University Center Abdelhafid Boussouf Mila
Institute of Mathematics and Computer Science
1st Year Mathematics (Algebra 1)

Solution to Tutorial Series No. 2
Sets, relations and functions

Solution0 1.

- *Union of A and B:*
$$A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$$
- *Intersection of B and C:*
$$B \cap C = \{4, 6\}$$
- *Set difference $A - B$:*
$$A - B = \{1, 2\}$$
- *Symmetric difference of A and C:*
$$A \Delta C = \{1, 3, 5, 8, 10\}$$

Solution0 2.

1. $a \in E$: *True. Since* $E = \{a, b, c\}$, *a is an element of E.*
2. $a \subset E$: *False. a is not a subset of E;* $\{a\}$ *is a subset of E.*
3. $\{a\} \subset E$: *True.* $\{a\}$ *is a subset of E because* $a \in E$.
4. $\emptyset \in E$: *False.* $\emptyset$ *(empty set) is not an element of E.*
5. $\emptyset \subset E$: *True. The empty set* $\emptyset$ *is a subset of every set, including E.*
6. $\{\emptyset\} \subset E$: *False.* $\{\emptyset\}$ *is not a subset of E because* $\emptyset \notin E$.

Solution0 3.

1. $A \setminus B = A \cap B^c$ *By definition:*
$$A \setminus B = \{x \in A \mid x \notin B\},$$
and on the other hand:
$$A \cap B^c = \{x \in A \mid x \in B^c\} = \{x \in A \mid x \notin B\}.$$
Thus:
$$A \setminus B = A \cap B^c.$$

2. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Using the distributive property of intersection over union:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

3. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Using the distributive property of union over intersection:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

This can also be verified using element-based reasoning: If $x \in A \cup (B \cap C)$*, then* $x \in A$ *or* $x \in B \cap C$*. If* $x \in B \cap C$*, then* $x \in B$ *and* $x \in C$*, so* $x \in A \cup B$ *and* $x \in A \cup C$*. Conversely, if* $x \in (A \cup B) \cap (A \cup C)$*, then* $x \in A \cup B$ *and* $x \in A \cup C$*. This implies* $x \in A$*, or* $x \in B$ *and* $x \in C$*, so* $x \in A \cup (B \cap C)$*.*

4. $A \triangle B = (A \cup B) \setminus (A \cap B)$

By the definition of symmetric difference:

$$A \triangle B = (A \setminus B) \cup (B \setminus A).$$

Using part (1):

$$A \setminus B = A \cap B^c \quad \text{and} \quad B \setminus A = B \cap A^c.$$

Thus:

$$A \triangle B = (A \cap B^c) \cup (B \cap A^c).$$

On the other hand:

$$(A \cup B) \setminus (A \cap B) = (A \cup B) \cap (A \cap B)^c.$$

Since $(A \cap B)^c = A^c \cup B^c$*, we have:*

$$(A \cup B) \setminus (A \cap B) = (A \cup B) \cap (A^c \cup B^c).$$

Using the distributive property:

$$(A \cup B) \cap (A^c \cup B^c) = [(A \cup B) \cap A^c] \cup [(A \cup B) \cap B^c].$$

Simplifying each term:

$$(A \cup B) \cap A^c = (A \cap A^c) \cup (B \cap A^c) = B \cap A^c,$$

$$(A \cup B) \cap B^c = (A \cap B^c) \cup (B \cap B^c) = A \cap B^c.$$

Thus:

$$(A \cup B) \setminus (A \cap B) = (A \cap B^c) \cup (B \cap A^c).$$

Therefore:

$$A \triangle B = (A \cup B) \setminus (A \cap B).$$

Solution0 4.

The Power Set $\mathcal{P}(E)$

The power set $\mathcal{P}(E)$ *of a set* E *is the set of all subsets of* E*, including the empty set and* E *itself. For* $E = \{a, b, c, d\}$*, the power set* $\mathcal{P}(E)$ *is:*

$$\mathcal{P}(E) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \\ \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, E\}.$$

In total, $\mathcal{P}(E)$ *contains* 2^n *subsets, where* $n = 4$ *is the number of elements in* E*. Thus,* $|\mathcal{P}(E)| = 2^4 = 16$.

Example of a Partition of E

A partition of E *is a collection of non-empty, pairwise disjoint subsets of* E *whose union equals* E*. An example of a partition of* E *is:*

$$\mathcal{P}_1 = \{\{a, b\}, \{c\}, \{d\}\}.$$

Verify:

- *Each subset is non-empty:* $\{a, b\}, \{c\}, \{d\} \neq \emptyset$,
- *The subsets are pairwise disjoint:*

  $$\{a, b\} \cap \{c\} = \emptyset, \quad \{a, b\} \cap \{d\} = \emptyset, \quad \{c\} \cap \{d\} = \emptyset,$$

- *The union of all subsets equals* E*:*

  $$\{a, b\} \cup \{c\} \cup \{d\} = \{a, b, c, d\} = E.$$

Thus, $\mathcal{P}_1 = \{\{a, b\}, \{c\}, \{d\}\}$ *is a valid partition of* E*.*

Solution0 5.

1. ***Images and Pre-images under*** $f(x) = \sin(x)$***:***

   (a) *The image of* $\mathbb{R}$ *under* $f(x) = \sin(x)$ *is:*

   $$f(\mathbb{R}) = [-1, 1],$$

   because the sine function oscillates between -1 *and* 1 *for all real* x*.*

   (b) *The image of* $[0, 2\pi]$ *under* $f(x) = \sin(x)$ *is:*

   $$f([0, 2\pi]) = [-1, 1],$$

   because $\sin(x)$ *completes one full cycle in the interval* $[0, 2\pi]$*.*

   (c) *The image of* $[0, \frac{\pi}{2}]$ *under* $f(x) = \sin(x)$ *is:*

   $$f([0, \frac{\pi}{2}]) = [0, 1],$$

   because the sine function is strictly increasing from 0 *to* 1 *in this interval.*

   (d) *The inverse image of* $[0, 1]$ *under* $f(x) = \sin(x)$ *is:*

   $$f^{-1}([0, 1]) = \bigcup_{k \in \mathbb{Z}} [2k\pi, 2k\pi + \pi],$$

   as sine is periodic with period 2π*.*

(e) *The inverse image of* $[3,4]$ *under* $f(x) = \sin(x)$ *is:*

$$f^{-1}([3,4]) = \emptyset,$$

because $\sin(x) \notin [3,4]$ *for any* $x \in \mathbb{R}$.

(f) *The inverse image of* $[1,2]$ *under* $f(x) = \sin(x)$ *is:*

$$f^{-1}([1,2]) = f^{-1}(\{1\}) = \bigcup_{k \in \mathbb{Z}} \left\{ \frac{\pi}{2} + 2k\pi \right\},$$

because $\sin(x) = 1$ *occurs only at* $x = \frac{\pi}{2} + 2k\pi$ *for* $k \in \mathbb{Z}$, *and* $\sin(x) \notin (1,2]$.

2. ***Comparison of*** $f(A \setminus B)$ ***and*** $f(A) \setminus f(B)$***:***
Let $f(x) = x^2 + 1$, $A = [-3,2]$, *and* $B = [0,4]$*:*

(a) *The set* $A \setminus B = [-3,0)$, *as* $B = [0,4]$ *removes* $[0,4]$ *from* A.

(b) *The image of* $A \setminus B$ *under* $f(x)$*:*

$$f(A \setminus B) = f([-3,0)) = (1,10],$$

because $f(x) = x^2 + 1$ *is increasing on* $[0,\infty)$ *and symmetric about* $x = 0$.

(c) *The image of* A *under* $f(x)$*:*

$$f(A) = f([-3,2]) = [1,10],$$

and the image of B *under* $f(x)$*:*

$$f(B) = f([0,4]) = [1,17].$$

(d) *The set* $f(A) \setminus f(B)$ *is:*

$$f(A) \setminus f(B) = [1,10] \setminus [1,17] = \emptyset.$$

Comparing:

$$f(A \setminus B) = [1,10), \quad f(A) \setminus f(B) = \emptyset.$$

Thus, $f(A \setminus B) \neq f(A) \setminus f(B)$.

3. ***Condition for*** $f(A \setminus B) = f(A) \setminus f(B)$***:***
For $f(A \setminus B) = f(A) \setminus f(B)$ *to hold, the function* f *must be **injective** (one-to-one). Injectivity ensures that elements in* $A \setminus B$ *map uniquely to* $f(A \setminus B)$, *without overlap from elements in* B.

Solution0 6.

1. $E = \mathbb{Z}$ *and* $x\mathcal{R}y \Leftrightarrow |x| = |y|$*:*

- ***Reflexive****: Yes, since* $|x| = |x|$ *for all* $x \in \mathbb{Z}$.
- ***Symmetric****: Yes, since* $|x| = |y| \implies |y| = |x|$.
- ***Antisymmetric****: No, because* $|x| = |y|$ *does not imply* $x = y$ *(e.g.,* $x = 3, y = -3$*).*
- ***Transitive****: Yes, since* $|x| = |y|$ *and* $|y| = |z|$ *imply* $|x| = |z|$.
- ***Type****: This is an* **equivalence relation***, not an order.*

2. $E = \mathbb{R} \setminus \{0\}$ and $x\mathcal{R}y \Leftrightarrow xy > 0$:

- ***Reflexive***: Yes, since $x \cdot x > 0$ for all $x \neq 0$.
- ***Symmetric***: Yes, since $xy > 0 \implies yx > 0$.
- ***Antisymmetric***: No, because $xy > 0$ does not imply $x = y$ (e.g., $x = 1, y = 2$).
- ***Transitive***: Yes, since $xy > 0$ and $yz > 0$ imply $xz > 0$.
- ***Type***: This is an **equivalence relation**, not an order.

3. $E = \mathbb{Z}$ and $x\mathcal{R}y \Leftrightarrow x - y$ is even:

- ***Reflexive***: Yes, since $x - x = 0$, which is even.
- ***Symmetric***: Yes, since $x - y$ even implies $y - x$ is even.
- ***Antisymmetric***: No, because $x - y$ even does not imply $x = y$ (e.g., $x = 2, y = 4$).
- ***Transitive***: Yes, since $x - y$ even and $y - z$ even imply $x - z$ is even.
- ***Type***: This is an **equivalence relation**, not an order.

Summary

- $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ are all ***equivalence relations***.
- None of them is an ***order*** because they fail antisymmetry.

Solution0 7.

1. $E = \mathbb{R}$ and $x\mathcal{R}y \Leftrightarrow x = -y$:

- ***Reflexive***: No, since $x = -x$ only holds for $x = 0$, so it is not reflexive.
- ***Symmetric***: Yes, since $x = -y \implies y = -x$.
- ***Antisymmetric***: No, because $x = -y$ and $y = -x$ do not imply $x = y$ (e.g., $x = 1, y = -1$).
- ***Transitive***: No, because $x = -y$ and $y = -z$ imply $x = -(-z) = z$, which contradicts the original definition unless $x = 0$ or $z = 0$.
- ***Type***: This is ***not an equivalence relation*** because it is not reflexive, and it is ***not an order*** because it is not antisymmetric.

2. $E = \mathbb{R}$ and $x\mathcal{R}y \Leftrightarrow \cos^2(x) + \sin^2(y) = 1$:

- ***Reflexive***: Yes, since $\cos^2(x) + \sin^2(x) = 1$ for all $x \in \mathbb{R}$.
- ***Symmetric***: Yes, since $\cos^2(x) + \sin^2(y) = 1 \implies \cos^2(y) + \sin^2(x) = 1$.
- ***Antisymmetric***: No, because $\cos^2(x) + \sin^2(y) = 1$ and $\cos^2(y) + \sin^2(x) = 1$ do not imply $x = y$.
- ***Transitive***: Yes, since $\cos^2(x)+\sin^2(y) = 1$ and $\cos^2(y)+\sin^2(z) = 1$ imply $\cos^2(x)+\sin^2(z) = 1$.
- ***Type***: This is ***an equivalence relation*** but ***not an order***.

3. $E = \mathbb{N}$ and $x\mathcal{R}y \Leftrightarrow \exists p, q \geq 1$ such that $y = px^q$ (where $p, q \in \mathbb{Z}$):

- ***Reflexive***: *Yes, since* $x = px^q$ *holds for* $p = 1, q = 1$, *implying* $x\mathcal{R}x$.
- ***Symmetric***: *No, since* $y = px^q$ *does not imply* $x = py^q$.
- ***Antisymmetric***: *Yes, because if* $y = px^q$ *and* $x = p'y^{q'}$, *then* $x = y$.
- ***Transitive***: *Yes, since if* $y = px^q$ *and* $z = p'y^{q'}$, *then* $z = (pp')x^{qq'}$.
- ***Type***: *This is a* ***partial order***, *not an equivalence relation.*

Solution0 8.

1. ***Relation*** $\sim_1$: $x \sim_1 y$ *if and only if* $x + y$ *is even.*

   Reflexive: *Yes, because* $x + x = 2x$ *is always even for any* $x \in \mathbb{Z}$.

   Symmetric: *Yes, because if* $x + y$ *is even, then* $y + x = x + y$, *which is also even.*

   Transitive: *Yes, because if* $x+y$ *is even and* $y+z$ *is even, then* $(x+y)+(y+z) = x+2y+z$ *is even, implying that* $x + z$ *is even.*

   Equivalence Classes: *The equivalence classes are:*

   $$\dot{0} = \{x \in \mathbb{Z} \mid x \text{ is even}\}, \quad \dot{1} = \{x \in \mathbb{Z} \mid x \text{ is odd}\}.$$

2. ***Relation*** $\sim_2$: $x \sim_2 y$ *if and only if* x *and* y *have the same remainder when divided by 5.*

   Reflexive: *Yes, because* $x \mod 5 = x \mod 5$ *for any* $x \in \mathbb{Z}$.

   Symmetric: *Yes, because if* $x \mod 5 = y \mod 5$, *then* $y \mod 5 = x \mod 5$.

   Transitive: *Yes, because if* $x \mod 5 = y \mod 5$ *and* $y \mod 5 = z \mod 5$, *then* $x \mod 5 = z \mod 5$.

   Equivalence Classes: *The equivalence classes are:*

   $$\dot{0} = \{x \in \mathbb{Z} \mid x \equiv 0 \pmod 5\}, \quad \dot{1} = \{x \in \mathbb{Z} \mid x \equiv 1 \pmod 5\}, \quad \dot{2} = \{x \in \mathbb{Z} \mid x \equiv 2 \pmod 5\},$$

   $$\dot{3} = \{x \in \mathbb{Z} \mid x \equiv 3 \pmod 5\}, \quad \dot{4} = \{x \in \mathbb{Z} \mid x \equiv 4 \pmod 5\}.$$

3. ***Relation*** $\sim_3$: $x \sim_3 y$ *if and only if* $x - y$ *is a multiple of 7.*

   Reflexive: *Yes, because* $x - x = 0$ *is a multiple of 7 for any* $x \in \mathbb{Z}$.

   Symmetric: *Yes, because if* $x - y$ *is a multiple of 7, then* $y - x = -(x - y)$ *is also a multiple of 7.*

   Transitive: *Yes, because if* $x-y$ *and* $y-z$ *are multiples of 7, then* $(x-y)+(y-z) = x-z$ *is also a multiple of 7.*

   Equivalence Classes: *The equivalence classes are:*

   $$\dot{0} = \{x \in \mathbb{Z} \mid x \equiv 0 \pmod 7\}, \quad \dot{1} = \{x \in \mathbb{Z} \mid x \equiv 1 \pmod 7\}, \quad \dot{2} = \{x \in \mathbb{Z} \mid x \equiv 2 \pmod 7\},$$

   $$\dot{3} = \{x \in \mathbb{Z} \mid x \equiv 3 \pmod 7\}, \quad \dot{4} = \{x \in \mathbb{Z} \mid x \equiv 4 \pmod 7\}, \quad \dot{5} = \{x \in \mathbb{Z} \mid x \equiv 5 \pmod 7\},$$

   $$\dot{6} = \{x \in \mathbb{Z} \mid x \equiv 6 \pmod 7\}.$$

Solution0 9.
We will prove the equivalence in two directions.

(1) If $x\mathcal{R}y$, then $\dot{x} = \dot{y}$:

Since $\mathcal{R}$ is an equivalence relation, it satisfies three properties: reflexivity, symmetry, and transitivity. By the definition of an equivalence relation, if $x\mathcal{R}y$, then x and y belong to the same equivalence class, denoted $\dot{x} = \dot{y}$. This means that the equivalence classes of x and y are identical.

$$x\mathcal{R}y \quad \Rightarrow \quad \dot{x} = \dot{y}.$$

(2) If $\dot{x} = \dot{y}$, then $x\mathcal{R}y$:

If $\dot{x} = \dot{y}$, then by the definition of equivalence classes, x and y belong to the same equivalence class. Therefore, by the properties of an equivalence relation, $x\mathcal{R}y$.

$$\dot{x} = \dot{y} \quad \Rightarrow \quad x\mathcal{R}y.$$

Thus, we have shown both directions, completing the proof.

Solution0 10.
Let $\mathbb{N}^*$ denote the set of positive integers. Define the relation $\mathcal{R}$ on $\mathbb{N}^*$ by $x\mathcal{R}y$ if and only if x divides y.

1. ***Show that $\mathcal{R}$ is a partial order relation on $\mathbb{N}^*$:***

   To show that $\mathcal{R}$ is a partial order, we need to verify that it is reflexive, antisymmetric, and transitive.

   - ***Reflexive:*** *For any $x \in \mathbb{N}^*$, x divides itself, i.e., $x\mathcal{R}x$.*
   - ***Antisymmetric:*** *If $x\mathcal{R}y$ and $y\mathcal{R}x$, then x divides y and y divides x. This implies that $x = y$, because the only way two distinct positive integers can divide each other is if they are equal.*
   - ***Transitive:*** *If $x\mathcal{R}y$ and $y\mathcal{R}z$, then x divides y and y divides z. This implies that x divides z, so $x\mathcal{R}z$.*

   Since $\mathcal{R}$ is reflexive, antisymmetric, and transitive, it is a partial order on $\mathbb{N}^*$.

2. ***Is $\mathcal{R}$ a total order relation?***

   A relation is a total order if it is a partial order and, for any two elements x and y in $\mathbb{N}^*$, either $x\mathcal{R}y$ or $y\mathcal{R}x$ holds. In this case, $\mathcal{R}$ is not a total order because, for example, 2 and 3 do not divide each other, so neither $2\mathcal{R}3$ nor $3\mathcal{R}2$ holds. Therefore, $\mathcal{R}$ is not a total order.

3. ***Describe the sets $\{x \in \mathbb{N}^* \mid x\mathcal{R}5\}$ and $\{x \in \mathbb{N}^* \mid 5\mathcal{R}x\}$:***

   - *The set $\{x \in \mathbb{N}^* \mid x\mathcal{R}5\}$ is the set of all positive integers that divide 5. The divisors of 5 are 1 and 5, so:*
     $$\{x \in \mathbb{N}^* \mid x\mathcal{R}5\} = \{1, 5\}.$$
   - *The set $\{x \in \mathbb{N}^* \mid 5\mathcal{R}x\}$ is the set of all positive integers divisible by 5. This set is:*
     $$\{x \in \mathbb{N}^* \mid 5\mathcal{R}x\} = \{5, 10, 15, 20, 25, \dots\}.$$

4. ***Does $\mathbb{N}^*$ have a least element? A greatest element?***

- ***Least element:*** *The least element in $\mathbb{N}^*$ with respect to the relation $\mathcal{R}$ is 1, because 1 divides all positive integers. Therefore, 1 is the least element.*
- ***Greatest element:*** *The greatest element in $\mathbb{N}^*$ with respect to the relation $\mathcal{R}$ does not exist because there is no single integer that is divisible by all positive integers. Thus, there is no greatest element.*

Solution0 11.
Let f be the function from $\mathbb{R}$ to $\mathbb{R}$ defined by $f(x) = x^2 + x - 2$.

1. ***Definition of** $f^{-1}(\{4\})$**:*** *The set $f^{-1}(\{4\})$ is the preimage of $\{4\}$ under f, i.e., it consists of all $x \in \mathbb{R}$ such that $f(x) = 4$.*

$$f(x) = 4 \quad \Rightarrow \quad x^2 + x - 2 = 4$$

Solving this equation:

$$x^2 + x - 6 = 0$$

Factorizing:

$$(x-2)(x+3) = 0$$

Thus, $x = 2$ or $x = -3$. Therefore, $f^{-1}(\{4\}) = \{2, -3\}$.

2. ***Is the function f bijective?***

Injectivity: f is not bijective because f is not injective.

Surjectivity: For surjectivity, we would need to show that for every $y \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that $f(x) = y$. However, since the function is quadratic and opens upwards, it is not surjective over $\mathbb{R}$. Specifically, $f(x) = x^2 + x - 2$ has a minimum value, but no maximum, meaning it cannot take all real values. Therefore, f is not surjective.

Since f is neither injective nor surjective, it is not bijective.

3. ***Definition of** $f([-1,1])$**:*** *The set $f([-1,1])$ is the image of the interval $[-1,1]$ under the function f, i.e., it is the set of all values $f(x)$ for $x \in [-1,1]$.*

To calculate $f([-1,1])$, we need to find the minimum and maximum values of $f(x) = x^2 + x - 2$ on the interval $[-1,1]$.

First, evaluate $f(x)$ at the endpoints of the interval:

$$f(-1) = (-1)^2 + (-1) - 2 = 1 - 1 - 2 = -2$$
$$f(1) = 1^2 + 1 - 2 = 1 + 1 - 2 = 0$$

Next, compute the derivative of $f(x)$:

$$f'(x) = 2x + 1$$

Setting $f'(x) = 0$ to find critical points:

$$2x + 1 = 0 \quad \Rightarrow \quad x = -\frac{1}{2}$$

Since $-\frac{1}{2} \in [-1,1]$, we evaluate f at $x = -\frac{1}{2}$:

$$f\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right) - 2 = \frac{1}{4} - \frac{1}{2} - 2 = -\frac{9}{4}$$

Thus, the minimum value of $f(x)$ on $[-1,1]$ is $-\frac{9}{4}$, and the maximum value is 0.

Therefore, $f([-1,1]) = \left[-\frac{9}{4}, 0\right]$.

4. ***Definition of*** $f^{-1}([-2,4])$***:*** *The set* $f^{-1}([-2,4])$ *is the preimage of the interval* $[-2,4]$*, i.e., it consists of all* $x \in \mathbb{R}$ *such that* $f(x) \in [-2,4]$*.*

   We need to solve for x *such that* $-2 \leq f(x) = x^2 + x - 2 \leq 4$*.*

   First, solve $f(x) \geq -2$*:*

$$x^2 + x - 2 \geq -2 \quad \Rightarrow \quad x^2 + x \geq 0$$

   Factoring:

$$x(x+1) \geq 0$$

   This inequality holds when $x \leq -1$ *or* $x \geq 0$*.*

   Next, solve $f(x) \leq 4$*:*

$$x^2 + x - 2 \leq 4 \quad \Rightarrow \quad x^2 + x - 6 \leq 0$$

   Factoring:

$$(x-2)(x+3) \leq 0$$

   This inequality holds when $-3 \leq x \leq 2$*.*

   Combining the two results, we have:

$$-3 \leq x \leq -1 \quad \textit{or} \quad 0 \leq x \leq 2$$

   Therefore, the set $f^{-1}([-2,4]) = [-3,-1] \cup [0,2]$*.*

Solution0 12.

1. ***Injectivity:***
   A function f *is injective if* $f(x_1) = f(x_2)$ *implies* $x_1 = x_2$*. Let's assume* $f(x_1) = f(x_2)$*, which gives:*

$$\frac{2x_1}{1+x_1^2} = \frac{2x_2}{1+x_2^2}.$$

   Simplifying this equation:

$$x_1(1+x_2^2) = x_2(1+x_1^2),$$

   which does not necessarily imply $x_1 = x_2$*. Therefore, the function is* ***not injective****.*

   Counterexample: *Take* $x_1 = 2$ *and* $x_2 = \frac{1}{2}$*:*

$$f(2) = \frac{4}{5}, \quad f\left(\frac{1}{2}\right) = \frac{4}{5}.$$

   Clearly, $f(2) = f\left(\frac{1}{2}\right)$*, but* $2 \neq \frac{1}{2}$*, proving that the function is not injective.*

2. ***Surjectivity:***
   A function f *is surjective if for every* $y \in \mathbb{R}$*, there exists an* $x \in \mathbb{R}$ *such that* $f(x) = y$*. We know that the function* $f(x) = \frac{2x}{1+x^2}$ *has a maximum at* $x = 1$ *where* $f(1) = 1$ *and a minimum at* $x = -1$ *where* $f(-1) = -1$*, and as* $x \to \pm\infty$*,* $f(x) \to 0$*. Therefore, the range of* $f(x)$ *is* $(-1,1)$*, and the function is* ***not surjective*** *because it cannot take values outside of this interval.*

3. ***Range of*** $f(x)$***:***
We now show that the range of $f(x) = \frac{2x}{1+x^2}$ *is* $[-1, 1]$*. To do this, we need to find the maximum and minimum values of* $f(x)$*.*

First, we calculate the derivative of $f(x)$*:*

$$f'(x) = \frac{(1+x^2)(2) - 2x(2x)}{(1+x^2)^2} = \frac{2(1-x^2)}{(1+x^2)^2}.$$

Setting $f'(x) = 0$ *gives:*

$$1 - x^2 = 0 \quad \Rightarrow \quad x = \pm 1.$$

Evaluating $f(x)$ *at* $x = 1$ *and* $x = -1$*:*

$$f(1) = \frac{2 \times 1}{1 + 1^2} = 1, \quad f(-1) = \frac{2 \times (-1)}{1 + (-1)^2} = -1.$$

As $x \to \pm\infty$*,* $f(x) \to 0$*. Therefore, the range of* $f(x)$ *is* $[-1, 1]$*, so we have shown that:*

$$f(\mathbb{R}) = [-1, 1].$$

4. ***Restriction*** $g(x) = f(x)$ ***on*** $[-1, 1]$***:***
Now, we need to show that the restriction of f *to* $[-1, 1]$*, which we denote by* $g(x) = f(x)$*, is a bijection.*

Injectivity: *Since the derivative* $f'(x)$ *is positive over the entire interval* $[-1, 1]$*, the function* $f(x) = \frac{2x}{1+x^2}$ *is strictly increasing on this interval.*

Therefore, the function $f(x)$ *is* ***injective*** *on* $[-1, 1]$*.*

Surjectivity: *The range of* $f(x)$ *on* $[-1, 1]$ *is* $[-1, 1]$*, so the restriction* $g(x)$ *is surjective.*

Since $g(x)$ *is both injective and surjective, it is a bijection.*

Solution0 13.
Let $f : E \to F$*,* $g : F \to G$*, and* $h = g \circ f$*.*

1. ***Injectivity of*** f***:*** *Show that if* h *is injective, then* f *is injective. Also, show that if* h *is surjective, then* g *is surjective.*

Proof:

1.1 Injectivity of f*: Assume that* $h = g \circ f$ *is injective. To show that* f *is injective, we need to prove that for any* $x_1, x_2 \in E$*, if* $f(x_1) = f(x_2)$*, then* $x_1 = x_2$*.*

Since $h(x_1) = g(f(x_1))$ *and* $h(x_2) = g(f(x_2))$*, if* $f(x_1) = f(x_2)$*, then*

$$h(x_1) = h(x_2).$$

Since h *is injective, it follows that*

$$x_1 = x_2.$$

Hence, f *is injective.*

1.2 Surjectivity of g*: Assume that* $h = g \circ f$ *is surjective. To show that* g *is surjective, we need to prove that for every* $y \in G$*, there exists some* $x \in F$ *such that* $g(x) = y$*.*

Since h *is surjective, for each* $y \in G$*, there exists* $x \in E$ *such that* $h(x) = g(f(x)) = y$*. Therefore, for every* $y \in G$*, we can find an* $x \in F$ *such that* $g(x) = y$*, which proves that* g *is surjective.*

2. ***Surjectivity of*** f***:*** *Show that if* h *is surjective and* g *is injective, then* f *is surjective.*

   Proof:

   Assume that h *is surjective and* g *is injective. To prove that* f *is surjective, we need to show that for every* $y \in F$*, there exists some* $x \in E$ *such that* $f(x) = y$*.*

   Since h *is surjective, for each* $y \in G$*, there exists* $z \in E$ *such that* $h(z) = g(f(z)) = y$*. Since* g *is injective, there exists a unique* $x \in F$ *such that* $f(x) = y$*, which implies that* f *is surjective.*

3. ***Injectivity of*** g***:*** *Show that if* h *is injective and* f *is surjective, then* g *is injective.*

   Proof:

   Assume that h *is injective and* f *is surjective. To show that* g *is injective, we need to prove that if* $g(x_1) = g(x_2)$*, then* $x_1 = x_2$*.*

   Since $h(x_1) = g(f(x_1))$ *and* $h(x_2) = g(f(x_2))$*, if* $g(x_1) = g(x_2)$*, we have*

   $$h(x_1) = h(x_2).$$

   Since h *is injective, it follows that*

   $$f(x_1) = f(x_2).$$

   Since f *is surjective, there exists some* $x \in F$ *such that* $f(x) = y$*, and therefore,* $g(x_1) = g(x_2)$*.*

   Hence, g *is injective.*