for any two points on the body are parallel lines, the motion is called ***rectilinear translation***. If the paths of motion are along curved lines, the motion is called ***curvilinear translation***.

Rotation about a fixed axis.

When a rigid body rotates about a fixed axis, all the particles of the body, except those which lie on the axis of rotation, move along

circular paths.

General plane motion.

When a body is subjected to general plane motion, it undergoes a combination of translation and rotation. The translation occurs within a reference plane, and the rotation occurs about an axis perpendicular to the reference plane.

To understand the general motion of a rigid body, we have taken a body of regular shape instead of an arbitrary shape as shown in Fig.2. We see that the body not only translates but also rotates in an arbitrary manner.

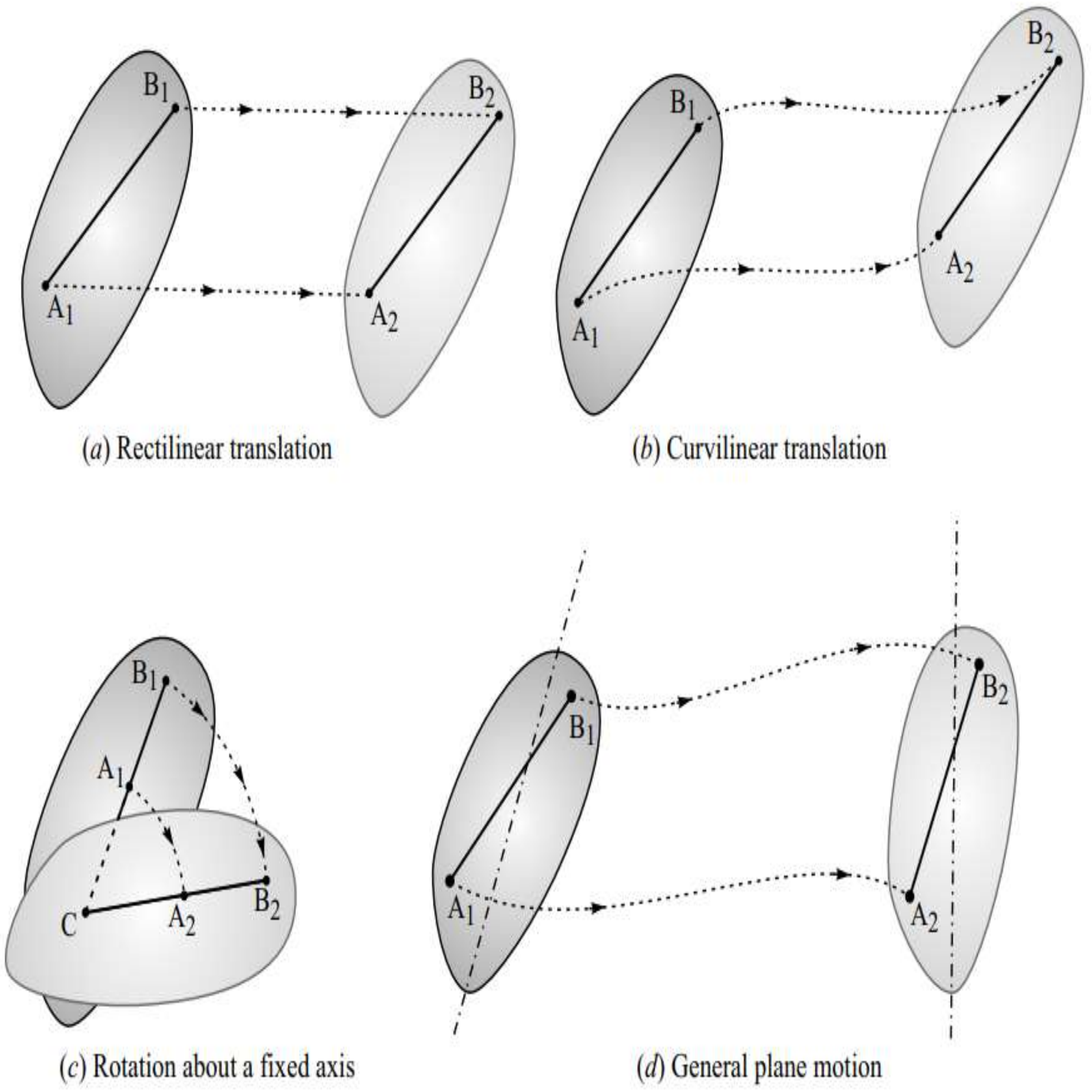


Fig.1.The classification of motion

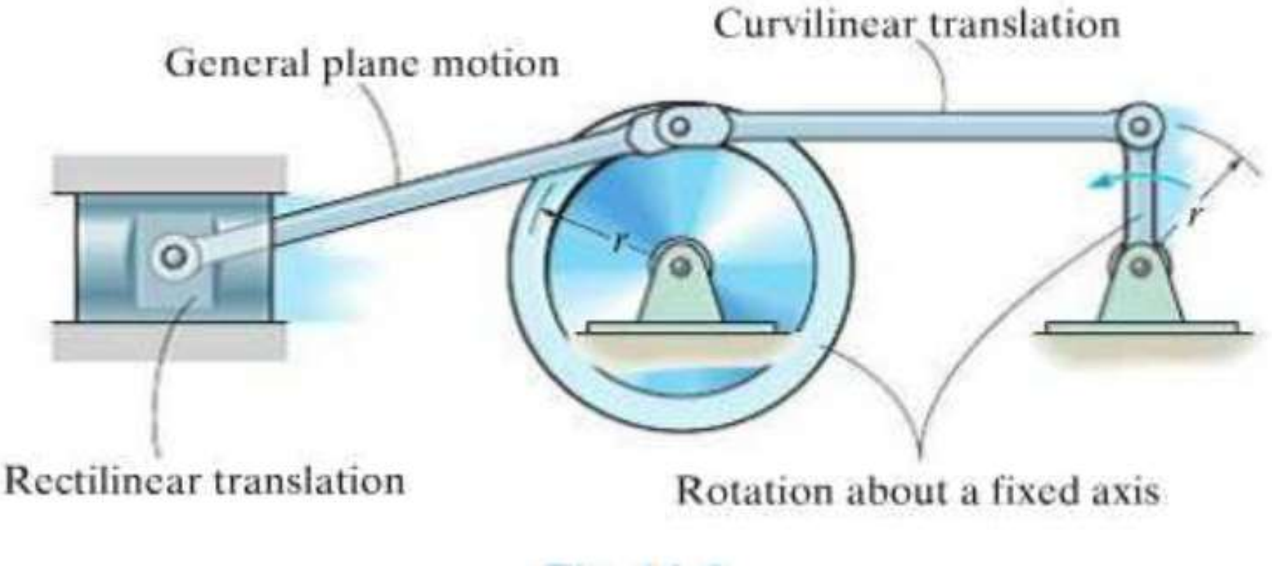


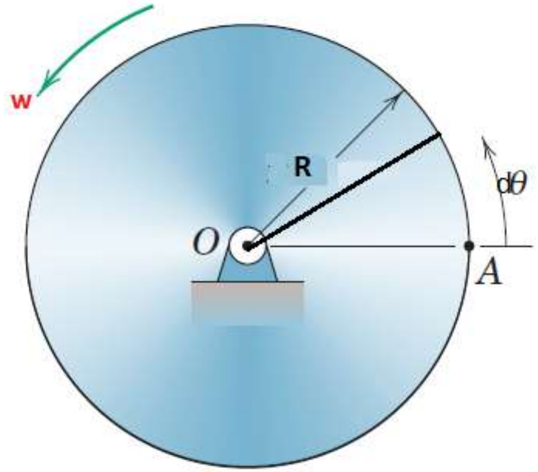
Fig.2.Diffrent motion of body.

Curvilinear

Rotational and Linear Kinematics

Rotational Motion	Quantity	Linear Motion
θ	Position	x
$\Delta\theta$	Displacement	Δx
ω	Velocity	v
α	Acceleration	a
t	Time	t

The angular velocity ω and angular acceleration α of a rigid body in plane rotation are respectively, the first and second time derivatives of the angular position coordinate θ of any line in the plane of motion of the body. These definitions give

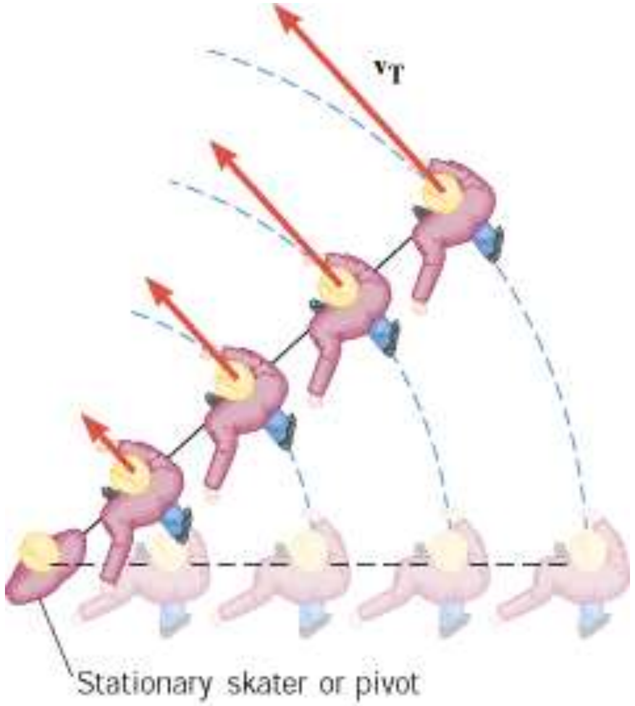


$$\omega = \frac{d\theta}{dt} = \dot{\theta}$$

$$\omega \, d\omega = \alpha \, d\theta$$

$$\alpha = \frac{d\omega}{dt} = \dot{\omega} \quad \text{or} \quad \alpha = \frac{d^2\theta}{dt^2} = \ddot{\theta}$$

Relating Rotational and Translational Speed



$$\left|\vec{v}\right| = v = \left|\omega\right| r \quad \text{(for radian measure only)}.$$

- The rotational speed ω is the same at any points
- The translational speed is different for the points with different distance from the rotational axis.

Relative motion analysis

The position of a point on a moving rigid body in the 2D *xy*-plane

In planar kinematics, we fully determine the position of a rigid body by fixing the position vectors of 2 points in the rigid body that can be freely chosen, like points *A* and *B* of the rectangle in Fig.3.

$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$$

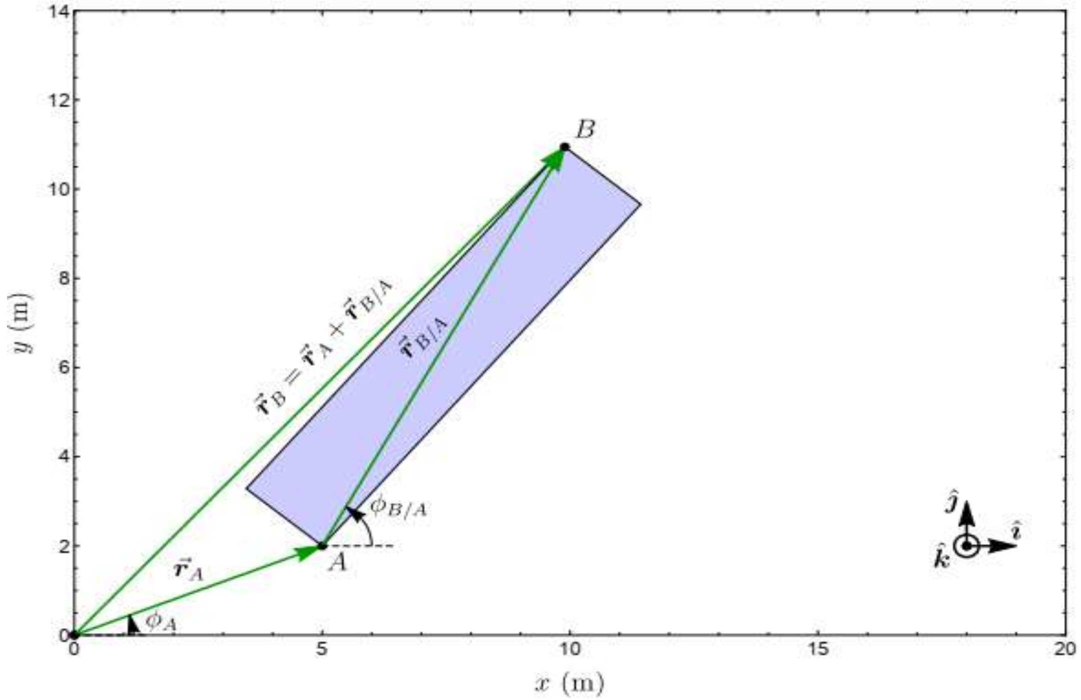


Figure .3 The orientation of a rigid body in the 2D *xy*-plane can uniquely be described by a position vector $\boldsymbol{r}_A$ and an angle $\phi_{B/A}$

Angular velocity vector

The angular velocity vector $\vec{\omega}$ a rigid body is a vector with magnitude $|\omega| = |\dot{\phi}_{B/A}|$ and a direction that is perpendicular to the plane in which the rigid body rotates. Its direction can be determined using the right hand rule.

The angular velocity vector (unit rad/s) of a rigid body that rotates in the *xy* plane is:

$$\vec{\omega}_{2D} = \omega\hat{\boldsymbol{k}} = \dot{\phi}_{B/A}\hat{\boldsymbol{k}}$$

We have 3 coordinates, namely x_A, y_A and $\phi_{B/A}$, where $\phi_{B/A}$ is the angle the relative position vector $\boldsymbol{r}_{B/A}$ makes with the *x*-axis. With x_A, y_A we can determine $\boldsymbol{r}_A$, and with $\phi_{B/A}$ and knowledge of the distance $|\boldsymbol{r}_{B/A}|$ we can determine $\boldsymbol{r}_{B/A}$:

$$
\begin{aligned}
\vec{\boldsymbol{r}}_{A,2D} &= x_A\hat{\boldsymbol{\imath}} + y_A\hat{\boldsymbol{\jmath}} \\
\vec{\boldsymbol{r}}_{B/A,2D} &= |\vec{\boldsymbol{r}}_{B/A}|\cos\phi_{B/A}\hat{\boldsymbol{\imath}} + |\vec{\boldsymbol{r}}_{B/A}|\sin\phi_{B/A}\hat{\boldsymbol{\jmath}}
\end{aligned}
$$

Now we determine $\boldsymbol{r}_B$ from the 3 coordinates by adding these two vectors as shown in

$$
\begin{aligned}
\vec{\boldsymbol{r}}_B &= \vec{\boldsymbol{r}}_A + \vec{\boldsymbol{r}}_{B/A} \\
\vec{\boldsymbol{r}}_{B,2D} &= (x_A + |\vec{\boldsymbol{r}}_{B/A}|\cos\phi_{B/A})\hat{\boldsymbol{\imath}} + (y_A + |\vec{\boldsymbol{r}}_{B/A}|\sin\phi_{B/A})\hat{\boldsymbol{\jmath}}
\end{aligned}
$$

Velocity of a point B in a rigid body

$$
\begin{aligned}
\vec{\boldsymbol{v}}_{B,2D} &= \frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{r}}_B = \frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{r}}_A + \frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{r}}_{B/A} \\
&= \vec{\boldsymbol{v}}_A + \dot{\phi}_{B/A}|\vec{\boldsymbol{r}}_{B/A}|\left(-\sin\phi_{B/A}\hat{\boldsymbol{\imath}} + \cos\phi_{B/A}\hat{\boldsymbol{\jmath}}\right) \\
&= \vec{\boldsymbol{v}}_A + \dot{\phi}_{B/A}|\vec{\boldsymbol{r}}_{B/A}|\hat{\boldsymbol{\phi}}_A
\end{aligned}
$$

The velocity of point B in can be split up in two parts: a vector $\boldsymbol{v}_{B,\mathrm{trans}}$ related to translation and a vector $\boldsymbol{v}_{B,\mathrm{rot}}$ related to rotation:

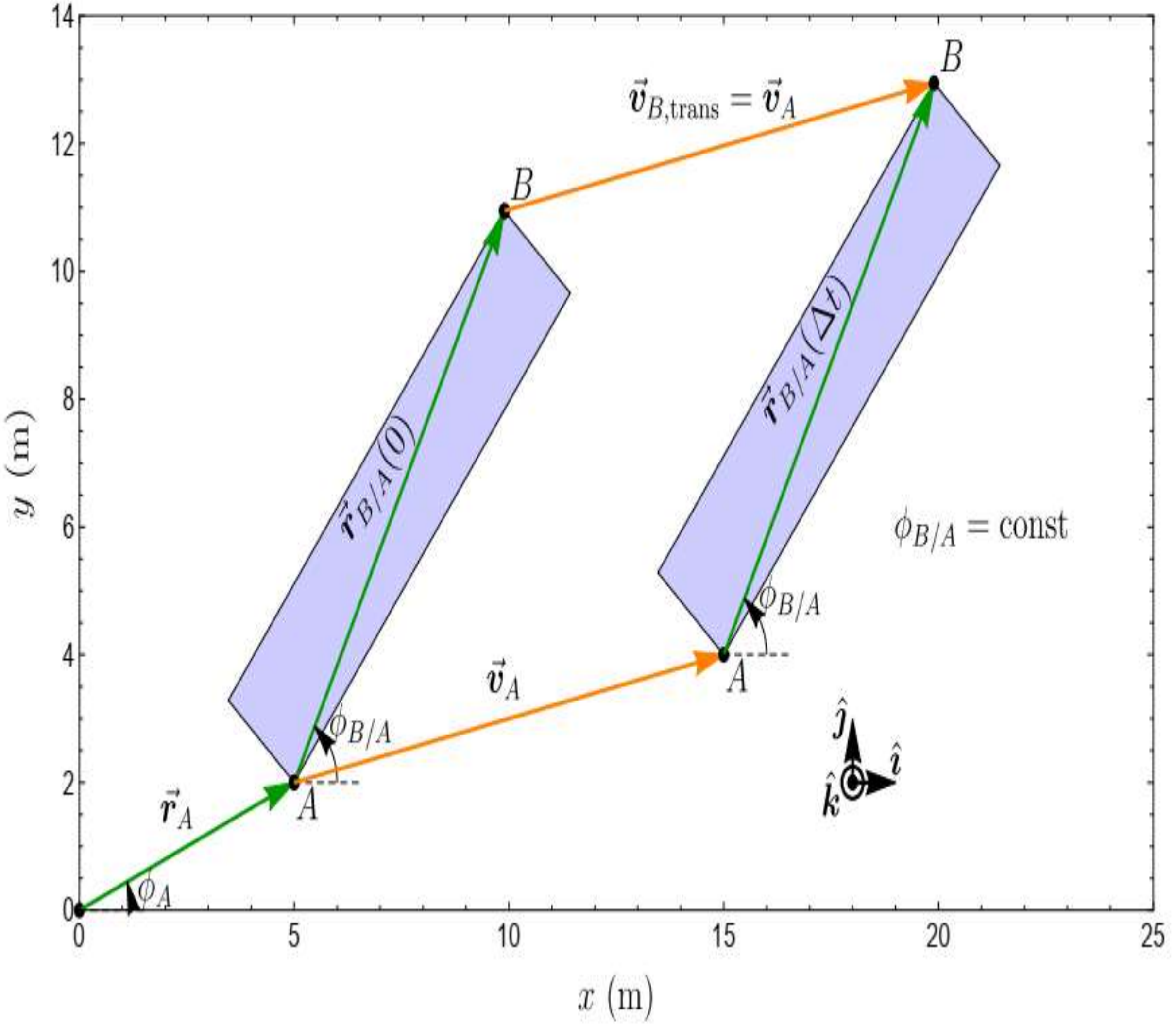


Figure .2: Pure translation of a rigid body: $\phi_{B/A}$ = constant .

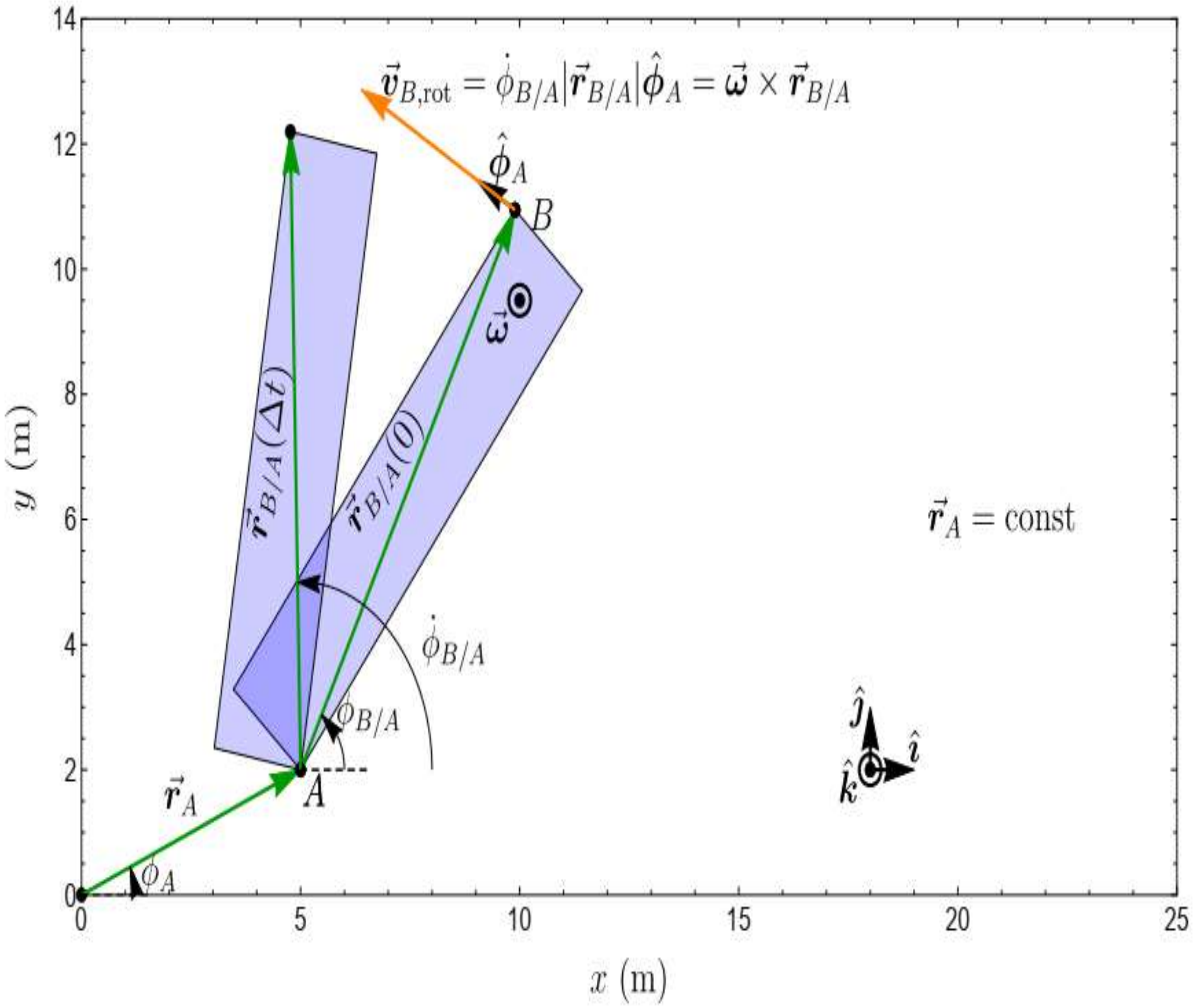


Figure .3: Pure rotation of a rigid body: r_A = constant.

General motion

In general, as shown in Fig.4, the motion of a point in a rigid body is a sum of translational and rotational motion.

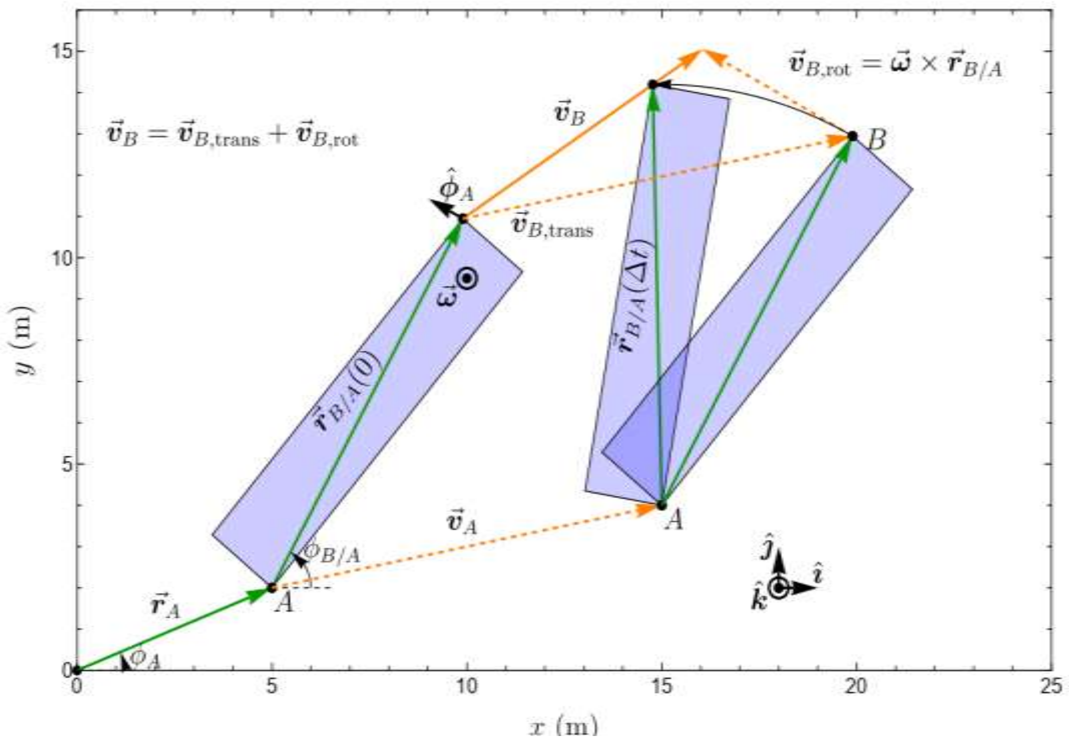
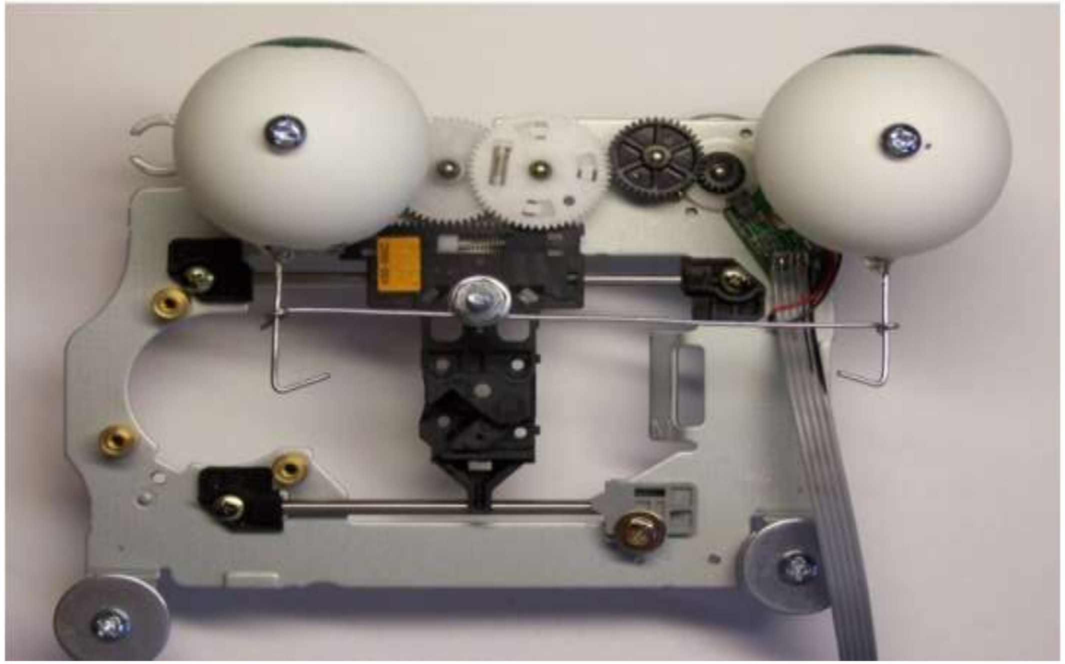


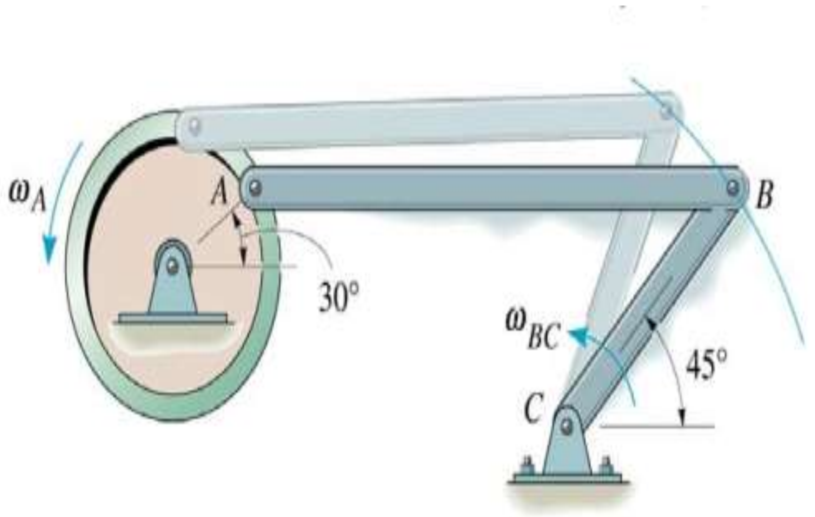
Figure .4: General motion of a rigid body: a combination of rotation and translation.

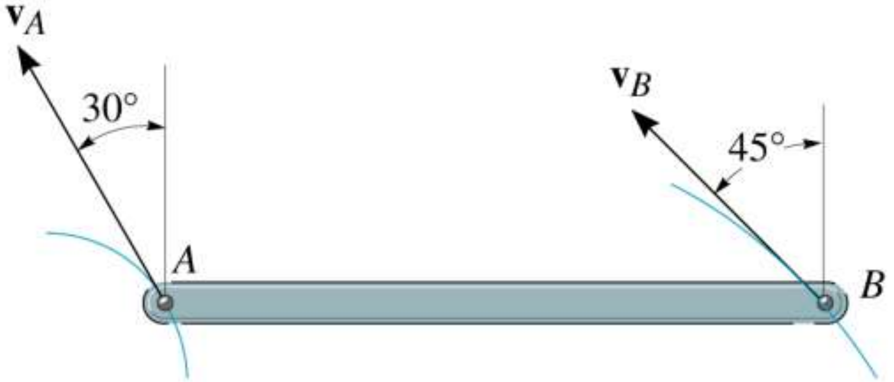


We obtain the most general equation and important equation for the velocity in a rigid body

$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A}$$

Example When using the relative velocity equation. Points A and B should generally be points on the body with a know motion. Often these points are pin connections in linkages. Here both points A and B have circular motion since the disk and link BC move in circular paths. The direction vA and vB are known since they are always tangent to the circular path of motion.





Angular acceleration of a rigid body

We take the derivative of the velocity vector of a point in a rigid body

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\vec{v}_B &= \frac{\mathrm{d}}{\mathrm{d}t}\vec{v}_A + \frac{\mathrm{d}\vec{\omega}}{\mathrm{d}t} \times \vec{r}_{B/A} + \vec{\omega} \times \frac{\mathrm{d}\vec{r}_{B/A}}{\mathrm{d}t} \\ \vec{a}_B &= \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} + \vec{\omega} \times \vec{v}_{B,\mathrm{rot}}\end{aligned}$$

In this derivation we defined the angular acceleration vector of the $\vec{\alpha}$ rigid body.
The angular acceleration vector of $\vec{\alpha}$ rigid body is the time derivative of its angular velocity vector.

$$\vec{\alpha} \equiv \frac{\mathrm{d}\vec{\omega}}{\mathrm{d}t}$$

The general vector expression for the acceleration vector of a point ***B*** in a rigid body that has translational and rotational acceleration:

$$\vec{a}_B = \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A})$$

This equation shows that the acceleration of a point *B* on a rigid body consists of three contributions that are shown in

1. The translational acceleration $\vec{a}_{B,\text{trans}} = \vec{a}_A$ to the acceleration of point *A*.
2. The angular acceleration $\vec{a}_{B,\text{ang}} = \vec{\alpha} \times \vec{r}_{B/A}$:o the angular acceleration vector.
3. The centripetal acceleration $\vec{a}_{B,\text{cptl}} = \vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A})$ e to the angular velocity vector .

Example.2

Velocity of a point on a rigid body in planar motion. A plate ABC, of an equilateral triangle geometry, is in motion in the *x-y* plane. At the instant shown in the figure, point *B* has velocity V_B = 0.3 i + 0.6 j and the plate has angular velocity ω= 2 rad/s k. Find the velocity of point A.

Solution We are given V_B and ω, and we need to find v_A, the velocity of point A on the same rigid body. We know that, Thus, to find v_A , $\vec{v}_A = \vec{v}_B + \vec{\omega} \times \vec{r}_{A/B}$ d $r_{A/B}$. Now,

$$\vec{r}_{A/B} = \vec{r}_A - \vec{r}_B = \vec{0} - (0.2\,\text{m}\hat{\imath}) = -0.2\,\text{m}\hat{\imath}$$

$$\vec{v}_A = \vec{v}_B + \vec{\omega} \times \vec{r}_{A/B}$$

$$\begin{aligned} \vec{v}_A &= \vec{v}_B + \vec{\omega} \times \vec{r}_{A/B} \quad \vec{r}_{A/B} = \vec{r}_A - \vec{r}_B = \vec{0} - (0.2\,\text{m}\hat{\imath}) = -0.2\,\text{m}\hat{\imath} \\ &= (0.3\hat{\imath} + 0.6\hat{\jmath})\,\text{m/s} + 2\,\text{rad/s}\hat{k} \times (-0.2\hat{\imath})\,\text{m} \\ &= (0.3\hat{\imath} + 0.6\hat{\jmath})\,\text{m/s} - 0.4\hat{\jmath}\,\text{m/s} \\ &= (0.3\hat{\imath} + 0.2\hat{\jmath})\,\text{m/s}. \end{aligned}$$

$$\boxed{\vec{v}_A = (0.3\hat{\imath} + 0.2\hat{\jmath})\,\text{m/s}}$$

The coordinate representation 2D

Relative Velocity

$$\mathbf{r}_A = \mathbf{r}_B + (x\mathbf{i} + y\mathbf{j})$$

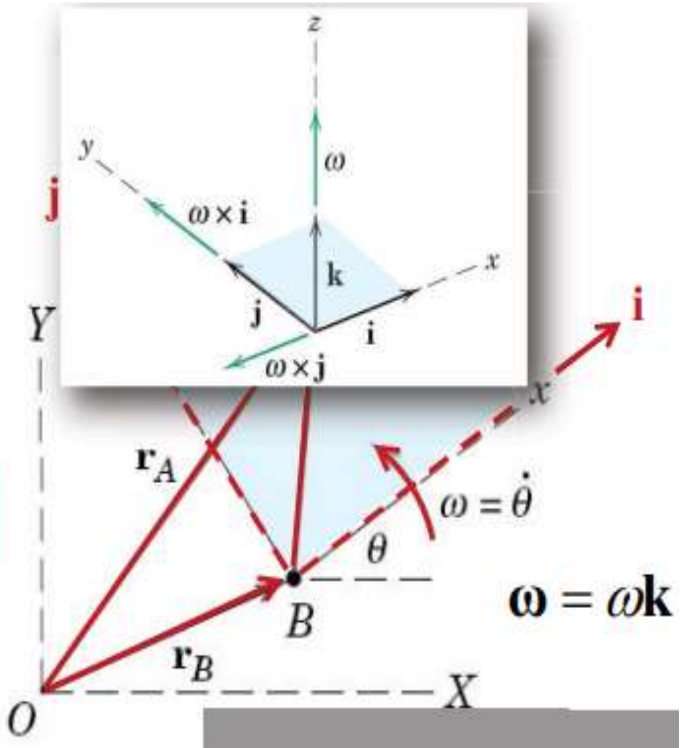
$$\boxed{\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r} + \mathbf{v}_{rel}}$$

$$\dot{\mathbf{r}}_A = \dot{\mathbf{r}}_B + \frac{d}{dt}(x\mathbf{i} + y\mathbf{j})$$

$$\dot{\mathbf{r}}_A = \dot{\mathbf{r}}_B + (x\dot{\mathbf{i}} + y\dot{\mathbf{j}}) + (\dot{x}\mathbf{i} + \dot{y}\mathbf{j})$$

$$\dot{\mathbf{r}}_A = \dot{\mathbf{r}}_B + (\boldsymbol{\omega} \times x\mathbf{i} + \boldsymbol{\omega} \times y\mathbf{j}) + (\dot{x}\mathbf{i} + \dot{y}\mathbf{j})$$

$$\dot{\mathbf{r}}_A = \dot{\mathbf{r}}_B + (\boldsymbol{\omega} \times (x\mathbf{i} + y\mathbf{j})) + (\dot{x}\mathbf{i} + \dot{y}\mathbf{j})$$



$$\boldsymbol{\omega} = \omega\mathbf{k}$$

$$\dot{\mathbf{i}} = \omega\mathbf{j} \qquad \dot{\mathbf{j}} = -\omega\mathbf{i}$$

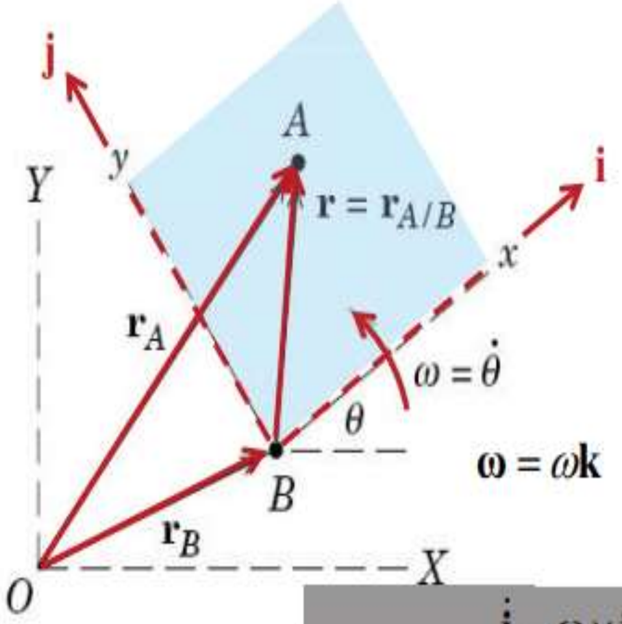
$$\dot{\mathbf{i}} = \boldsymbol{\omega} \times \mathbf{i} \qquad \dot{\mathbf{j}} = \boldsymbol{\omega} \times \mathbf{j}$$

Relative Acceleration

$$\mathbf{r}_A = \mathbf{r}_B + (x\mathbf{i} + y\mathbf{j})$$

$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r} + \mathbf{v}_{rel}$$

$$\mathbf{a}_A = \mathbf{a}_B + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \dot{\mathbf{r}} + \dot{\mathbf{v}}_{rel}$$



$$\boldsymbol{\omega} = \omega\mathbf{k}$$

$$\mathbf{a}_A = \mathbf{a}_B + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r} + \mathbf{v}_{rel}) + \dot{\mathbf{v}}_{rel}$$

$$\mathbf{a}_A = \mathbf{a}_B + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \boldsymbol{\omega} \times \mathbf{v}_{rel} + \dot{\mathbf{v}}_{rel}$$

$$\boxed{\mathbf{a}_A = \mathbf{a}_B + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\boldsymbol{\omega} \times \mathbf{v}_{rel} + \mathbf{a}_{rel}}$$

$$\dot{\mathbf{i}} = \boldsymbol{\omega} \times \mathbf{i}$$

$$\dot{\mathbf{j}} = \boldsymbol{\omega} \times \mathbf{j}$$

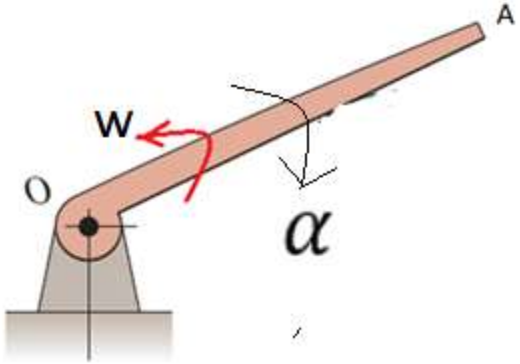
$$\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r} + \mathbf{v}_{rel}$$

$$\dot{\mathbf{v}}_{rel} = \boldsymbol{\omega} \times \mathbf{v}_{rel} + \mathbf{a}_{rel}$$

$$v = r\omega$$

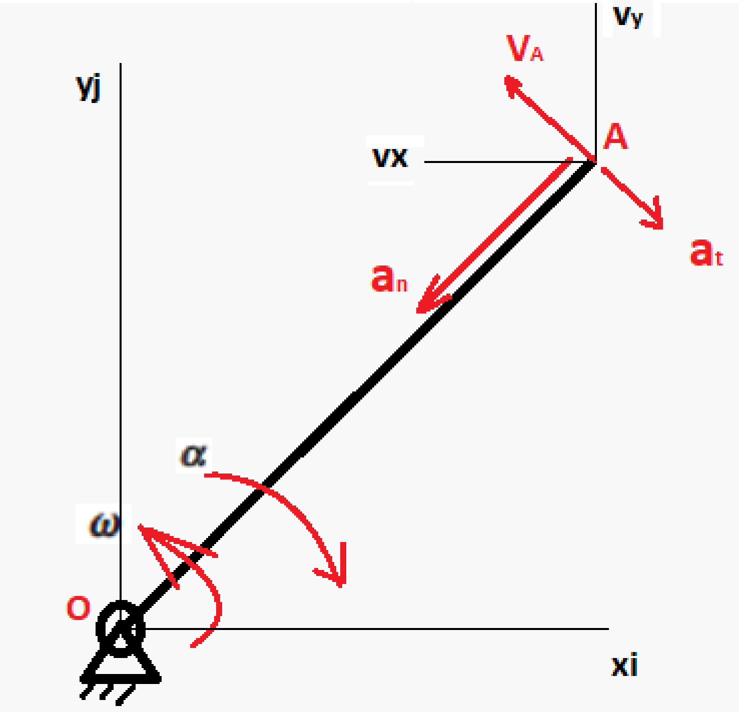
$$a_n = r\omega^2 = v^2/r = v\omega$$

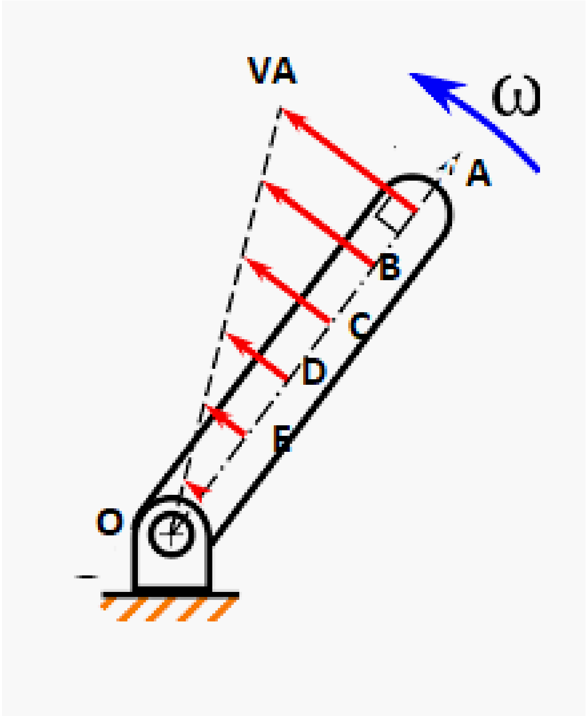
$$a_t = r\alpha$$



$\vec{V}_A = V_x\, i + V_y\, j$

$\vec{a}_A = a_x\, i + a_y\, j$





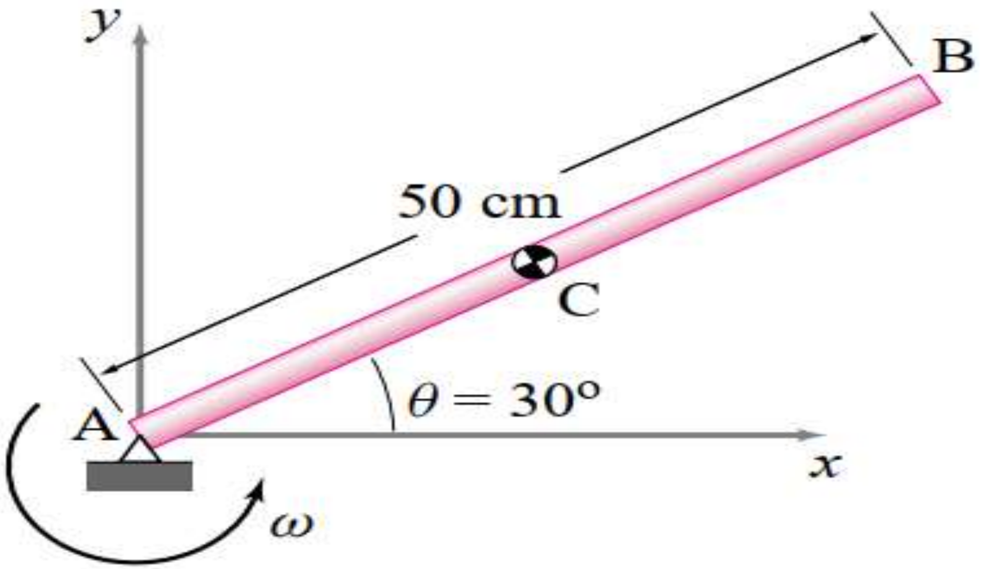
Example.2

A uniform bar AB of length l= 50 cm rotates counterclockwise about point A with constant angular speed ω. At the instant shown in Fig .3 the linear speed v_C of the center of mass C is 7:5 cm/ s.

(a) What is the angular speed of the bar?

(b) What is the angular velocity of the bar?

(c) What is the linear velocity of end B?



Relative Acceleration

Translation and Fixed Axis Rotation

A Translation

- Consider rigid body in translation:

- direction of any straight line inside the body is constant,

- all particles forming the body move in parallel lines.

- For any two particles in the body, $\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$

  Differentiating with respect to time $\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A}$

  All particles have the same velocity. $\vec{v}_B = \vec{v}_A$

- Differentiating with respect to time again,

$$\ddot{\vec{r}}_B = \ddot{\vec{r}}_A + \ddot{\vec{r}}_{B/A} = \ddot{\vec{r}}_A$$

All particles have the same acceleration. $\vec{a}_B = \vec{a}_A$

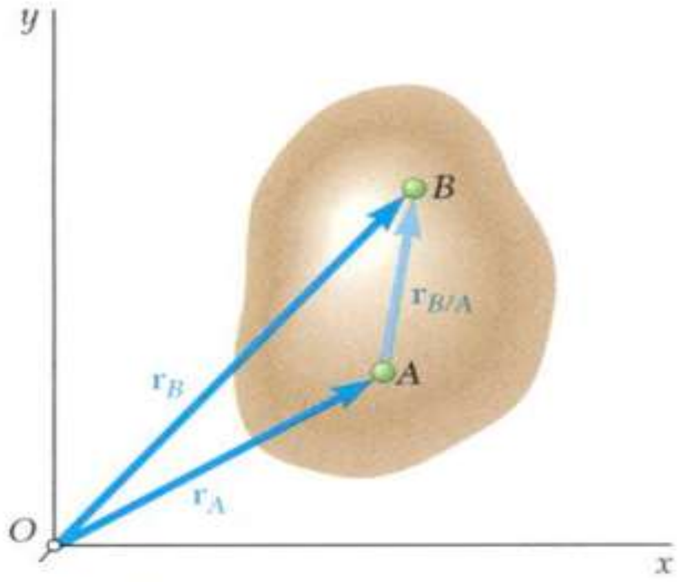


Figure .5: Translation of a rigid body.

Rotation About a Fixed Axis.

When a body rotates about a fixed axis, any point P in the body travels along a circular path.
The change in angular position, *d* , is called the angular displacement, which units of either radians or revolutions. They are related by
1 revolution = 2 π radians.
Angular velocity, ω, is obtained by taking the time derivating of angular displacement
Similarly, angular acceleration is

$$\omega = d\theta/dt \ (rad/s) \ +$$

$$\alpha = d^2\theta/dt^2 = d\omega/dt \ \text{or} \ \alpha = \omega(d\omega/d\theta) + \ rad/s^2$$

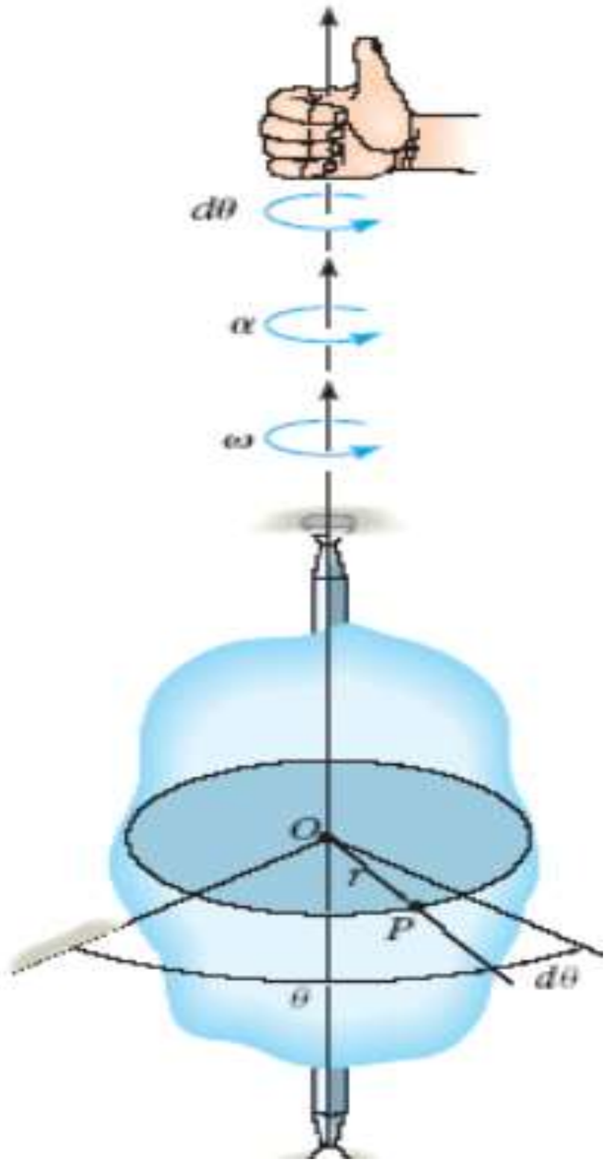


Figure .6: Rotation of a rigid body.

Velocity

Consider rotation of rigid body about a fixed axis AA

- The same result is obtained from

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$$

$$\vec{\omega} = \omega\vec{k} = \dot{\theta}\vec{k} = angular\ velocity$$

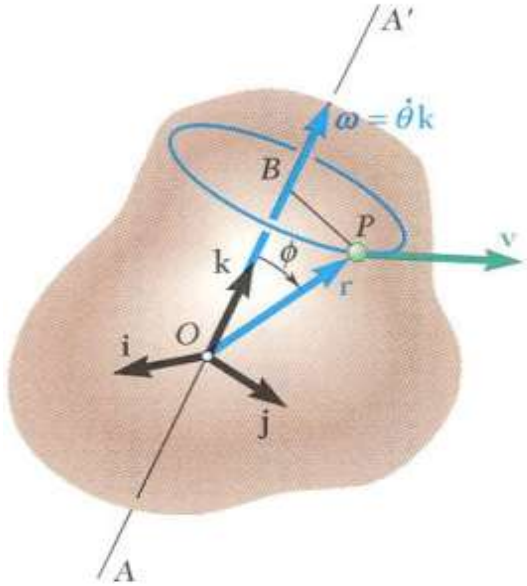


Figure .7: Rotation of a rigid body.

Acceleration

Differentiating to determine the acceleration,

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{\omega} \times \vec{r})$$

$$= \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}$$

$$= \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \vec{v}$$

and

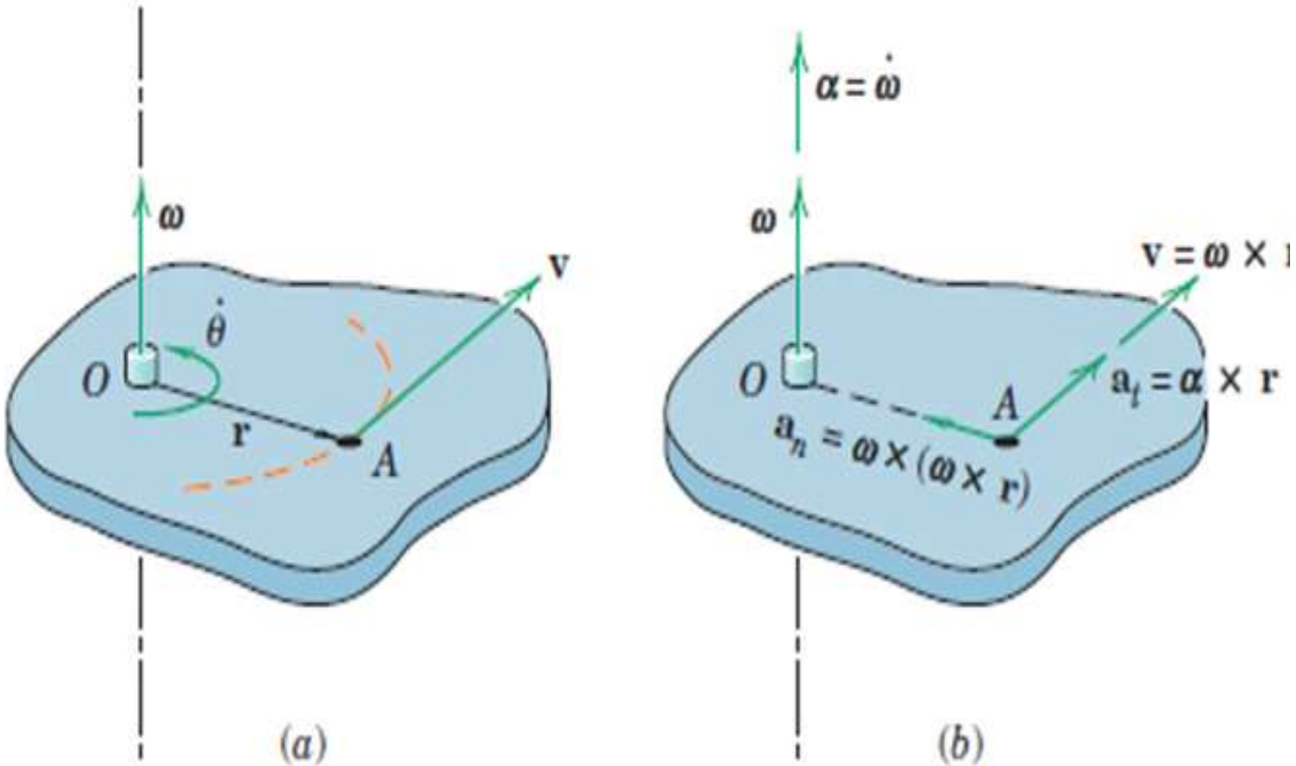
$$\frac{d\vec{\omega}}{dt} = \vec{\alpha} = angular\ acceleration$$

$$= \alpha\vec{k} = \dot{\omega}\vec{k} = \ddot{\theta}\vec{k}$$

$$\vec{\mathbf{v}} = \dot{\vec{\mathbf{r}}} = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}$$

$$\vec{\boldsymbol{\omega}} = \omega\, k \qquad \vec{\mathbf{r}} = xi + yj$$

The order of the vectors to be crossed must be retained. The reverse order gives $\mathbf{r} \times \boldsymbol{\omega} = -\mathbf{v}$.



From the definition of the cross product, using a *right-handed coordinate system*, we get

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i} \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k} \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$$

General Plane Motion

A general plane motion can always be considered as the sum of a translation and a rotation

Example 1.

Consider, for example, a wheel rolling on a straight track (Fig. 8). Over a certain interval of time, two given points A and B will have moved, respectively, from A1 to A2 and from B1 to B2. The same result could be obtained through a translation which would bring A and B into A2 and B'1 (the line AB remaining vertical), followed by a rotation about A bringing B into B2.

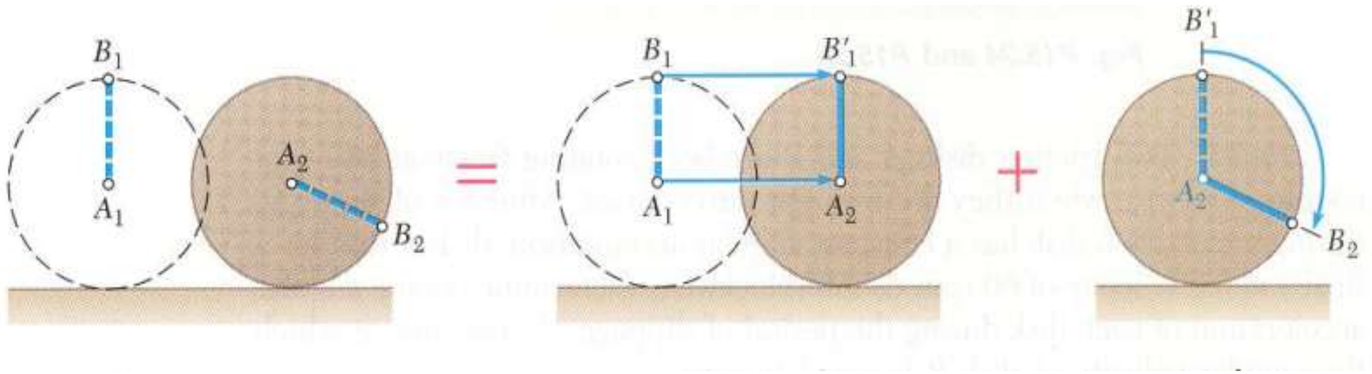


Fig.8 A General plane motion ,a combination of Translation & Rotation

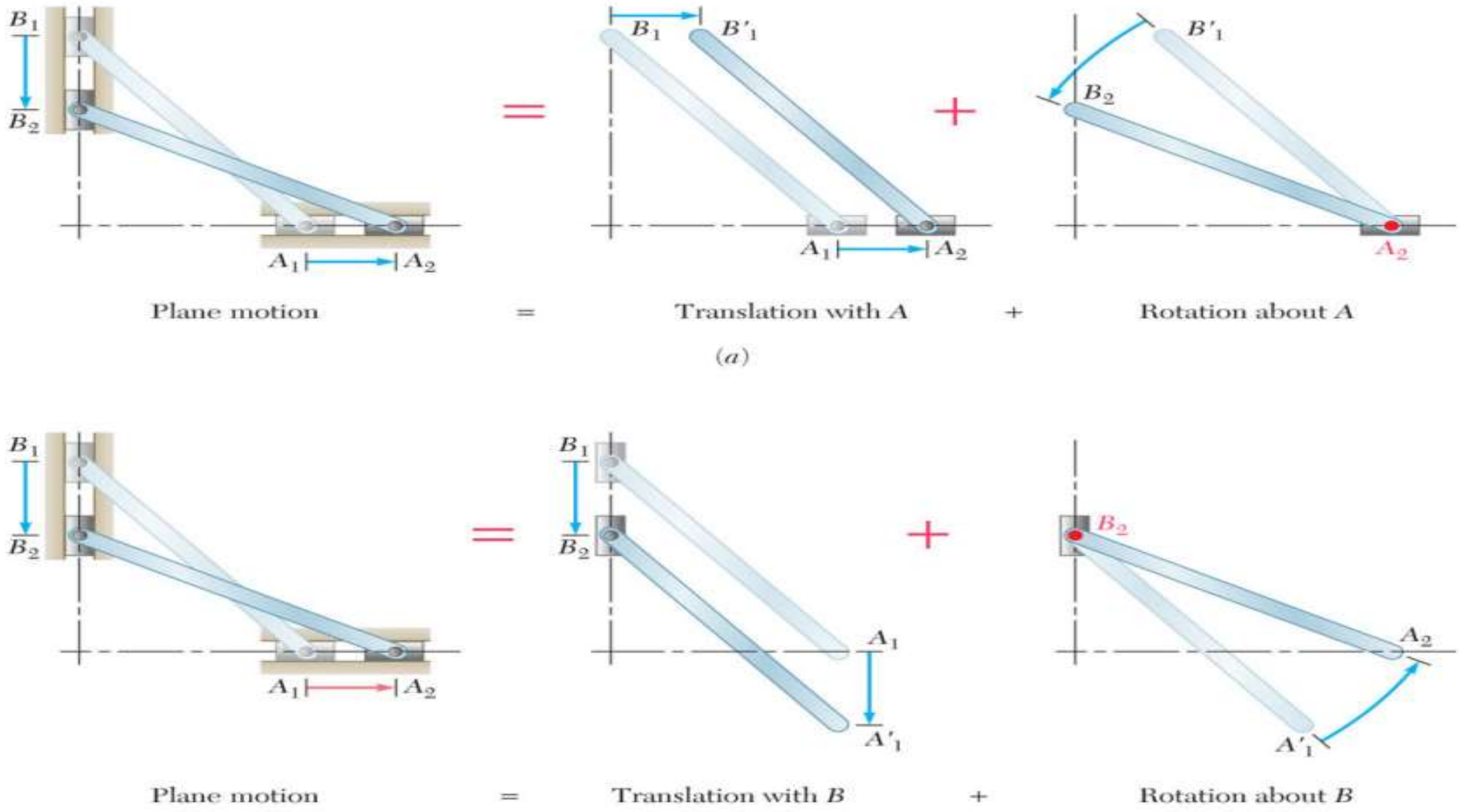


Fig.9 A General plane motion ,with sliders

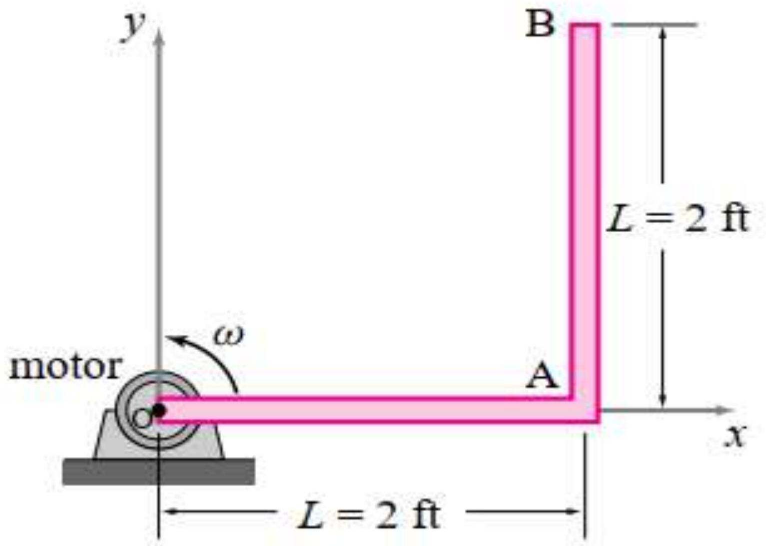
Example.3

Test the acceleration formula on something you know. Consider the 'L 'shaped bar. At the instant shown, the bar is rotating at 4 rad/s and is slowing down at the rate of 2 rad/s2

(i) Find the acceleration of point A.

(ii) Find the relative acceleration $\vec{a}_{B/A}$ point B with respect to point A and use the result to find the absolute acceleration of point B $(\vec{a}_B = \vec{a}_A + \vec{a}_{B/A})$.

(iii) Find the acceleration of point B directly and verify the result obtained in (ii).



Special topic in planar kinematics

Pure rolling in 2-D

In this section, we would like to add to the vocabulary of special motions by considering *pure rolling*. Most commonly, one discusses pure rolling of round objects on flat ground, like wheels and balls, but we will also mention more advanced topics.

2-D rolling of a round wheel on level ground

The simplest case, ***the no-slip rolling*** of a round wheel, is an instructive starting point. First, we define the geometric and kinematic variables as shown in Fig. 10. For convenience, we pick a point ***D*** which was at $x_D = 0$ at the start of rolling, when $x_C = 0$. The key to the kinematics is that:

The arc length traversed on the wheel is the distance traveled by the wheel center.

That is

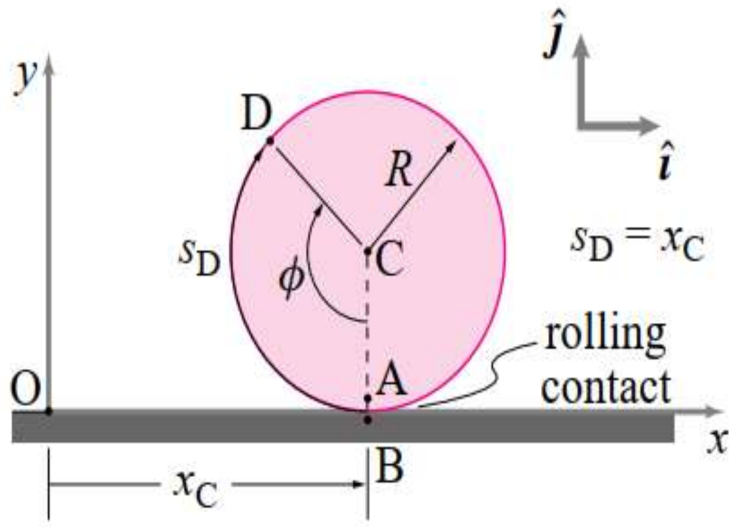
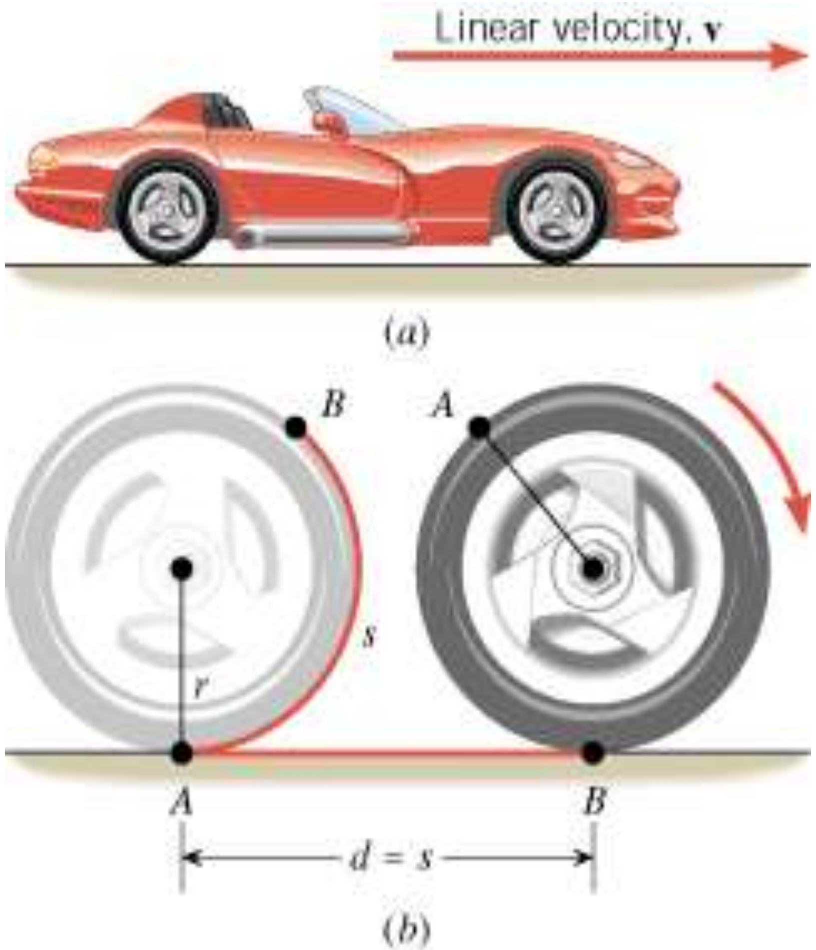


Figure .10: Pure rolling of a round wheel on a level support

Rolling Motion



$$v = r\omega$$

- (*a*) An automobile moves with a linear speed *v*.
- (*b*) If the tires roll and do not slip, the distance *d*, through which an axle moves, equals the circular arc length ***s*** along the outer edge of a tire.

To represent the translational motion of a rigid body in space, we require three independent coordinates, namely, x, y and z, and to represent its rotational motion, we require three more coordinates, namely, angular coordinates θ_x, θ_y and θ_z. Hence, six independent coordinates are required to represent the motion of a rigid body. Thus, we say that a rigid body has six degrees of freedom.

For a system of rigid bodies, we can establish a local Cartesian coordinate system for each rigid body. Transformation matrices are used to describe the relative motion between rigid bodies.

Eulerian Angles

Three independent direction angles define the orientation of a set of ***xyz*** axes. Eulerian angles treat this matter as a specific sequence of rotations.

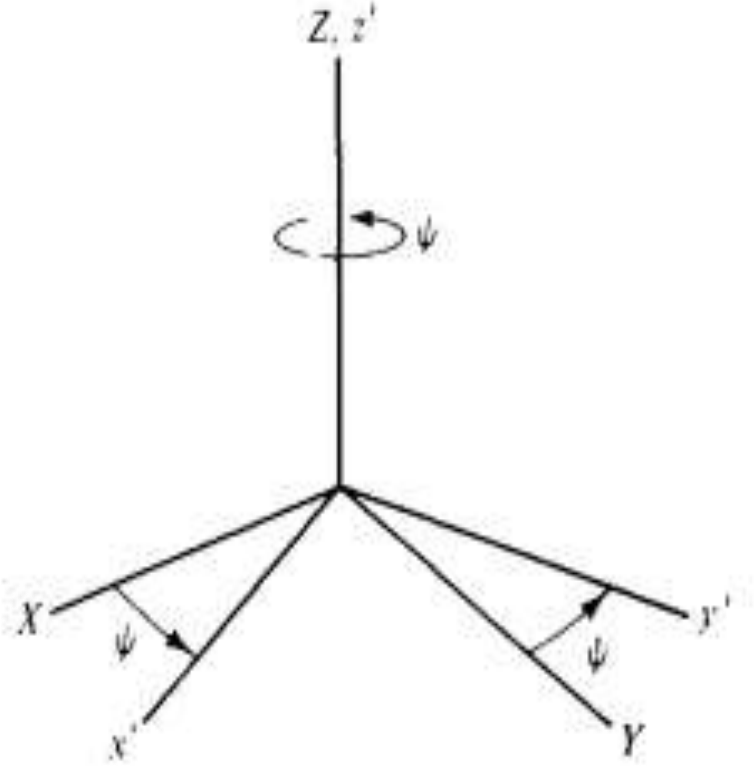


Figure.11 Procession.

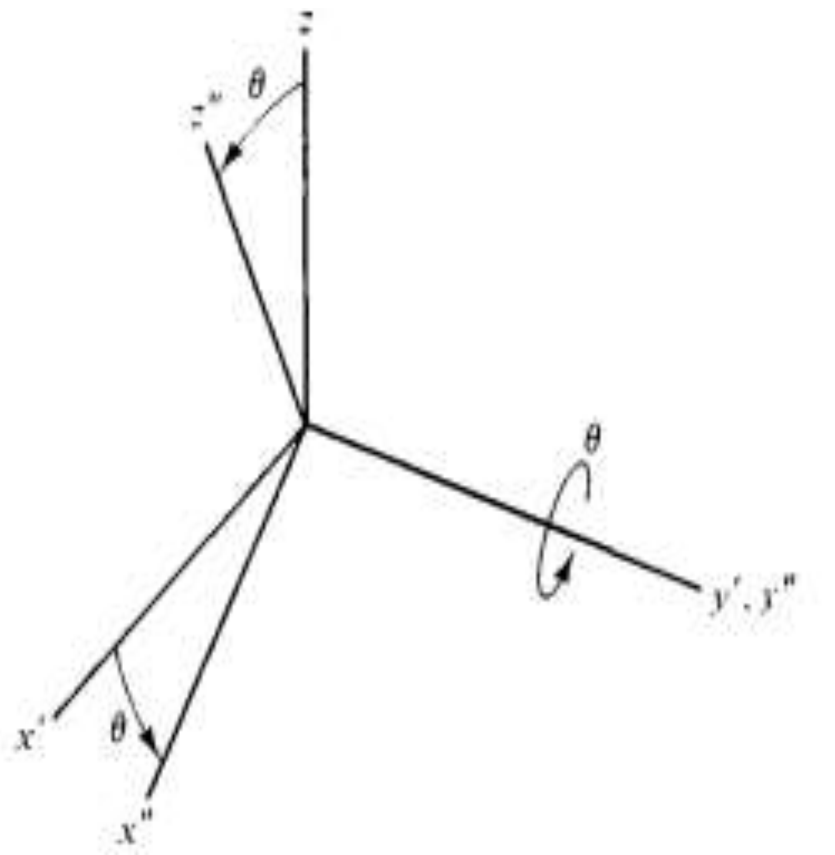


Figure.12 Nutation.

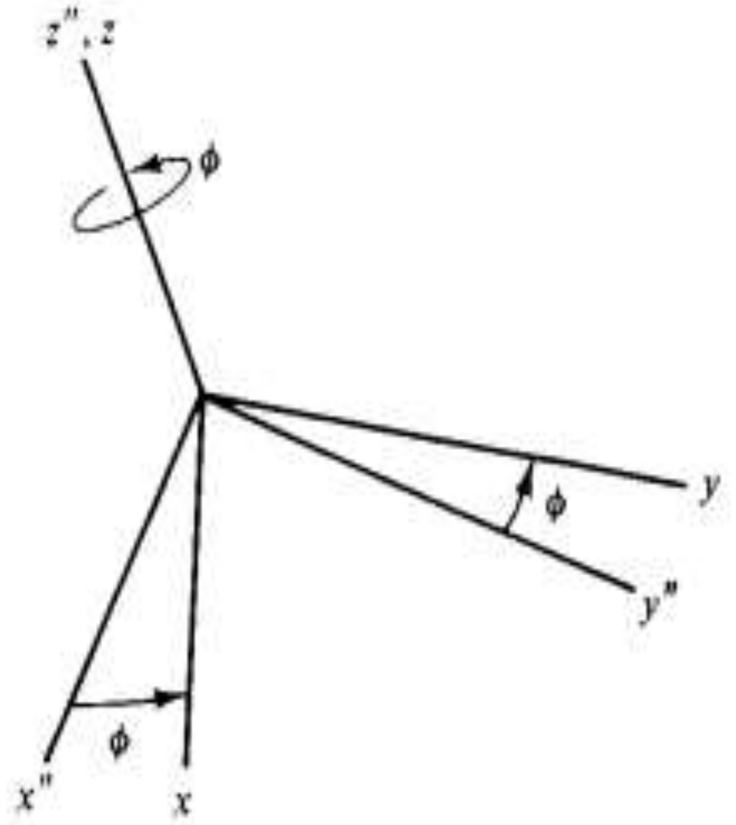


Figure.13 Spin.

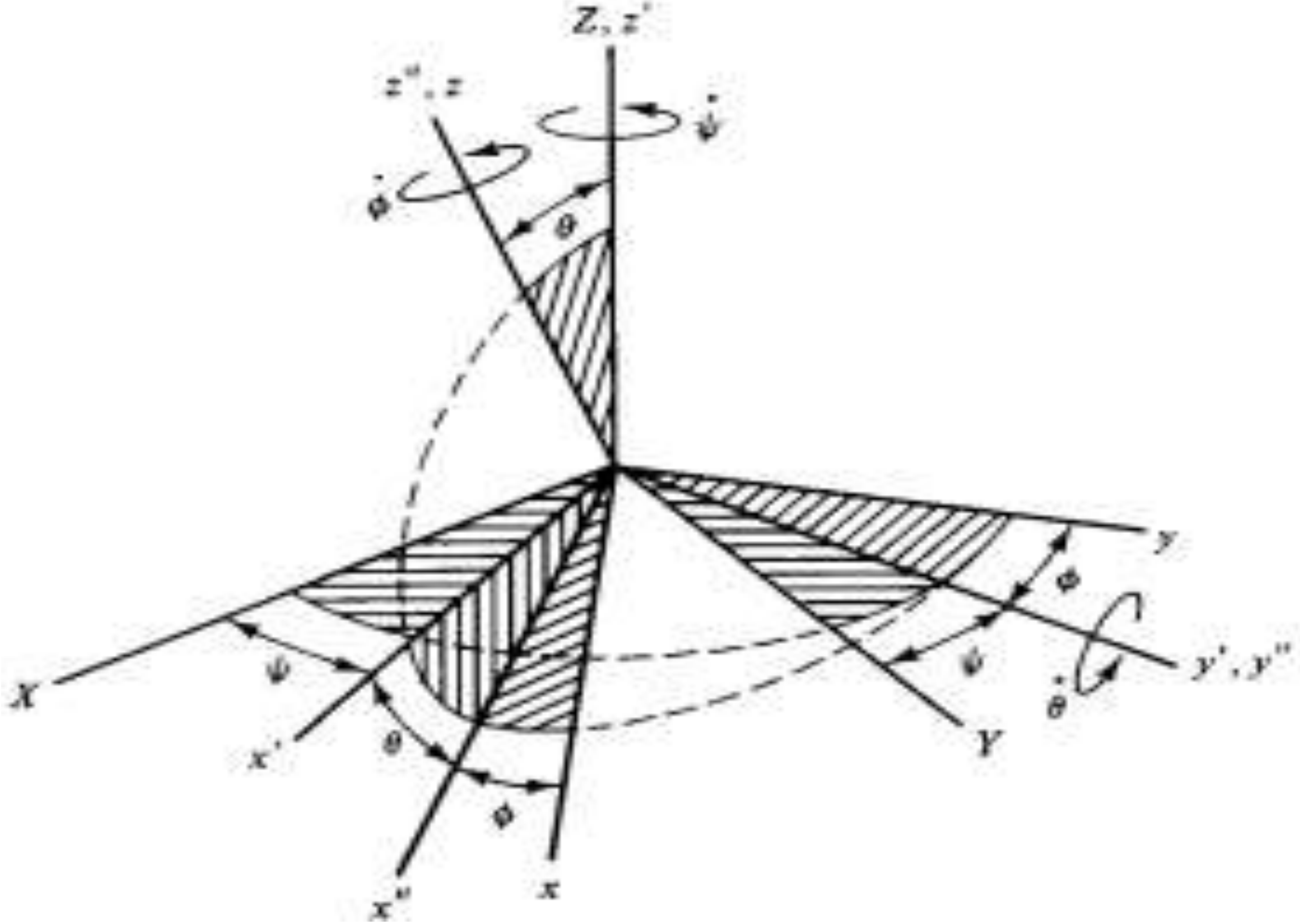


Figure. 14. Eulerian Angles

The coordinate representation of special rotations 3D

- The coordinate representation for the rotation of the point about a Cartesian coordinates axis.
- The transformation from XYZ to $x'y'z'$ ay be found from to be

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = [R_\psi] \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}, \qquad [R_\psi] = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- The second transformation is given by

$$\begin{Bmatrix} x'' \\ y'' \\ z'' \end{Bmatrix} = [R_\theta] \begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix}, \qquad [R_\theta] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

- The last transformation is given by

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = [R_\phi] \begin{Bmatrix} x'' \\ y'' \\ z'' \end{Bmatrix}, \qquad [R_\phi] = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The transformation matrix from first reference frame to the last one is given by or the rotation matrix for Euler rotations is the composition of the matrices of three rotations.

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = [R] \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}, \quad [R] = [R_\phi][R_\theta][R_\psi]$$

The angular velocity and angular acceleration are readily expressed in terms of the angles of precession, nutation, and spin by adding the rotation rates about the respective axes. The angular velocity of xyz is the (vector) sum of the individual rotation rates, so

$$\bar{\omega} = \dot{\psi}\bar{K} + \dot{\theta}\bar{j}' + \dot{\phi}\bar{k}.$$

- The angular velocity of x'y'z' reference frame is:

$$\bar{\omega}' = \dot{\psi}\bar{K}.$$

- We can use the expressions for $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ to obtain angular acceleration as

$$\bar{\alpha} = \ddot{\psi}\bar{K} + \ddot{\theta}\bar{j}' + \dot{\theta}\dot{\bar{j}}' + \ddot{\phi}\bar{k} + \dot{\phi}\dot{\bar{k}}$$
$$= \ddot{\psi}\bar{K} + \ddot{\theta}\bar{j}' + \dot{\theta}(\bar{\omega}' \times \bar{j}') + \ddot{\phi}\bar{k} + \dot{\phi}(\bar{\omega} \times \bar{k}).$$

- Expressions for unit vectors in terms of body coordinate vector are:

$$\bar{K} = \sin\theta[-(\cos\phi)\bar{i} + (\sin\phi)\bar{j}] + (\cos\theta)\bar{k},$$
$$\bar{j}' = (\sin\phi)\bar{i} + (\cos\phi)\bar{j}.$$

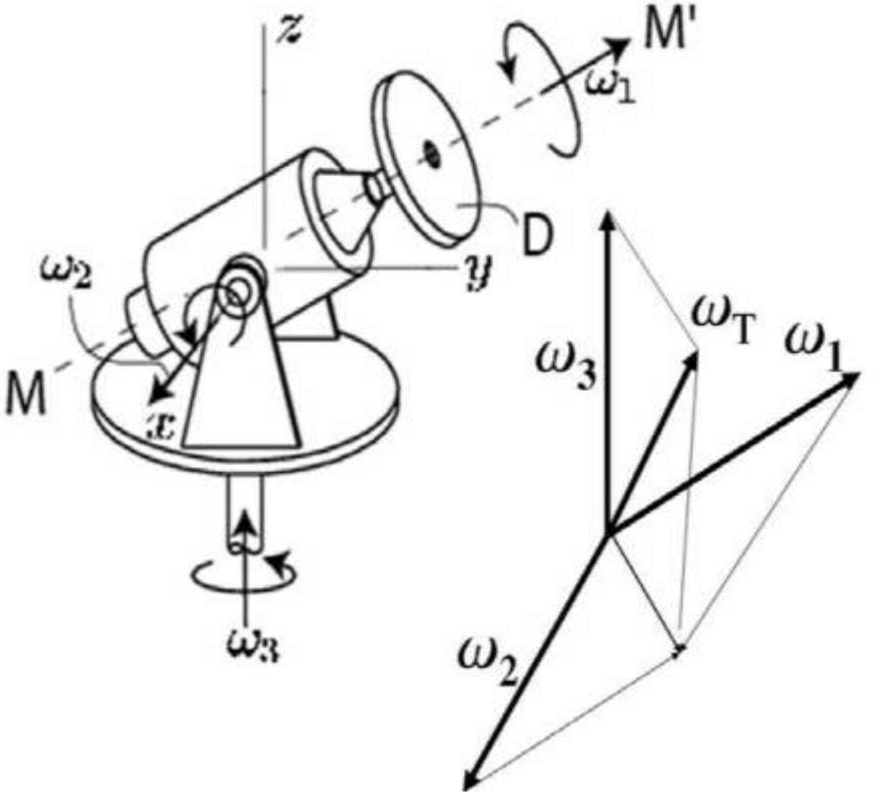
- Thus, the angular velocity and angular acceleration are

$$\bar{\omega} = (-\dot{\psi}\sin\theta\cos\phi + \dot{\theta}\sin\phi)\bar{i}$$
$$+ (\dot{\psi}\sin\theta\sin\phi + \dot{\theta}\cos\phi)\bar{j} + (\dot{\psi}\cos\theta + \dot{\phi})\bar{k},$$

$$\bar{\alpha} = (-\ddot{\psi}\sin\theta\cos\phi + \ddot{\theta}\sin\phi - \dot{\psi}\dot{\theta}\cos\theta\cos\phi + \dot{\psi}\dot{\phi}\sin\theta\sin\phi + \dot{\phi}\dot{\theta}\cos\phi)\bar{i}$$
$$+ (\ddot{\psi}\sin\theta\sin\phi + \ddot{\theta}\cos\phi + \dot{\psi}\dot{\theta}\cos\theta\sin\phi + \dot{\psi}\dot{\phi}\sin\theta\cos\phi - \dot{\theta}\dot{\phi}\sin\phi)\bar{j}$$
$$+ (\ddot{\psi}\cos\theta + \ddot{\phi} - \dot{\psi}\dot{\theta}\sin\theta)\bar{k}.$$

The angular velocity is thus a vector and for a complex configuration, the various components can be vectorially added to obtain the total angular velocity. Consider the complex rotating configuration shown below. We want to determine the angular velocity of the disc D.

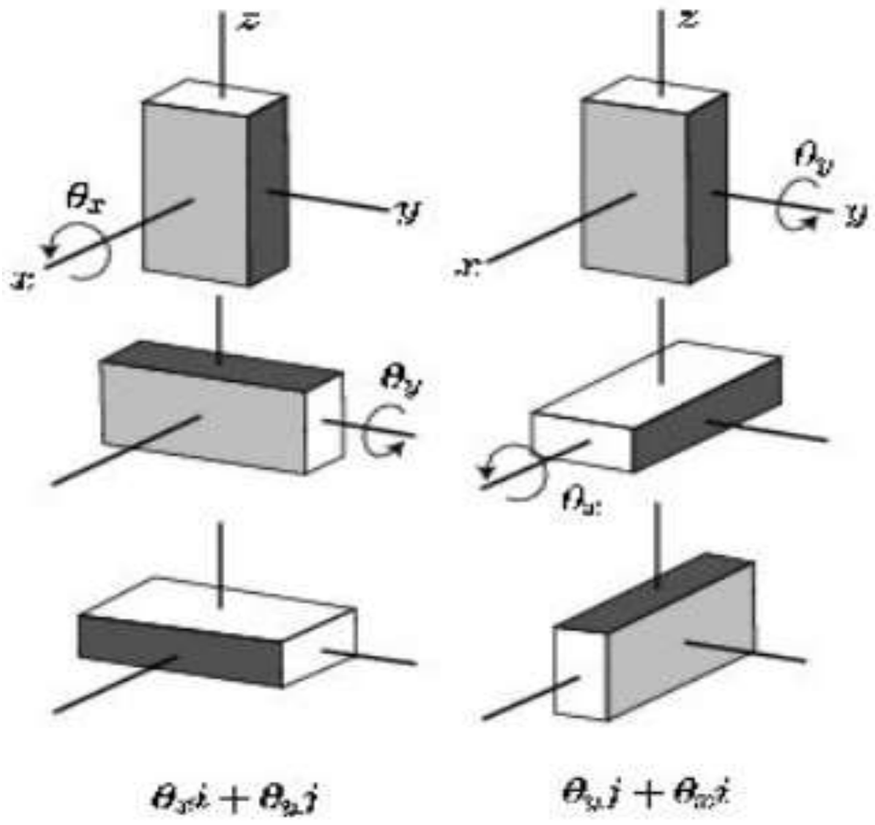
First, we note that the disc is rotating with angular velocity $\boldsymbol{\omega}_1$ about the axis MM'. In turn, this axis is rotating with angular velocity $\boldsymbol{\omega}_2$ about the horizontal axis, which is at this instant aligned with the $\boldsymbol{x}$ axis. At the same time, the whole assembly is rotating about the $\boldsymbol{z}$ axis with angular velocity $\boldsymbol{\omega}_3$. Therefore, the total angular velocity of the disc is vector sum of the individual angular velocity vectors. The resultant vector is shown in the figure: $\boldsymbol{\omega}_{total} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \boldsymbol{\omega}_3$. Expressed in the fixed $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ system for which the configuration is instantaneously aligned as shown, we have



$$\boldsymbol{\omega} = \omega_2\, \boldsymbol{i} + \omega_1 \cos\phi\, \boldsymbol{j} + (\omega_1 \sin\phi + \omega_3)\, \boldsymbol{k}\ .$$

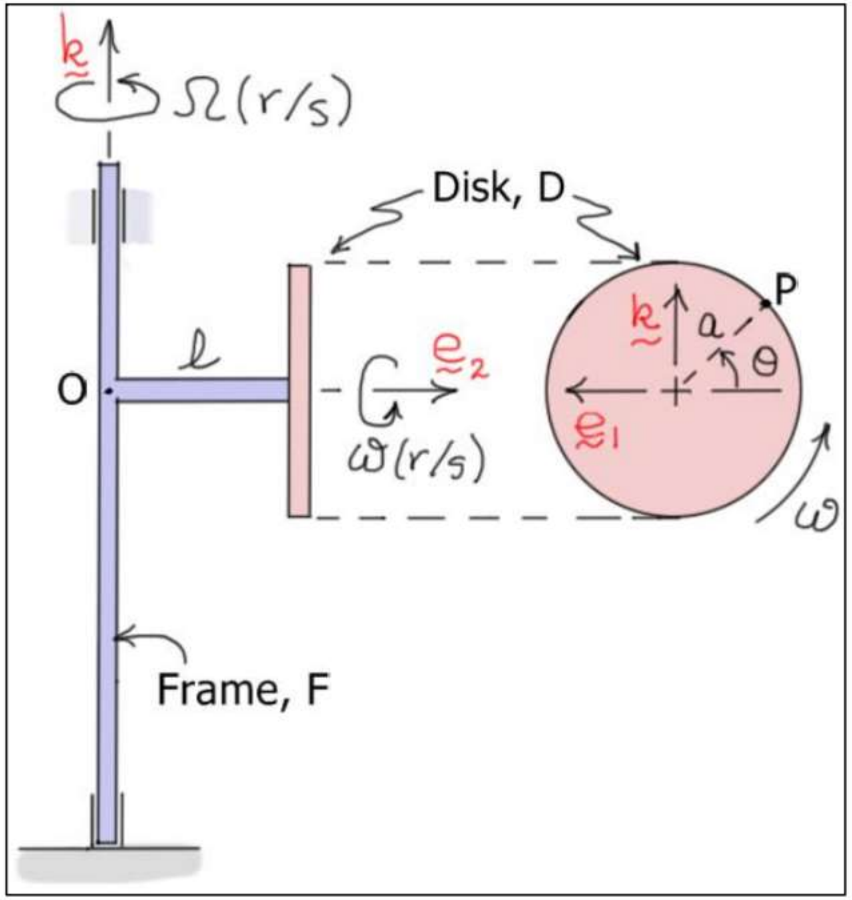
Here, φ is the angle between MM' and the y axis

At first sight, it seems that we should be able to express a rotation as a vector which has a direction along the axis of rotation and a magnitude that is equal to the angle of rotation. Unfortunately, if we consider two such rotation vectors, $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$, not only would the combined rotation $\boldsymbol{\theta}$ be different from $\boldsymbol{\theta}_1+\boldsymbol{\theta}_2$, but in general $\boldsymbol{\theta}_1+\boldsymbol{\theta}_2 \neq \boldsymbol{\theta}_2+\boldsymbol{\theta}_1$. This situation is illustrated in the figure, in which we consider a 3D rigid body undergoing two 90° rotations about the x and y axis.



Example 4:

The system shown consists of two connected bodies – the frame ***F*** and the disk ***D***. Frame *F* rotates at a rate of Ω(rad/s) about the fixed vertical direction (annotated by the unit vector k). Disk ***D*** is affixed-to and rotates relative to ***F*** at a rate of ω(rad/s) about the horizontal arm of *F* (annotated by the rotating unit vector e_2).



Reference frames:

$$R:(\underset{\sim}{i},\ \underset{\sim}{j},\ \underset{\sim}{k})$$

$$F:(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{k})$$

Find: (express the results using unit vectors fixed in F)

a) the angular velocity of disk ***D*** in ***R*** $\quad {}^{R}\underset{\sim}{\omega}_{D}$

b) the angular acceleration of disk ***D*** in ***R*** $\quad {}^{R}\underset{\sim}{\alpha}_{D}$

Solution:

a) Using the summation rule: $\boxed{{}^{R}\underset{\sim}{\omega}_{D} = {}^{F}\underset{\sim}{\omega}_{D} + {}^{R}\underset{\sim}{\omega}_{F} = \omega\, \underset{\sim}{e}_{2} + \Omega\, \underset{\sim}{k}}$ (rad/s)

b) The angular acceleration is found by ***direct differentiation***.

$$
\begin{aligned}
{}^{R}\underset{\sim}{\alpha}_{D} &= \frac{{}^{R}d}{dt}\left({}^{R}\underset{\sim}{\omega}_{D}\right) = \dot{\omega}\, \underset{\sim}{e}_{2} + \omega \frac{{}^{R}d}{dt}\left(\underset{\sim}{e}_{2}\right) + \dot{\Omega}\, \underset{\sim}{k} + \underbrace{\Omega \frac{{}^{R}d}{dt}\left(\underset{\sim}{k}\right)}_{\text{zero}} \\
&= \dot{\omega}\, \underset{\sim}{e}_{2} + \omega\left({}^{R}\underset{\sim}{\omega}_{F} \times \underset{\sim}{e}_{2}\right) + \dot{\Omega}\, \underset{\sim}{k} = \dot{\omega}\, \underset{\sim}{e}_{2} + \omega\left(\Omega\, \underset{\sim}{k} \times \underset{\sim}{e}_{2}\right) + \dot{\Omega}\, \underset{\sim}{k}
\end{aligned}
$$

So,

$$\boxed{{}^{R}\underset{\sim}{\alpha}_{D} = -\omega\, \Omega\, \underset{\sim}{e}_{1} + \dot{\omega}\, \underset{\sim}{e}_{2} + \dot{\Omega}\, \underset{\sim}{k}} \ (\text{rad/s}^2)$$

Example 5

Velocity and acceleration in 3-D: The rod shown in the figure rotates about the y-axis at angular speed 10 rad / s and accelerates at the rate of 2 rad/s2. The dimensions of the rod are $L = h = 2$ m and $r = 1$ m. There is a small mass $\boldsymbol{P}$ glued to the rod at its free end.

At the instant shown, the three segments of the rod are parallel to the three axes.

(a) Find the velocity of point $\boldsymbol{P}$ at the instant shown.

(b) Find the acceleration of point $\boldsymbol{P}$ at the instant shown