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1st Year Mathematics (Algebra 1)

Solution to Tutorial Series No. 1
Logic concepts

Solution0 1. *(Exercise 1)*

1. *The sentence "Paris is in France or Madrid is in China" is a proposition.*
   - ***Truth Value:** True.*
2. *The sentence "Open the door" is not a proposition because it is a command.*
3. *The sentence "The moon is a satellite of the Earth" is a proposition.*
   - ***Truth Value:** True.*
4. *The equation* $x + 5 = 7$ *is not a proposition.*
5. *The inequality* $x + 5 > 9$ *for every real number* x *is a proposition.*
   - ***Truth Value:** False.*

Solution0 2.
Determine whether each of the following implications is true or false.

1. *If* 0.5 *is an integer, then* $1 + 0.5 = 3$.
   True
2. *If* $5 > 2$, *then cats can fly.*
   False
3. *If* $3 \times 5 = 15$, *then* $1 + 2 = 3$.
   True
4. *For any real* $x \in \mathbb{R}$, *if* $x \leq 0$, *then* $(x - 1) < 0$.
   True

Solution0 3.
Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ *be a function. Negate the following propositions:*

1. $\exists x \in \mathbb{R}$ *such that* $f(x) = 0$.
   Negation: $\forall x \in \mathbb{R},\ f(x) \neq 0$.
2. $\exists M > 0,\ \forall A > 0,\ \exists x \geq A\ :\ f(x) \leq M$.
   Negation: $\forall M > 0,\ \exists A > 0,\ \forall x \geq A : f(x) > M$.
3. $\exists x \in \mathbb{R},\ f(x) > 0$.
   Negation: $\forall x \in \mathbb{R},\ f(x) \leq 0$.
4. $\forall \epsilon > 0,\ \exists \eta > 0,\ \forall (x, y) \in I^2,\ (|x - y| \leq \eta \Rightarrow |f(x) - f(y)| > \epsilon)$.
   Negation: $\exists \epsilon > 0,\ \forall \eta > 0,\ \exists (x, y) \in I^2,\ |x - y| \leq \eta \wedge |f(x) - f(y)| \leq \epsilon$.

Solution0 4.

1. ***Contrapositive:*** *"For all integers a and b, if a or b is odd, then $a+b$ is odd."*
2. ***Converse:*** *"For all integers a and b, if a and b are even, then $a+b$ is even."*
3. ***Negation:*** *"There exist integers a and b such that $a+b$ is even and a is odd or b is odd."*
4. ***Original Statement:*** *False (Counterexample: $a=1, b=1$, $a+b=2$, but both are odd.)*
5. ***Contrapositive:*** *False (The contrapositive is logically equivalent to the original statement.)*
6. ***Converse:*** *True (The sum of two even numbers is always even.)*
7. ***Negation:*** *True*

Solution0 5.
Prove that if n is an even integer, then n^2 is also an even integer.

Proof: *Suppose n is an even integer. By definition, this means $n=2k$ for some integer k. Now, consider n^2:*

$$n^2 = (2k)^2 = 4k^2$$

Since k is an integer, $4k^2$ is clearly divisible by 2 (specifically, $4k^2 = 2(2k^2)$). Therefore, n^2 is even. □

Solution0 6.
Suppose, for the sake of contradiction, that $\sqrt{2}$ is rational. This means we can express $\sqrt{2}$ as $\frac{p}{q}$, where p and q are integers with no common factors (i.e., $\frac{p}{q}$ is in its simplest form).

Then,

$$\sqrt{2} = \frac{p}{q}$$

Squaring both sides gives:

$$2 = \left(\frac{p}{q}\right)^2 = \frac{p^2}{q^2}$$

Multiplying through by q^2 gives:

$$2q^2 = p^2$$

This implies that p^2 is even (since $2q^2$ is even). From this, we conclude that p itself must be even (because the square of an odd number is odd, and the square of an even number is even).

Let $p=2k$ for some integer k. Substituting in $p^2=(2k)^2=4k^2$, we get:

$$2q^2 = 4k^2 \implies q^2 = 2k^2$$

This shows that q^2 is even, hence q must also be even (similar reasoning as for p). Now, both p and q are even. However, this contradicts our initial assumption that $\frac{p}{q}$ is in its simplest form (no common factors). If both p and q are even, then $\frac{p}{q}$ can be further reduced by dividing both numerator and denominator by 2, which contradicts the assumption that $\frac{p}{q}$ is already in its simplest form.

Therefore, our initial assumption that $\sqrt{2}$ is rational must be false. Hence, $\sqrt{2}$ is irrational. □

Solution0 7.

(1) Prove that if n^2 ***is an even integer, then*** n ***is also an even integer.***

We will prove this statement by contrapositive.

Contrapositive: *The contrapositive of "If* n^2 *is even, then* n *is even" is: "If* n *is odd, then* n^2 *is odd."*

Proof: *Let* n *be an odd integer. By the definition of odd integers,* $n = 2k + 1$ *for some integer* k*.*

Now, calculate n^2*:*

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

Since $2k^2 + 2k$ *is an integer, the expression* $2(2k^2 + 2k) + 1$ *is odd.*

Thus, if n *is odd, then* n^2 *is odd. This proves the contrapositive, and hence, if* n^2 *is even, then* n *is even.*

(2) Prove that if a ***and*** b ***are integers and*** ab ***is odd, then both*** a ***and*** b ***are odd.***

We will prove this statement by contrapositive.

Contrapositive: *The contrapositive of "If* ab *is odd, then both* a *and* b *are odd" is: "If either* a *or* b *is even, then* ab *is even."*

Proof: *Assume that either* a *or* b *is even. Without loss of generality, assume* a *is even. Then,* $a = 2k$ *for some integer* k*.*

Therefore, the product ab *becomes:*

$$ab = (2k) \times b = 2(kb).$$

Since $2(kb)$ *is divisible by 2,* ab *is even.*

Thus, if either a *or* b *is even, then* ab *is even. This proves the contrapositive, and hence, if* ab *is odd, then both* a *and* b *must be odd.*

Solution0 8.

1. *Prove that for all positive integers* n*,* $1 + 3 + 5 + \cdots + (2n - 1) = n^2$*.*

   Proof: *We will prove the statement by mathematical induction.*

   Base Case: *For* $n = 1$*,*

   $$1 = 1^2$$

   which is true.

   Inductive Step: *Assume the statement holds for some arbitrary positive integer* k*, i.e.,*

   $$1 + 3 + 5 + \cdots + (2k - 1) = k^2.$$

   We need to prove it holds for $k + 1$*:*

   $$1 + 3 + 5 + \cdots + (2k - 1) + (2(k + 1) - 1) = (k + 1)^2.$$

   Adding $2(k + 1) - 1 = 2k + 1$ *to both sides gives:*

   $$k^2 + 2k + 1 = (k + 1)^2.$$

   Hence, the statement holds for $k + 1$*.*

   Therefore, by mathematical induction, for all positive integers n*,* $1+3+5+\cdots+(2n-1) = n^2$*.* □

2. *Prove that for all positive integers n, $2^n > n$.*

Proof: *We will prove the statement by mathematical induction.*

Base Case: *For $n = 1$,*

$$2^1 = 2 > 1,$$

which is true.

Inductive Step: *Assume the statement holds for some arbitrary positive integer k, i.e.,*

$$2^k > k.$$

We need to prove it holds for $k + 1$:

$$2^{k+1} = 2 \cdot 2^k > 2 \cdot k = k + k \geq k + 1.$$

Therefore, $2^{k+1} > k + 1$.

Hence, by mathematical induction, for all positive integers n, $2^n > n$. □

Solution0 9.

We are tasked with proving that for all $x \in \mathbb{R}$, the inequality

$$|x - 1| \leq x^2 - x + 1$$

holds. We'll use a ***proof by disjunction of cases****.*

Case 1: $x - 1 \geq 0$ ***(i.e.,*** $x \geq 1$

If $x - 1 \geq 0$, the absolute value becomes:

$$|x - 1| = x - 1.$$

Substituting $|x - 1| = x - 1$ into the inequality:

$$x - 1 \leq x^2 - x + 1.$$

Rearranging terms:

$$x^2 - 2x + 2 \geq 0.$$

Factoring $x^2 - 2x + 2$ (or completing the square):

$$x^2 - 2x + 2 = (x - 1)^2 + 1.$$

Since $(x - 1)^2 \geq 0$ and $1 > 0$, it follows that:

$$x^2 - 2x + 2 \geq 0.$$

Thus, the inequality holds in this case.

Case 2: $x - 1 < 0$ ***(i.e.,*** $x < 1$

If $x - 1 < 0$, the absolute value becomes:

$$|x - 1| = -(x - 1) = 1 - x.$$

Substituting $|x - 1| = 1 - x$ into the inequality:

$$1 - x \leq x^2 - x + 1.$$

Rearranging terms:

$$x^2 \geq 0.$$

This inequality is always true because $x^2 \geq 0$ for all $x \in \mathbb{R}$.

Conclusion:

In both cases, the inequality $|x - 1| \leq x^2 - x + 1$ holds for all $x \in \mathbb{R}$.

Solution0 10.
Prove that the following statement is false: "Every positive integer is the sum of three squares."
Proof: *We will provide a counterexample to show that not every positive integer can be expressed as the sum of three squares.*

Consider the integer $n = 7$.

Now, we will check if 7 *can be expressed as* $a^2 + b^2 + c^2$ *where* a, b, c *are integers.*

Testing possible combinations:

$$1^2 + 1^2 + 1^2 = 1 + 1 + 1 = 3,$$
$$1^2 + 1^2 + 2^2 = 1 + 1 + 4 = 6,$$
$$1^2 + 2^2 + 2^2 = 1 + 4 + 4 = 9.$$

None of these combinations yield 7.

Therefore, 7 *cannot be expressed as the sum of three squares.*

Since we have found at least one positive integer (specifically 7*) that cannot be written as the sum of three squares, the statement "Every positive integer is the sum of three squares" is false.* □