

Series of Tutorial No. 2
Sets, relations and functions

Exercise 1.

Consider the following sets: $A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5, 6, 7\}$ and $C = \{2, 4, 6, 8, 10\}$ Find the following sets: $(A \cup B)$, $(B \cap C)$, $(A - B)$ and $(A \Delta C)$

Exercise 2.

Let $E = \{a, b, c\}$ be a set. Can we write: 1) $a \in E$, 2) $a \subset E$, 3) $\{a\} \subset E$, 4) $\emptyset \in E$, 5) $\emptyset \subset E$, 6) $\{\emptyset\} \subset E$?

Exercise 3.

Let A , B , and C be three subsets of a set E . Prove that:

1. $A \setminus B = A \cap B^c$.
2. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
3. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
4. $A \Delta B = (A \cup B) \setminus (A \cap B)$.

Exercise 4. (Power set)

Let $E = \{a, b, c, d\}$. Find the power set $\mathcal{P}(E)$ of E , which is the set of all subsets of E . Give an example of a partition of E , which is a collection of non-empty disjoint subsets of E whose union equals E .

Exercise 5.

1. What is the image of the sets \mathbb{R} , $[0, 2\pi]$, $[0, \frac{\pi}{2}]$, and the inverse image of the sets $[0, 1]$, $[3, 4]$, $[1, 2]$ under the function $f(x) = \sin(x)$?
2. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x^2 + 1$. Consider the sets $A = [-3, 2]$ and $B = [0, 4]$. Compare $f(A \setminus B)$ and $f(A) \setminus f(B)$.
3. What condition must function f satisfy so that $f(A \setminus B) = f(A) \setminus f(B)$?

Exercise 6.

Determine whether the following relations are reflexive, symmetric, antisymmetric, and transitive:

1. $E = \mathbb{Z}$ and $x \mathcal{R} y \Leftrightarrow |x| = |y|$.
2. $E = \mathbb{R} \setminus \{0\}$ and $x \mathcal{R} y \Leftrightarrow xy > 0$.
3. $E = \mathbb{Z}$ and $x \mathcal{R} y \Leftrightarrow x - y$ is even.

Identify among the above examples which relations are orders and which are equivalence relations.

Exercise 7.

Determine whether the following relations are reflexive, symmetric, antisymmetric, and transitive:

1. $E = \mathbb{R}$ and $x\mathcal{R}y \Leftrightarrow x = -y$.
2. $E = \mathbb{R}$ and $x\mathcal{R}y \Leftrightarrow \cos^2(x) + \sin^2(y) = 1$.
3. $E = \mathbb{N}$ and $x\mathcal{R}y \Leftrightarrow \exists p, q \geq 1$ such that $y = px^q$ (where p and q are integers).

Identify among the above examples which relations are orders and which are equivalence relations.

Exercise 8.

Let $E = \mathbb{Z}$, the set of all integers. Consider the following relations on E :

1. Relation \sim_1 defined by $x \sim_1 y$ if and only if $x + y$ is even.
2. Relation \sim_2 defined by $x \sim_2 y$ if and only if x and y have the same remainder when divided by 5.
3. Relation \sim_3 defined by $x \sim_3 y$ if and only if $x - y$ is a multiple of 7.

For each relation \sim_i (where $i = 1, 2, 3$):

1. Determine if \sim_i is an equivalence relation on \mathbb{Z} . Explain why or why not.
2. If \sim_i is an equivalence relation, identify the equivalence classes of \mathbb{Z} under \sim_i .

Exercise 9.

Let \mathcal{R} be an equivalence relation on a non-empty set E . Show that

$$\forall x, y \in E, \quad x\mathcal{R}y \Leftrightarrow \dot{x} = \dot{y}.$$

Exercise 10.

Let \mathbb{N}^* denote the set of positive integers. Define the relation \mathcal{R} on \mathbb{N}^* by $x\mathcal{R}y$ if and only if x divides y .

1. Show that \mathcal{R} is a partial order relation on \mathbb{N}^* .
2. Is \mathcal{R} a total order relation?
3. Describe the sets $\{x \in \mathbb{N}^* \mid x\mathcal{R}5\}$ and $\{x \in \mathbb{N}^* \mid 5\mathcal{R}x\}$.
4. Does \mathbb{N}^* have a least element? A greatest element?

Exercise 11.

Let f be the function from \mathbb{R} to \mathbb{R} defined by $f(x) = x^2 + x - 2$.

1. Give the definition of $f^{-1}(\{4\})$. Calculate $f^{-1}(\{4\})$.
2. Is the function f bijective?
3. Give the definition of $f([-1, 1])$. Calculate $f([-1, 1])$.
4. Give the definition of $f^{-1}([-2, 4])$. Calculate $f^{-1}([-2, 4])$.

Exercise 12.

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = \frac{2x}{1+x^2}$.

1. Is f injective? Is f surjective?
2. Show that $f(\mathbb{R}) = [-1, 1]$.
3. Show that the restriction $g : [-1, 1] \rightarrow [-1, 1]$ defined by $g(x) = f(x)$ is a bijection.

Exercise 13.

Let $f : E \rightarrow F$, $g : F \rightarrow G$, and $h = g \circ f$.

1. Show that if h is injective, then f is injective. Also, show that if h is surjective, then g is surjective.
2. Show that if h is surjective and g is injective, then f is surjective.
3. Show that if h is injective and f is surjective, then g is injective.