Waves and vibrations

November 2024

Teaching objectives   Teaching objectives Introduce the student to the phenomena of mechanical vibrations restricted to low amplitude oscillations for 1 or 2 degrees of freedom as well as to the study of the propagation of mechanical waves. Recommended prior knowledge : 1st year Mathematics and Physics concepts

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Chapter 1

Vibrations

1.1 GENERALITIES “Introduction to Lagrange’s equations

1.1.1 General information on vibrations

Definition of a periodic movement

A movement is said to be periodic if it repeats identically to itself during equal intervals of time.

Examples: The movement of revolution of the Moon: The moon makes a complete cycle of revolution around the earth in approximately 29 days.

Heartbeats: the heartbeat is a succession of contractions and relaxations of the cardiac muscles that activate valves and cause the circulation of blood in the body.

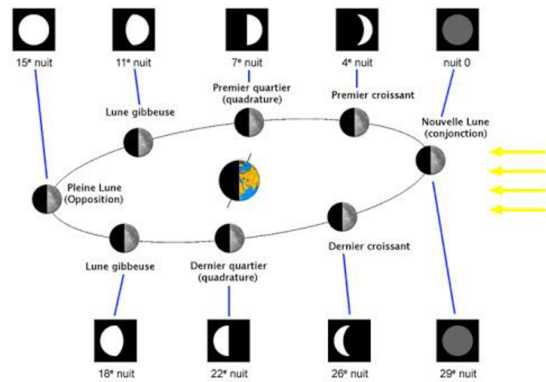

FIGURE 1.1: Movement of revolution of the moon

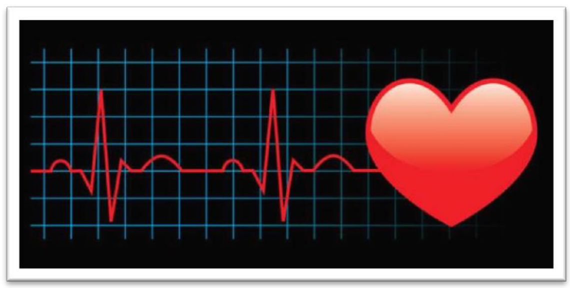

FIGURE 1.2: Electrocardiogram

Definition of an oscillation

Oscillation: any movement of a body that moves alternately from one side to the other from an equilibrium position. Oscillation refers to any repetitive motion around an equilibrium point. It can occur in mechanical systems (like a pendulum) or in other systems, such as electrical circuits (AC current) or biological rhythms. - Type of Motion: Oscillations typically describe smooth, periodic motions where the system swings back and forth in a regular pattern. - Examples: - A pendulum swinging back and forth. - The oscillation of an electrical signal in an alternating current (AC). - Seasonal cycles in nature or heartbeats in biology. - Broader Context : Oscillation can occur in "physical, chemical, electrical, or biological systems". It may or may not involve physical movement; for example, **light waves** oscillate but do not "vibrate" in the mechanical sense.

Subdivided of Oscillations:

Free oscillations an oscillator is free if it oscillates without external interventions (without friction) during its return to equilibrium -Damped oscillations the oscillator is subjected to friction forces that dissipate energy the oscillation damps and eventually stops. -Forced oscillations an oscillator is forced if an external action communicates energy to it. -Damped forced oscillations the external periodic force (excitation) compensates for the losses of snow removal by friction, the oscillations thus maintained do not dampen.

Definition of vibration:

Definition: Vibration specifically refers to the rapid, mechanical oscillations of a material or object. It involves the back-and-forth movement of particles or structures, often in response to an external force. - Type of Motion: Vibration is generally associated with fast, small-amplitude movements around an equilibrium position, often creating sound or heat as energy dissipates. - Examples: - The vibration of a guitar string when plucked. - The shaking of a mobile phone during a notification. - The vibrating motion of an engine or machinery. - Mechanical Nature: Vibration typically involves **mechanical systems** and is usually associated with physical objects. It is a subset of oscillation, specifically relating to mechanical systems.

Key differences:

1. Scope:
   - Oscillation is a more general term and can apply to any repetitive motion (mechanical, electrical, biological, etc.).
   - Vibration specifically refers to mechanical oscillations of a structure or object.

2. Speed and Amplitude:
   - Vibration is often faster and involves smaller movements, whereas oscillation can be slower and cover a larger range of motion.

3. Systems:
   - Oscillation applies to systems as varied as electrical circuits and mechanical objects.
   - Vibration is usually limited to mechanical or physical systems.
   - Oscillation is a general concept of periodic motion, applicable to various systems.
   - Vibration is a specific type of oscillation that refers to the mechanical movement of objects.

Definition of a periodic motion

Periodic motion is a type of motion that repeats itself at regular intervals of time. In other words, an object in periodic motion returns to the same position and state after a fixed time period, known as the period. - Examples: The swinging of a pendulum, the vibration of a guitar string, the orbit of planets around the sun, and the oscillations of a mass on a spring.

Mathematical Form:
- Periodic motion is often described mathematically by sine or cosine functions, which reflect the repetitive nature of the movement. For fast motions we use the frequency (f) expressed in hertz (HZ) it is related to the period by:

$$f = \frac{1}{T} \tag{1.1}$$

The number of revolutions per second is called pulsation ω (noted , measured in rad/s)

$$\omega = 2\pi f = \frac{2\pi}{T} \tag{1.2}$$

1Hz = 1 period per second
1KHz = $10^3 Hz$

$1MHz = 10^6 Hz$

$1GHz = 10^9 Hz$

Note:

1. An oscillator is said to be harmonic if the system evolves according to a periodic law of sinusoidal form (Figure 1-3).

$$x(t) = A\cos(t + \varphi) \tag{1.3}$$

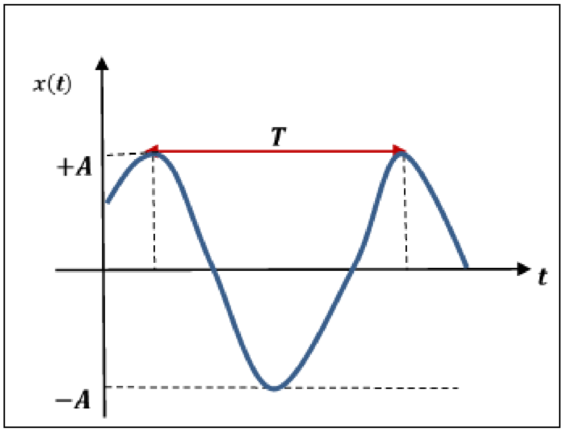

FIGURE 1.3: Representation of a harmonic oscillation

A: Amplitude of the oscillation (Maximum value of the displacement)

ω : Pulsation of the oscillation

φ : the initial phase (t = 0)

Speed: v The speed of the oscillating point M is the derivative of its displacement.

2- The oscillator is said to be non-harmonic if the system evolves according to a periodic law of any non-sinusoidal form figure 1.4.

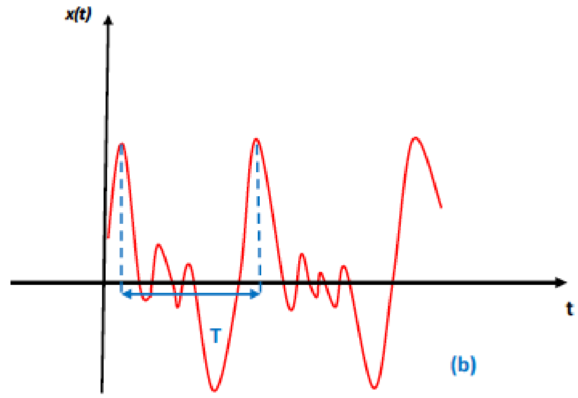

FIGURE 1.4: Representation of an anharmonic oscillation

1.1.2 Concept of energy

Key Concepts of Energy in Vibration and Oscillation:] The concept of energy in vibration and oscillation involves the continuous exchange between two primary forms of mechanical energy: kinetic energy (KE) and potential energy (PE). In systems undergoing vibration or oscillation, energy moves back and forth between these two forms as the object or system moves through its cycle.

Key Concepts of Energy in Vibration and Oscillation:

Total Mechanical Energy:

The total mechanical energy in a vibrating or oscillating system remains constant (assuming no energy loss due to friction or damping). This energy is the sum of kinetic and potential energy at any given point in the motion.

$$E_{Tot} = KE + PE = Const \tag{1.4}$$

Kinetic Energy (KE):

Kinetic energy is the energy of motion. It is at its maximum when the object moves the fastest, which typically occurs when the object passes through its equilibrium

position (the center of its motion).
Formula for kinetic energy:

$$KE = \frac{1}{2}mv^2 \tag{1.5}$$

$$v = \frac{\partial x}{\partial t} = \dot{x} \tag{1.6}$$

$$\Rightarrow KE = \frac{1}{2}m\dot{x}^2 \tag{1.7}$$

Where m is the mass of the object and v is its velocity.
- In an oscillating system, kinetic energy is highest when the velocity is greatest and zero at the turning points of motion.

Potential Energy (PE):

- Potential energy is the energy stored due to the position or configuration of the object. In oscillating systems, this can be elastic potential energy (in springs) or gravitational potential energy (in pendulums). - In an oscillating system, potential energy is maximum when the object is at the extreme points of its motion (the turning points) and zero at the equilibrium position. - Formula for potential energy in a mass-spring system:

$$PE = \frac{1}{2}kx^2 \tag{1.8}$$

Where k is the spring constant and x is the displacement from the equilibrium position.

Energy Exchange:

- During the motion, there is a continuous exchange between kinetic and potential energy.
- At the equilibrium position, the velocity is at its maximum, so kinetic energy is maximum and potential energy is zero.

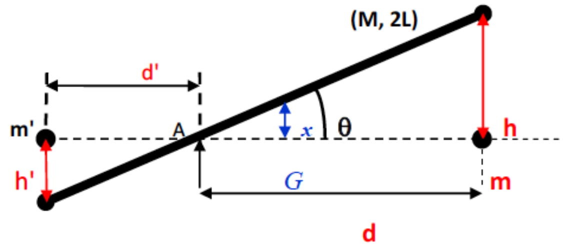

FIGURE 1.5: Double pendulum

- At the extreme points of the motion (maximum displacement), the velocity is zero, so kinetic energy is zero and potential energy is maximum.
- This energy transfer is what drives the oscillation or vibration.
- Damping and Energy Loss:
- In real systems, damping (due to friction or air resistance) can cause energy loss in the form of heat, leading to a gradual decrease in the amplitude of the oscillation.
- In a damped system, total mechanical energy decreases over time, eventually bringing the motion to a stop.

Examples:
- Simple Harmonic Oscillator: In a mass-spring system, the mass moves back and forth, converting potential energy stored in the spring into kinetic energy and vice versa.
- Pendulum: In a pendulum, gravitational potential energy and kinetic energy exchange as the pendulum swings from side to side.
- Vibration of a string: When a guitar string vibrates, the tension in the string stores potential energy, while the motion of the string translates that into kinetic energy.

Practical example Consider a mechanical system figure 1.5, below made up of two point masses (m and m') fixed to the free ends of a rod of mass M and length 2L. This system is a rotational movement relative to or fixed point A. Calculate the kinetic energy and the potential energy of the system: **Solution : The system is made up of 3 masses, so there are 3 kinetic**

energies and 3 potential energies:

1- The kinetic energy KE

$$KE_{Tot} = KE_M + KE_m + KE_{m'}$$

$$KE_M = \frac{1}{2} J_{M/A} \theta^2$$

$$J_{M/A} = \frac{1}{12} M(2L^2) + M(AG)^2$$

$$AG = \frac{L}{2}$$

$$J_{M/A} = \frac{1}{12} M(2L^2) + M(\frac{L}{2})^2 = \frac{7}{12} ML^2$$

$$\Rightarrow KE_M = \frac{1}{2}\frac{7}{12} ML^2\theta^2$$

$$T_m = \frac{1}{2} J_{m/A} \theta^2$$

$$J_{m/A} = [0 + md^2]$$

$$d = L + \frac{L}{2} = \frac{3L}{2}$$

$$J_{m/A} = m\left(\frac{3L}{2}\right)^2$$

$$\Rightarrow T_m = \frac{1}{2}\left[\frac{4}{9} m\right] L^2\theta^2$$

$$T_{m'} = \frac{1}{2} J_{m'/A} \theta^2$$

$$J_{m'/A} = [0 + m'd^2]$$

$$d = \frac{L}{2}$$

$$J_{m'/A} = m'\left(\frac{L}{2}\right)^2$$

$$\Rightarrow T_{m'} = \frac{1}{2}\left[\frac{1}{4}m'\right]L^2\theta^2$$

$\mathrm{T}_{Tot} = \frac{1}{2}\frac{7}{12}ML^2\theta^2 + \frac{1}{2}\left[\frac{4}{9}m\right]L^2\theta^2 + \frac{1}{2}\left[\frac{4}{9}m'\right]L^2\theta^2$

$$\Rightarrow T_{Tot} = \frac{1}{2}\left[\frac{7}{12}M + \frac{9}{4}m + \frac{1}{4}m'\right]L^2\theta^2$$

2- The Potential energy PE

$$PE_{Tot} = PE_M + PE_m + PE_{m'}$$

$\mathrm{PE}_M = Mgx$

$$x = \frac{1}{2}L\sin\theta$$

$\mathrm{PE}_M = \frac{1}{2}MgL\sin\theta$

PE$_m = mgh$

h= d$\sin\theta = \frac{3}{2}L\sin\theta$

PE$_m = \frac{3}{2}mgL\sin\theta$

$$PE_{m'} = -m'gh'$$
$$h' = \frac{d}{3}\sin\theta = \frac{1}{2}L\sin\theta$$

PE$_{m'} = -\frac{1}{2}m'gL\sin\theta$

PE$_{Tot} = \frac{1}{2}MgL\sin\theta + \frac{3}{2}mgL\sin\theta - \frac{1}{2}m'gL\sin\theta$

PE$_{Tot} = \frac{1}{2}gL\sin\theta(M + 3m - m')$

Equilibrium Conditions

The equilibrium condition set if the equilibrium

$$F = 0$$

if

$$x = x_0 \Rightarrow F\Big|_{x=x_0}$$

For a force derived from a potential,

$$F = -\frac{\partial U}{\partial x}$$

The equilibrium condition is written as:

$$\frac{\partial U}{\partial x}\Big|_{x=x_0} = 0 \tag{1.11}$$

There are two types of equilibrium:

1. Stable Equilibrium :
   Once the system is displaced from its equilibrium position, it returns to it. In this case, the restoring force is:

$$f = -C_x$$

with
C $\succ$ 0

$$C = -\frac{\partial f}{\partial x} = -\frac{\partial}{\partial x}\left(-\frac{\partial U}{\partial x}\right) = \frac{\partial^2 U}{\partial x^2} \tag{1.13}$$

$$\frac{\partial U}{\partial x}\Big|_{x=x_0} > 0 \tag{1.14}$$

This condition of stable equilibrium is the condition for oscillation.

2- Unstable Equilibrium :
The system does not return to its equilibrium when displaced. In this case, the unstable equilibrium condition is written as:

$$\frac{\partial U}{\partial x}\Big|_{x=x_0} < 0 \tag{1.15}$$

In the case of rotation, we replace:

$$x \to \theta$$

$$x_0 \to \theta_0$$

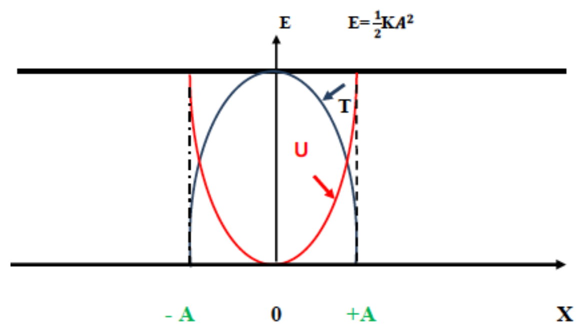

FIGURE 1.6: the variation of energies as a function of displacement

It is possible to graphically represent the evolution of three energies : potential energy kinetic energy and total (mechanical) energy, The figure1-6 . [1]

[1]When kinetic energy decreases, potential energy increases, and the opposite is also true. This phenomenon is known as the conservation of total energy in a system.

1.1.3 Lagrange Formalism

Generalized Coordinates

Generalized coordinates are the set of independent or related real viable coordinates allowing to describe and configure all the elements of a system at any time t.

Exemple

The position of a point M in space can be determined by 3 coordinates Along the axes (x, y, z)
The position of a solid body in space can be defined by six coordinates:

1. 03 coordinates relative to the center of gravity

2. 03 coordinates related to the Euler angles (θ, ϕ, ψ)

We designate by $q_1(t), q_2(t), q_3(t)....q_N(t)$**: The generalized coordinates.**
$\dot{q_1}, \dot{q_2}, \dot{q_3}.....\dot{q_N}$**: The generalized speeds.**

Degree of freedom

This is the number of independent coordinates needed to determine the position of each element of a system during its motion at any time: We write:

$$d = N - r \tag{1.16}$$

with:
d : Degree of freedom
N :Generalized number of coordinates

r :Number of relations, between the generalized coordinates (number of links)
Exemple: let a mechanical system consist of two points M_1 and M_1 connected by a rod of length L. Find the number of degrees of freedom.

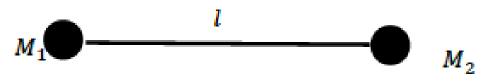

FIGURE 1.7: Exemple

$$\begin{cases} M_1(x_1, y_1, z_1) \\ M_2(x_2, y_2, z_2) \end{cases} \Rightarrow N = 6$$

$$l = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} = Cst \Rightarrow r = 1$$

so :

$$d = N - r = 6 - 1 = 5 \rightarrow d = 5$$

Lagrange formalism

This formalism is based on the Lagrange function

$$L = KE - PE$$

The set of equations of motion is written as:

$$\sum_{i=1}^{n} \left\{ \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \left(\frac{\partial L}{\partial q_i} \right) \right\} = 0 \tag{1.17}$$

Where:
L: Lagrange Function or Lagrangian;
KE The Kinetic Energy of the System;

PE The Potential Energy of the System;

q_i The generalized coordinate and is the generalized velocity of the system.

For a system with one degree of freedom (N=1 or dof=1)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)-\left(\frac{\partial L}{\partial q}\right)=0$$

For a one-dimensional system, Lagrange's equation is written as :

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)-\left(\frac{\partial L}{\partial x}\right)=0$$

For a rotational movement θ :

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\left(\frac{\partial L}{\partial \theta}\right)=0$$

Chapter 2

Linear systems with a single degree of freedom

2.1 Introduction : Study of undamped free oscillations

A system that oscillates without the influence of any external force is known as a free oscillator. Such systems are considered conservative.

2.1.1 Complex representation and Definition of Fourier series

To facilitate calculations we transform the sinusoidal quantities into exponential form using the Euler form:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

The periodic quantity can be expressed by the sums of the sine and cosine functions in order to manipulate it physically and mathematically. This sum is called a

Fourier series. The Fourier series of a periodic function $f(t)$ with a period T is defined by:

$$f(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)]$$

- Where a_0, a_n, and b_n are the Fourier coefficients.

$$a_0 = \frac{1}{T} \int_0^T f(t)\, dt$$

$$a_n = \frac{2}{T} \int_0^T f(t) \cos(n\omega t)\, dt$$

$$b_n = \frac{2}{T} \int_0^T f(t) \sin(n\omega t)\, dt$$

The angular frequency (ω) is called the fundamental frequency :

$$T = \frac{2\pi}{\omega}$$

- The frequencies $(n\omega)$, multiples of ω, are called harmonics.

2.1.2 Study of the mechanical system

To Obtained the differential equation can be determined by: 1. second Newton's law 2. Conservation of energy 3. Lagrange's method.

Example : A mass m attached to the free end of a spring and moving without friction in a vertical direction.

Newton's dynamic principle applied to mass

$$\sum \overrightarrow{F} = m\overrightarrow{a}$$

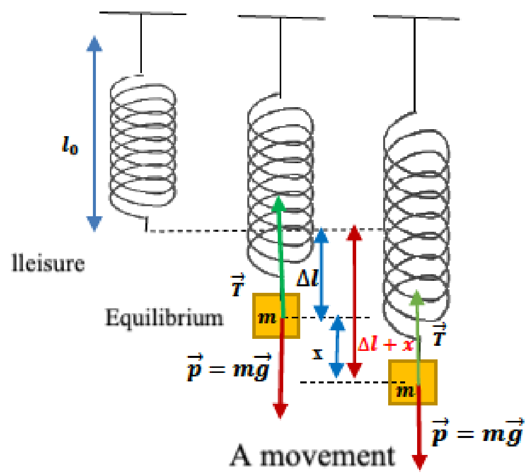

FIGURE 2.1: Harmonic oscillator

Projection on the ox axis:

$$m\vec{g} + \vec{T} = m\vec{a}$$

$$m\ddot{x} = mg - T$$

$$\Leftrightarrow m\ddot{x} = mg - k(x + \Delta l)$$

$$\Leftrightarrow m\ddot{x} = mg - kx - k\Delta l$$

In equilibrium

$$\sum \overrightarrow{F} = \overrightarrow{0}$$

$$m\vec{g} + \vec{T} = \vec{0}$$

$$mg - k(\Delta l) = 0$$

Equilibrium conditions $\Rightarrow$

$$m\ddot{x} = -kx + \underbrace{mg - k\Delta l}_{0}$$

$$m\ddot{x} = -kx$$

$$\Leftrightarrow m\ddot{x} + kx = 0$$

$$\Rightarrow \ddot{x} + \omega_0^2 x = 0$$

This is the differential equation of motion

Energy Conservation

$$E_{Tot} = T + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

In a free system, the mechanical (or total) energy is conserved, thus:

$$\frac{\mathrm{d}E_{Tot}}{\mathrm{d}t} = 0 \Rightarrow m\ddot{x}\dot{x} + k\dot{x}x = 0$$

$$\Rightarrow \ddot{x} + \frac{k}{m}x = 0 \Leftrightarrow \ddot{x} + \omega_0^2 x = 0$$

Lagrange Method

The Lagrangian of the system L is:

$$L = T - U$$

Where T is the kinetic energy and U is the potential energy. So L equal :

$$L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

The Lagrangian formalism:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = 0 \tag{2.1}$$

By substituting (I.18) in .

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = & \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\mathrm{d}}{\mathrm{d}t}(m\ddot{x}) \quad = m\ddot{x} \\ \frac{\partial L}{\partial x} & = -kx \end{cases}$$

2.1.3 Solution of the Differential Equation

The solution of the equation:

$$\ddot{x} + \frac{k}{m}x = 0 \Leftrightarrow \ddot{x} + \omega_0^2 x = 0$$

is of the form :

$$x(t) = Ae^{rt}$$

where r is a real number and A is a positive constant.

$$\dot{x} = Are^{rt}///\ddot{x} = Ar^2e^{rt} \Leftrightarrow (r^2 - \omega_0^2)Ae^{rt} = 0$$

$$\begin{Bmatrix} r_1 = & i\omega_0 \\ r_2 = & -i\omega_0 \end{Bmatrix}$$

We have two solutions:

$$x_1(t) = A_1e^{r_1} = A_1e^{i\omega_0 t}///x_2(t) = A_2e^{r_2} = A_2e^{-i\omega_0 t}$$

$$x(t) = A(e^{i\omega_0 t} + e^{-}i\omega_0 t)$$

The general solution of the equation of motion :

$$x(t) = x_1(t) + x_2(t) = A(e^{i\omega_0 t} + e^{-}i\omega_0 t)$$

According to Euler's relation :

$$e^{\pm}i\omega_0 t = \cos(\omega_0 t) \pm i\sin(\omega_0 t)$$

SO :

$$x(t) = A_1(\cos(\omega_0 t) + i\sin(\omega_0 t)) + A_2(\cos(\omega_0 t) - i\sin(\omega_0 t))$$

$$x(t) = (A_1 + A_2)\cos(\omega_0 t + (A_1 - A_2)i\sin(\omega_0 t)$$

$$x(t) = B\cos\omega_0 t + C\sin\omega_0 t$$

Where :

$$\begin{cases} B = & \cos\theta \\ C = & \sin\theta \end{cases}$$

So:

$$x(t) = D\cos\theta\cos\omega_0 t + D\sin\theta\sin\omega_0 t = D\cos(\omega_0 t + \varphi)$$

D and φ are constants deduced from the initial conditions. The oscillations are

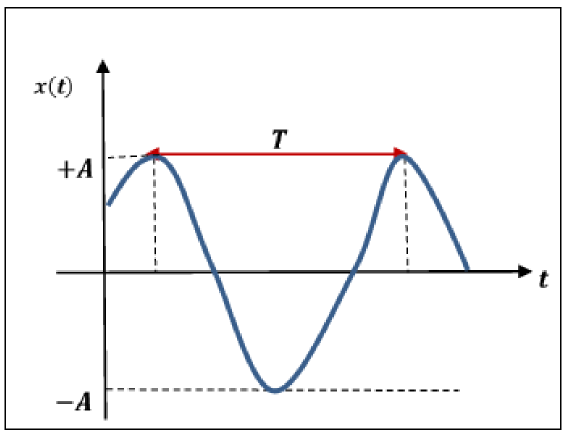

FIGURE 2.2: Sinusoidal motion

sinusoidal in amplitude and natural period:

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$$

Study of the electrical system

Let us consider an electrical circuit $C\ L$

$$\sum_i V_i = 0$$

$$V_C + V_L = 0$$

$$L\frac{di}{dt} + \frac{q}{c} = 0$$

where :

$$i = \frac{dq}{qi} \Rightarrow L\ddot{q} + \frac{q}{c} = 0 \Leftrightarrow \ddot{q} + \frac{1}{LC}q = 0$$

$$\Rightarrow \ddot{q} + \omega_0^2 = 0$$

$$q(t) = Q\cos(\omega_0 t + \varphi)$$

with :

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Energetic Aspects

$$E_m = T + U$$

$$\begin{cases} T = \frac{1}{2}m\dot{x}^2 \\ U = \frac{1}{2}kx^2 \end{cases}$$

$$x(t) = A\cos(\omega_0 t + \varphi)//\dot{x(t)} = -A\omega_0\sin(\omega_0 t + \varphi)$$

$$\begin{cases} T = \frac{1}{2}m(-A\omega_0\sin(\omega_0 t + \varphi))^2 \\ U = \frac{1}{2}k(A\cos(\omega_0 t + \varphi))^2 \end{cases}$$

From an energetic point of view, this oscillator transforms elastic energy into kinetic energy and vice versa.

$$E_m = \frac{1}{2}kA^2\sin^2(\omega_0 t + \varphi) + \frac{1}{2}kA^2\cos^2(\omega_0 t + \varphi) \Leftrightarrow E_m = \frac{1}{2}kA^2$$

$$\omega_0^2 = \frac{k}{m}$$

- The mechanical energy of an oscillator is proportional to the square of the amplitude (see Chapter 1: Presentation of Mechanical Energy).

2.1.4 Electro-mechanical Analogy

By comparing the differential equations of the elastic pendulum and the LC electrical circuit, the following electromechanical analogy can be established:

Potential Energy: corresponds to the energy in the capacitor.

Electrical System	Mechanical System
$\ddot{x} + \frac{k}{m}x = 0$	$\ddot{q} + \frac{1}{LC}q = 0$
Elongation: x	Charge : q
Mass : m	Inductance : L
Ressort : k	Inverse de la capacité $\frac{1}{C}$
Kinetic energy: $\frac{1}{2}m\dot{x}^2$	Energy of the coil $\frac{1}{2}L\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2$
Potential energy: $\frac{1}{2}kx^2$	Capacitor energy $\frac{1}{2}\left[\frac{q^2}{c}\right]$

TABLE 2.1: Electromechanical Analogy (Free Motion)

Chapter 3

Single degree of freedom damped free linear systems

3.1 Introduction to Damped Free Oscillation Types of Friction

"A part of the oscillator's energy is transferred to the external environment (dissipated through friction or radiation). The amplitude of the oscillations decreases over time, and the oscillator eventually comes to a stop."

3.2 Types of Friction

The equations of motion depend on the nature of friction. The solution to the equation of motion is only possible with certain types of friction.

3.2.1 Solid Friction

The force of friction is proportional $\sim$ to the normal reaction of the support.

$$F_f = -sgn(\nu)\mu R \tag{3.1}$$

With:
F : Restoring force $F = -kx$
μ : Coefficient of dynamic friction $\neq$ static friction.

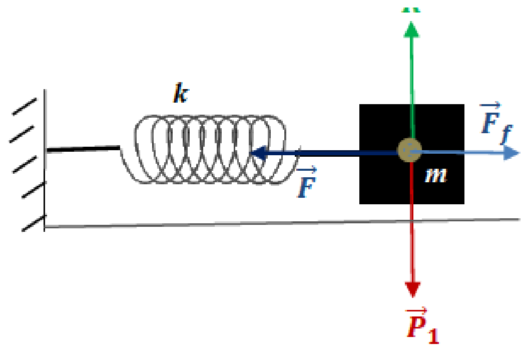

FIGURE 3.1: Solid Friction

3.2.2 Fluid or Viscous Friction

The fluid friction force is proportional $\neq$ to and opposite to the velocity.

$$F_f = -\alpha \upsilon \tag{3.2}$$

With $\alpha \succ 0$

3.2.3 Friction in Highly Viscous Media

The friction in highly viscous media is proportional $\neq$ to the square of the velocity. The equation of motion is non-linear and generally does not have an analytical solution.

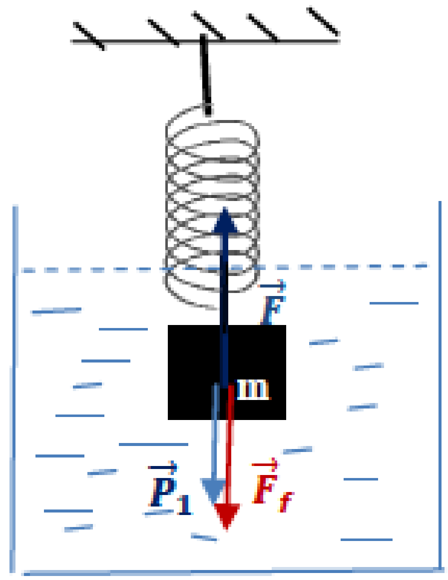

FIGURE 3.2: Viscous Friction

3.2.4 Other Complex Types of Friction

In this course, we will limit ourselves to viscous friction forces that are proportional to velocity. The expression for the viscous friction force is as follows:

$$F_q = -\alpha\dot{q} \tag{3.3}$$

With: α The coefficient of viscous friction $\alpha : [N.\frac{s}{m}$ $q :$ The generalized coordinate of the system $\dot{q} :$ The generalized velocity of the system. In a one-dimensional x motion, the force is expressed as:

$$\overrightarrow{f} = -\alpha\overrightarrow{\nu} = -\alpha\dot{x}\overrightarrow{u}$$

In mechanics, the damper is represented by figure 3.3.

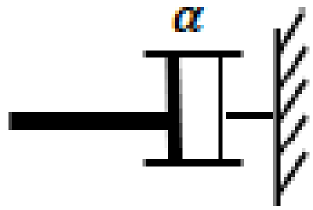

FIGURE 3.3: Represent of damper

3.3 Lagrange's equation in a damped system

"If friction $f = -\alpha\dot{q}$ exists, the Lagrange equation becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = F_q$$

Under the action of friction forces, the system dissipates (loses) mechanical energy in the form of heat, so there is a relationship between the forceF_q and the dissipation function D on one side, and the viscous friction coefficient on the other side α.

$$F_q = -\frac{\partial D}{\partial \dot{q}} \tag{3.4}$$

with :

$$D = \frac{1}{2}\alpha\dot{q}^2$$

The Lagrange equation for a damped system becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = -\frac{\partial D}{\partial \dot{q}}$$

3.4 Mass-Spring-Damper System Differential Equation

3.4.1 The equation of motion

To study the motion of such a system, we can use the fundamental relation of dynamics (FRD) and the Lagrange relations.

A- Fundamental Relation of Dynamics FRD :

$$\sum \overrightarrow{F}_{ext} = m\overrightarrow{\gamma}$$

$$\overrightarrow{P} + \overrightarrow{F}_R + \overrightarrow{F}_\alpha = m\overrightarrow{\gamma}$$

Projection on ox:

$$mg - k(x + x_0) - \alpha\dot{x} = m\ddot{x}$$

$$\underbrace{mg - kx_0}_{=0} + kx - \alpha\dot{x} = m\ddot{x}$$

$$\Rightarrow m\ddot{x} + \alpha\dot{x} + kx = 0$$

$$\ddot{x} + \frac{\alpha}{m}\dot{x} + \frac{k}{m}x = 0$$

This is homogeneous second degree linear differential equation.

B-Lagrange method

Kinetic energy:

$$T = \frac{1}{2}m\dot{x}^2$$

Potential energy:

$$U = \frac{1}{2}kx^2$$

Dissipation function:

$$D = \frac{1}{2}\alpha\dot{x}^2$$

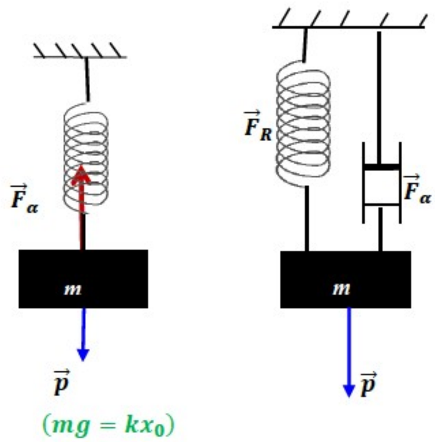

FIGURE 3.4: Mass-spring-damper system In balance and in motion

Lagrange function:

$$L = T - U$$

$$\Rightarrow L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

Lagrange formalism:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = -\frac{\partial D}{\partial \dot{x}}$$

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) & = m\ddot{x} \\ \frac{\partial L}{\partial x} & = -kx \\ \frac{\partial D}{\partial \dot{x}} & = \alpha\ddot{x} \end{cases}$$

By replacing in equation (III.1) we will have:

$$m\ddot{x} + kx = -\alpha\dot{x}$$

$$\Rightarrow \ddot{x} + \frac{\alpha}{m}\dot{x} + \frac{k}{m}x = 0$$

This is the differential equation of motion in the case of a free damped system.

As in the case of the undamped harmonic oscillator, the natural frequency of the system is $\omega_0 = \frac{k}{m}$. However, a new term associated with the damping parameter appears $\left(\frac{\alpha}{m}\right)$. This coefficient is equal to $\lambda = \frac{\alpha}{2m}$, with λ being the damping factor.

Therefore, the equation of motion is written as:

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = 0 \tag{3.5}$$

3.4.2 Solution of the Equation of Motion

Equation (3.5) is a second-order differential equation without an external term. The set of solutions to this equation forms a 2-dimensional vector space. The general solution of this equation can be expressed as a linear combination of two solutions that form a basis. This basis can be found by focusing on exponential time solutions.

$$\begin{cases} x(t) = Ae^{rt} \\ \dot{x(t)} = Are^{rt} \\ \ddot{(x(t))} = Ar^2e^{rt} \end{cases}$$

We substitute the three terms into equation (3.5), which gives:

$$Ar^2e^{-i\omega t} + 2\lambda Are^{-i\omega t} + \omega_0^2 Ae^{-i\omega t} = 0$$

By factoring $Ae^{-i\omega t}$, we obtain: (3.6)

$$Ae^{-i\omega t}(r^2 + 2\lambda r + \omega_0^2) = 0 \tag{3.6}$$

Equation (3.6) is known as the characteristic equation. It is a second-degree equation and can yield either two distinct real roots, a double root (within the real numbers), or two complex roots. The reduced discriminant is calculated as follows:

$$\Delta = \lambda^2 - \omega_0^2$$

Three regimes are to be studied:

Heavily Damped (Aperiodic) Regime

In this regime, the discriminant of the characteristic equation is positive, leading to two distinct real roots. This indicates that the system returns to equilibrium without oscillating, meaning it is aperiodic. The solution to the equation of motion is therefore a sum of two exponentially decaying terms, each associated with one of the real roots.

$$\Delta > 0 \Rightarrow \lambda > \omega_0$$

$$\begin{cases} r_1 = & \frac{-2\lambda+\sqrt{\Delta}}{2} = & -\lambda + \sqrt{\lambda^2 - \omega_0^2} \\ r_2 = & \frac{-2\lambda-\sqrt{\Delta}}{2} = & -\lambda - \sqrt{\lambda^2 - \omega_0^2} \end{cases}$$

$$x(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t} \Rightarrow x(t) = A_1 e^{-\lambda+\sqrt{\lambda^2-\omega_0^2}t} + A_2 e^{-\lambda-\sqrt{\lambda^2-\omega_0^2}t}$$

$$x(t) = e^{-i\omega t}\left[A_1 e^{-\lambda+\sqrt{\lambda^2-\omega_0^2}t} + A_2 e^{-\lambda-\sqrt{\lambda^2-\omega_0^2}t}\right]$$

The coefficients A_1 and A_2 are determined by the initial replacement and speed conditions.

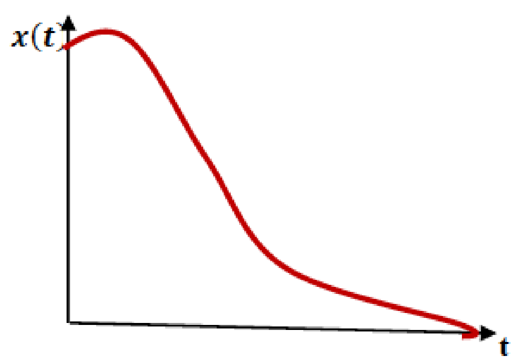

FIGURE 3.5: The aperiodic regime

By displacing the system from its equilibrium position, it no longer oscillates and comes to a complete stop after a certain amount of time, which depends on the damping coefficient. The larger the damping coefficient, the shorter the stopping time. This regime is called aperiodic, and the damping is heavy.

The term $\sqrt{\lambda^2 - \omega_0^2}$ is not considered an angular frequency because, in the case of a heavily damped regime, there is no oscillation around the equilibrium position.

Critical Aperiodic Regime

This corresponds to the case where the reduced discriminant is zero.

$$\Delta = 0 \Leftrightarrow \lambda = \omega_0$$

The characteristic equation has a real double root.

$$r_1 = r_2 = -\lambda$$

Thus, the function $e^{-\lambda t}$ is a solution of the differential equation.
The second solution can be obtained by noting that $te^{-\lambda t}$ it is also a solution. The general solution is then written as:

$$x(t) = Ate^{-\lambda t} + Be^{-\lambda t} \Rightarrow x(t) = (At + B)e^{-\lambda t}$$

In the case of the "critical aperiodic regime", where the discriminant of the characteristic equation is zero, we have a double real root. For a second-order differential equation of the form:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

its characteristic equation is:

$$r^2 + 2\zeta\omega_0 r + \omega_0^2 = 0$$

where:

- $\omega_0 = \sqrt{\frac{k}{m}}$ is the undamped natural frequency, - $\zeta = \frac{b}{2\sqrt{km}}$ is the damping ratio.

In the critical damping case, $\zeta = 1$, so the discriminant is zero, and the characteristic equation has a double root $r = -\omega_0$. The general solution is then:

$$x(t) = (A + Bt)e^{-\omega_0 t}$$

where A and B are constants determined by initial conditions.

$$\begin{cases} \lambda = & \frac{\alpha}{m} \\ \omega_0 = & \sqrt{\frac{k}{m}} \end{cases} \Rightarrow \lambda = \omega_0 \Rightarrow \frac{\alpha^2}{4m^2} = \frac{k}{m}$$

Where C: critical $\alpha_C = 2\sqrt{km}$. The critical aperiodic regime is the one in which the system returns to its equilibrium position more quickly than in any other aperiodic regime.

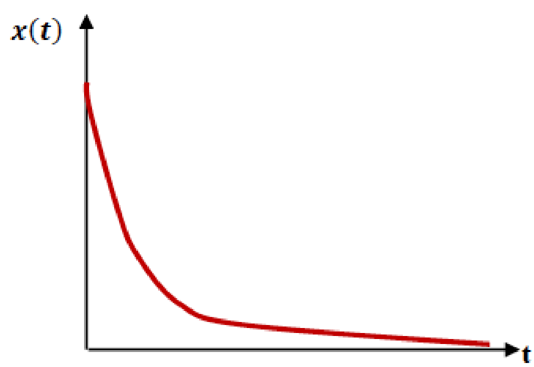

FIGURE 3.6: Critical aperiodic regime

The damping is known as critical damping. The critical regime plays an important role in certain practical applications and the design of measuring instruments, as after a disturbance, the system returns to its rest position as quickly as possible without overshooting it.

Note
For strong damping ($\lambda \geq \omega_0$), the system returns to its equilibrium position without oscillating; thus, a damped oscillator does not always oscillate.

Pseudo-periodic regime

A pseudo-periodic regime is a term often used in the context of dynamic systems, particularly in physics, engineering, and mathematics. It describes a behavior that appears to have a regular, periodic structure but doesn't precisely repeat in the same way as true periodic behavior. In other words, while the system may seem to exhibit a repetitive pattern, the intervals or amplitudes might vary slightly or drift over time, preventing exact repetition.
Corresponds to the case where the reduced discriminant is negative:

$$\Delta < 0 \Leftrightarrow \lambda < \omega_0$$

$$\Delta\acute{=}(-1)(\omega_0^2 - \lambda^2) = i^2(\omega_0^2 - \lambda^2)$$

With:ω_α the damping pulsation or the pseudo-pulsation : $\omega_\alpha = \sqrt{\omega_0^2 - \lambda^2}$ Thus, the solutions to the differential equation are:

$$x(t) = A_1 e^{(-\lambda - i\sqrt{\Delta})t} + A_2 e^{(-\lambda + i\sqrt{\Delta})t}$$

$$x(t) = e^{-\lambda t}(A_1 e^{-i\sqrt{\Delta}t} + A_2 e^{+i\sqrt{\Delta}t})$$

The solution to the equation:

$$x(t) = Ae^{-\lambda t}(A\cos\omega_\alpha t + B\sin\omega_\alpha t)$$

$$x(t) = A\cos(\omega_\alpha t - \varphi)$$

The constants A and φ are determined by the initial conditions. The system performs oscillations of decreasing amplitudes and of "pseudo-periodic regime" given by:

$$T_\alpha = \frac{2\pi}{\omega_\alpha}$$

$$T_\alpha = \frac{2\pi}{\omega_\alpha} \Rightarrow T_\alpha = \frac{2\pi}{\sqrt{\omega_0^2 - \lambda^2}}$$

$$\omega_\alpha^2 = \omega_0^2 \sqrt{1 - \zeta^2}$$

$$T_\alpha = \frac{2\pi}{\sqrt{\omega_0^2 - \lambda^2}} = \frac{T_0}{\sqrt{1 - \zeta^2}}$$

$$T_\alpha = \frac{T_0}{\sqrt{1 - \zeta^2}} \Rightarrow T_0 < T_\alpha$$

if $\lambda \ll \omega_0 \Rightarrow \zeta^2 = 1$ so $T - \alpha \simeq T_0$ The curve $x(t)$ is enveloped by the two exponentials $Ae^{-\lambda t}$ and $-Ae^{-\lambda t}$ since, in modulus, $\cos(\omega_\alpha t - \varphi)$ cannot exceed one. We see that x becomes zero as t tends towards its equilibrium position (see figure 3.7). There is a pseudo-frequency, and the motion is described as pseudo-periodic. The damping is weak. It should be noted that the pseudo-frequency ω_α is less than the

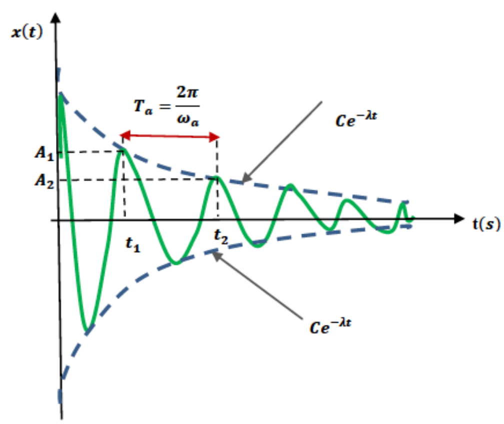

Figure 3.7: Oscillations (pseudo-period)

natural frequency ω_0, and the pseudo-period T_α is greater than the period T_0 of the corresponding undamped oscillator.

In mechanical vibration, a *pseudo-periodic regime* is generally represented by mathematical models that combine periodic and quasi-periodic terms or include non-linear effects. Such models often use coupled differential equations or introduce slowly varying terms to capture deviations from true periodicity. Here are some commonly used equations and formulations for describing pseudo-periodic behavior in mechanical systems:

Harmonic Oscillator with a Slowly Varying Frequency or Amplitude

A basic model for pseudo-periodic vibration can start with a harmonic oscillator with slowly varying parameters:

$$x(t) = A(t)\cos(\omega(t)t + \phi)$$

where: - $A(t)$ is a slowly varying amplitude function, - $\omega(t)$ is a time-dependent angular frequency, - ϕ is a phase constant.

In the pseudo-periodic regime, $A(t)$ and $\omega(t)$ change slowly over time, introducing small variations in the otherwise periodic motion.

Damped Forced Oscillator with Two Competing Frequencies

A damped oscillator driven by two or more incommensurate frequencies can also exhibit pseudo-periodic behavior:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F_1\cos(\omega_1 t) + F_2\cos(\omega_2 t)$$

where: - m is the mass, - c is the damping coefficient, - k is the stiffness, - F_1 and F_2 are forces applied at two different frequencies ω_1 and ω_2.

When ω_1 and ω_2 are not integer multiples of each other, the system will exhibit a pseudo-periodic regime due to the quasi-periodic forcing.

Nonlinear Oscillator with Weak Nonlinearity

A weakly nonlinear oscillator can also exhibit pseudo-periodic behavior, particularly if there's a resonance or near-resonance between different vibrational modes:

$$\frac{d^2x}{dt^2} + \omega_0^2 x + \alpha x^3 = 0$$

where α is a small nonlinear term. This equation is known as the *Duffing equation*.

In the pseudo-periodic regime, nonlinear effects (here through αx^3) introduce slight irregularities in the oscillation patterns, causing a deviation from simple periodicity.

3.4.3 The logarithmic decrement

(or *logarithmic decrease*) of a pseudo-periodic regime is a measure of the rate at which the amplitude of oscillations decays in a damped vibrational system. It quantifies the rate of energy dissipation over each cycle, helping to characterize damping in systems that do not oscillate with a true periodicity.

In the pseudo-periodic regime, where the oscillations are damped but not strictly periodic, the logarithmic decrement can still be applied to approximate the decay behavior. It's defined by the natural logarithm of the ratio of successive peak amplitudes in the damped oscillation.

Definition of Logarithmic Decrement

The logarithmic decrement δ is defined as:

$$\delta = \ln\left(\frac{x(t)}{x(t+T_\alpha)}\right)$$

where: - $x(t)$ is the amplitude at a given time t, - $x(t+T_\alpha)$ is the amplitude after one pseudo-period T_α.

-Equation for Logarithmic Decrement in Damped Systems

For a lightly damped system in a pseudo-periodic regime, the logarithmic decrement can be approximated as:

$$\delta = \frac{2\pi\lambda}{\omega_\alpha}$$

where: - λ is the damping coefficient (associated with the decay rate of the envelope $Ae^{-\lambda t}$), - ω_α is the pseudo-frequency (the damped angular frequency).

- Decay of Amplitude in Pseudo-Periodic Motion

The amplitude $A(t)$ of the pseudo-periodic motion decreases over time according to the exponential envelope:

$$A(t) = A_0 e^{-\lambda t}$$

The motion $x(t)$ in the pseudo-periodic regime is given by:

$$x(t) = A_0 e^{-\lambda t} \cos(\omega_\alpha t + \phi)$$

In this equation: - A_0 is the initial amplitude, - ϕ is the phase shift, - $\cos(\omega_\alpha t + \phi)$ represents the oscillatory component with the pseudo-frequency ω_α.

The logarithmic decrement, δ, provides a way to quantify how quickly the oscillations decay over time and is particularly useful for identifying damping characteristics in lightly damped systems. For a damped system:

$$x(t) = Ce^{-\lambda t} sin(\omega_\alpha t + \varphi)$$

$$\Rightarrow \delta = \ln \frac{Ce^{-\lambda t} sin(\omega_\alpha t + \varphi)}{Ce^{-\lambda(t_1+T\alpha)} sin(\omega_\alpha(t_1 + T_\alpha) + \varphi)}$$

$$\Rightarrow \delta = \ln e^{\lambda T_\alpha} = \lambda T_\alpha$$

$$\lambda T_\alpha = \lambda \frac{T_\alpha}{\sqrt{1-\zeta^2}} = \zeta\omega_0 \frac{T_\alpha}{1-\zeta^2} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

with :

$$\delta = \ln \frac{x(t_1)}{x(t_2)} = 2\pi \frac{\zeta}{\sqrt{1-\zeta^2}} = \lambda T_\alpha$$

Note:

For multiple periods:

$T = T_\alpha(t_1 = t_2 + nT_\alpha) \Rightarrow \delta \ln(\frac{x(t_1)}{x(t_1+nT_\alpha)}) = 2\pi \frac{n\zeta}{\sqrt{1-\zeta^2}}$

The pseudo-period and the logarithmic decrement are only meaningful if the regime is pseudo-periodic

3.4.4 Total Energy of a Damped Harmonic Oscillator

We consider that:

$$x(t) = Ae^{-\lambda t} \sin(\omega t + \varphi)$$

$$\dot{x(t)} = Ae^{-\lambda t}\omega \cos(\omega t + \varphi) - A\lambda e^{-\lambda t} \sin(\omega t + \varphi)$$

In the case of very weak damping $\lambda \to 0$, the pseudo-frequency is approximately equal to the natural frequency of the system, meaning:

$$\omega = \omega_0 \Rightarrow \omega = \sqrt{\frac{k}{m}}$$

The total energy is written as:

$$E_T(T) = U + T \Rightarrow$$

$$E_T(r) = \frac{1}{2} kA^2 e^{-i\omega t} \sin^2(\omega t + \varphi) \frac{1}{2} mA^2 \left[e^{-2\lambda t}\omega^2 \cos^2(\omega t + \varphi) + e^{-2\lambda t} \sin^2(\omega t + \varphi)\right]$$

$$-2\lambda\omega \sin(\omega t + \varphi) \cos(\omega t + \varphi)$$

If we make the second and third terms of the kinetic energy tend towards 0, we obtain for the total energy the following three expressions:

$$E_T(t) = \frac{1}{2}kA^2e^{-2\lambda t}$$

Where:

$$E_T(t) = \frac{1}{2}m\omega_0^2e^{-2\lambda t}$$

3.4.5 The quality factor

In a damped harmonic oscillator, there is a dissipation of mechanical energy. This dissipation is characterized by the coefficient or quality factor,Q, which accounts for the oscillator's efficiency or quality. The quality factor, denoted as Q, is a measure of an oscillator's efficiency in retaining its energy. A high Q-factor indicates that the oscillator effectively conserves its energy, experiencing minimal energy loss per cycle. In a damped harmonic oscillator (such as a pendulum or an oscillating electrical circuit), damping causes a gradual loss of mechanical energy due to friction or resistance. The quality factor Q is inversely related to this energy dissipation: the higher the Q-factor, the better the oscillator can sustain oscillations over a longer period without significant energy loss.

$$Q = 2\pi\frac{E_m}{\Delta E}$$

$$Q = 2\pi\frac{E_T(t)}{\left[\frac{1}{2}kA^2e^{-i\omega t} - \frac{1}{2}kA^2e^{-2\lambda(t+T)}\right]}$$

$$\Rightarrow Q = 2\pi\frac{\frac{1}{2}kA^2e^{-2\lambda t}}{\left[\frac{1}{2}kA^2e^{-i\omega t} - \frac{1}{2}kA^2e^{-2\lambda(t+T)}\right]}$$

$$\Leftrightarrow Q = 2\pi\frac{1}{1-e^{-2\lambda t}}$$

It is assumed that the damping is very low.
$\lambda \to 0$ and $e^{-2\lambda t} = 1 - 2\lambda T$

$$Q = \frac{\pi}{\lambda T}$$

Either :

$$Q = \frac{\omega}{2\lambda}$$

$$Q = \frac{\omega_0}{2\lambda}$$

$$\Delta \geq \Rightarrow \lambda \geq \omega_0$$

$$Q \leq \frac{1}{2}$$

There are no oscillations, the regime is aperiodic, the pseudo periodic is written:

$$T = \frac{2\lambda}{\omega_0\sqrt{1 - \frac{1}{4Q^2}}}$$

$$T_0 = \frac{T_a}{\sqrt{1 - \frac{1}{4Q^2}}}$$

3.4.6 Electric Harmonic Oscillator

The oscillating circuit, in addition to having an inductance L and a capacitance C, also includes an ohmic resistance R. In this type of circuit, the inductance L and capacitance C allow oscillations of electric charge or current, while the resistance R introduces damping. This damping causes the oscillations to gradually decrease in amplitude over time, resulting in energy dissipation through the resistor.

$$U_R + U_c + U_L = 0$$

$$Ri(t) + \frac{1}{c}q + L\frac{\mathrm{d}i}{\mathrm{d}t} = 0$$

$$R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{c}q + L\frac{\mathrm{d}^2q}{\mathrm{d}t^2} = 0$$

$$R\dot{q} + \frac{1}{C}q + L\ddot{q} = 0$$

$$\ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC}q = 0$$

$$\Rightarrow \ddot{q} + 2\lambda\dot{q} + \omega_0^2 q = 0$$

With:

$$\begin{cases} \lambda = & \frac{R}{2L} \\ \omega_0^2 = & \frac{1}{LC} \end{cases}$$

SO :

$$\ddot{q} + 2\lambda\dot{q} + \omega_0^2 q = 0 \Rightarrow \begin{cases} \lambda = & \frac{R}{2L} \\ \omega_0 = & \sqrt{\frac{1}{LC}} \end{cases}$$

Note

For critical damping $\lambda = \omega_0 \Rightarrow \frac{R}{2L} = \sqrt{\frac{1}{LC}}$ So : $R = R_c = 2\sqrt{\frac{L}{C}}$

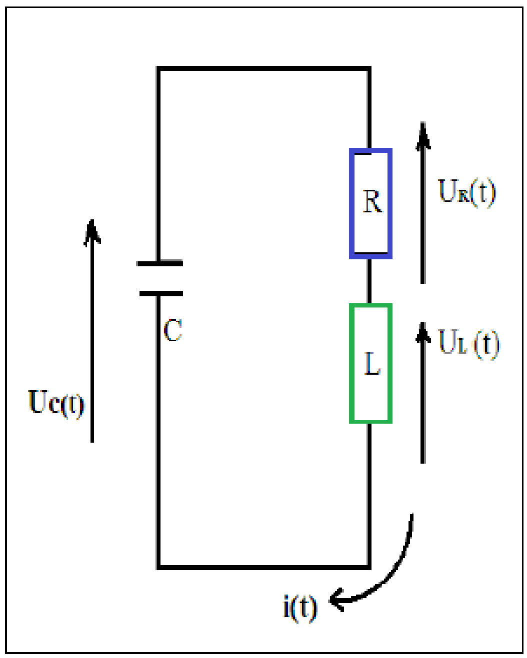

FIGURE 3.8: Electric harmonic oscillator

Feature	Mechanical oscillations	Electrical oscillations
Equation of motion	$m\ddot{x} + \lambda\dot{x} + kx = 0$	$L\ddot{q} + R\dot{q} + \frac{1}{C}q = 0$
Own pulse	$\omega_0 = \sqrt{\frac{k}{m}}(rds^{-1})$	$\omega_0 = \sqrt{\frac{1}{LC}}$
Coefficient of viscous friction	$\lambda(N.s.m^{-1})$	$R(\Omega)$
Damping coefficient	$\alpha = \frac{\lambda}{2m}(s^{-1})$	$\alpha = \frac{R}{2L}$
Damped pulsation	$\omega_\alpha = \sqrt{\frac{k}{m} - \frac{2\lambda}{4m^2}}(rd.s^{-1})$	$\omega_\alpha = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$
Quality factor	$Q = \sqrt{\frac{mk}{\lambda}}$	$Q = \frac{l}{R}\sqrt{\frac{L}{R}}$
Kinetic energy	$T = E_K = \frac{1}{2}m\dot{x}^2(J)$	$E_{BO} = \frac{1}{2}Li^2$
Potential energy	$E_p = \frac{1}{2}kx^2(J)$	$E_{co} = \frac{q^2}{C}$
Dissipation function	$D = \frac{1}{2}\alpha\dot{x}^2$	$E_D = \frac{1}{2}Ri^2$

Table 3.1: Analogy between mechanical and eclectic oscillations

Chapter 4

Forced linear system with one degree of freedom

Resonance is a phenomenon that occurs when a system naturally oscillates at a specific frequency, called its **natural frequency**. When an external force or energy source drives the system at this exact frequency, the system's oscillations get amplified significantly.

Imagine pushing someone on a swing. If you push them at just the right moments, matching the swing's natural rhythm, each push adds more energy, and they go higher and higher. This is resonance: your pushes (the external force) are timed perfectly with the swing's natural frequency, making the motion much larger.

In other contexts, like in bridges or buildings, resonance can be dangerous. If vibrations from wind, traffic, or other sources match the structure's natural frequency, it can cause large oscillations and even structural failure. That's why engineers carefully design structures to avoid resonance with common external forces.

4.1 Definition of Forced Oscillation

To maintain continuous motion, energy must be regularly supplied to the oscillator. This is achieved through an external driving force, which keeps the system oscillating. After a transient period, the system oscillates at the same frequency as the external driving force.

4.2 Lagrange's Equation for Forced Systems

When there is an external driving force $F(t)$, Lagrange's equation is written as: - For translational motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = F(t)$$

- For rotational motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \mu(t)$$

where $\mu(t) = F_{\text{ext}} \cdot L \cdot \frac{\partial r}{\partial \theta}$.

- F_{ext}: The applied external force. - L: Lever arm distance (distance from the axis of rotation to the point of force application). - r: The distance traveled by the mass in the direction of the force.

4.2.1 Example: Mass-Spring-Damper System

Consider the case of a vertical elastic pendulum, as shown in the figure:

It consists of a spring with stiffness constant k and a mass m. The mass is subjected to a damping force $\vec{F}_d = -\alpha\dot{x}$ and an external driving force $\vec{F}_{\text{ext}} = F_0 \sin(\Omega t)$.

The Lagrange equation is written as:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = -\frac{\partial D}{\partial \dot{x}} + F_{\text{ext}} \cdot t$$

- Kinetic Energy of the System: $T = \frac{1}{2}m\dot{x}^2$
- Potential Energy of the System: $U = \frac{1}{2}kx^2$
- Dissipation Function of the System: $D = \frac{1}{2}\alpha\dot{x}^2$
- Lagrangian Function:$L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

By substituting into Equation (IV.1), we have:

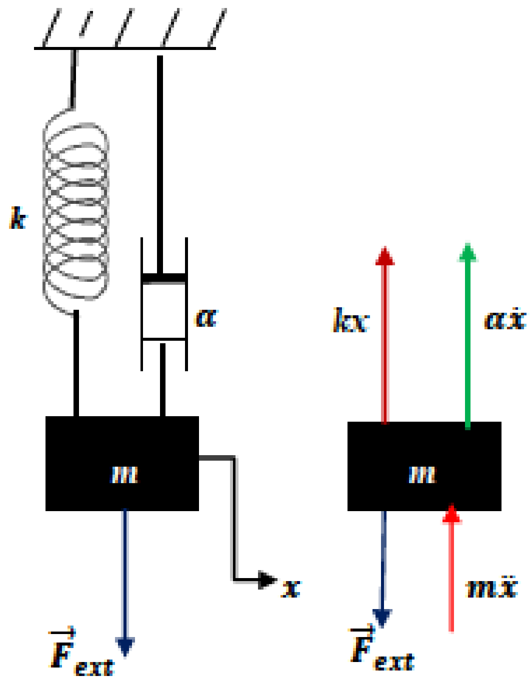

FIGURE 4.1: Mass-Spring-Damper System

Given:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

$$\frac{\partial L}{\partial x} = -kx$$

$$\frac{\partial D}{\partial \dot{x}} = \alpha\dot{x}$$

Substituting these into the Lagrange equation:

$$m\ddot{x} + kx = -\alpha\dot{x} + F_0 \sin(\Omega t)$$

Dividing by m:

$$\ddot{x} + \frac{\alpha}{m}\dot{x} + \frac{k}{m}x = \frac{F_0}{m}\sin(\Omega t)$$

Defining $2\lambda = \frac{\alpha}{m}$ and $\omega_0^2 = \frac{k}{m}$, we get:

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = \frac{F_0}{m}\sin(\Omega t)$$

This is a second-order differential equation with a driving term.

4.3 Solution to the Differential Equation

The solution to this differential equation is the sum of the solution without the driving term (homogeneous solution) $x_H(t)$ and a particular solution of the equation with the driving term $x_p(t)$, such that:

$$x_G(t) = x_H(t) + x_p(t)$$

- $x_H(t)$: The transient response. - $x_p(t)$: The steady-state response.

4.3.1 Transient Response

This is the solution to the homogeneous equation:

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = 0$$

For weak damping ($\lambda < \omega_0$), the solution is:

$$x_h(t) = Ce^{-\lambda t}\cos(\omega_d t + \theta)$$

where $\omega_d = \sqrt{\omega_0^2 - \lambda^2}$.

Solving **forced oscillations** involves analyzing a system where an external force is driving the oscillation. This force is typically periodic, such as $F(t) = F_0 \sin(\Omega t)$, where F_0 is the amplitude and Ω is the driving frequency. The general approach to solving forced oscillations combines the natural response of the system (the solution without the driving force) with the particular solution due to the external force.

Here's a step-by-step method for solving forced oscillations, especially for a damped mass-spring system:

Step 1: Set Up the Differential Equation

For a damped mass-spring system with a mass m, damping coefficient c, spring constant k, and external driving force $F(t) = F_0 \sin(\Omega t)$, the equation of motion is:

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin(\Omega t)$$

This can be rewritten as:

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = \frac{F_0}{m}\sin(\Omega t)$$

where: - $\lambda = \frac{c}{2m}$ is the damping coefficient. - $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural frequency of the undamped system.

Step 2: Find the Homogeneous Solution (Natural Response)
The **homogeneous equation** is the equation without the driving force:

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = 0$$

The solution to this equation, $x_h(t)$, depends on the damping ratio $\zeta = \frac{\lambda}{\omega_0}$:

1. Underdamped Case ($\zeta < 1$):

$$x_h(t) = e^{-\lambda t} \left(C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t) \right)$$

   where $\omega_d = \omega_0 \sqrt{1 - \zeta^2}$ is the damped natural frequency.

2. Critically Damped Case ($\zeta = 1$):

$$x_h(t) = (C_1 + C_2 t) e^{-\lambda t}$$

3. Overdamped Case** ($\zeta > 1$):

$$x_h(t) = C_1 e^{-(\lambda + \omega_r)t} + C_2 e^{-(\lambda - \omega_r)t}$$

   where $\omega_r = \sqrt{\lambda^2 - \omega_0^2}$.

The constants C_1 and C_2 are determined by initial conditions.

Step 3: Find the Particular Solution (Forced Response)
To find the particular solution, $x_p(t)$, which results from the driving force, assume a solution with the same form as the driving force. For a sinusoidal force $F(t) = F_0 \sin(\Omega t)$, try:

$$x_p(t) = A \sin(\Omega t) + B \cos(\Omega t)$$

Substitute $x_p(t)$ and its derivatives into the original differential equation and solve for A and B. This leads to:

$$A = \frac{F_0/m}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\lambda\Omega)^2}}$$

and the phase shift ϕ, where:

$$\tan \phi = \frac{2\lambda\Omega}{\omega_0^2 - \Omega^2}$$

The particular solution can then be written as:

$$x_p(t) = A\sin(\Omega t + \phi)$$

Step 4: Write the General Solution

The total solution $x(t)$ is the sum of the homogeneous and particular solutions:

$$x(t) = x_h(t) + x_p(t)$$

This includes both the transient (homogeneous) response, which decays over time if there is damping, and the steady-state (particular) response, which oscillates at the driving frequency Ω indefinitely.

Step 5: Apply Initial Conditions (If Necessary)

If initial conditions are provided, use them to determine the constants C_1 and C_2 in the homogeneous solution. This will give the complete solution tailored to the specific starting conditions of the system.

Step 6: Analyze the Solution for Resonance

Resonance occurs when the driving frequency Ω is close to the natural frequency ω_0 of the system. In this case, the amplitude of the particular solution $x_p(t)$ can become very large, as the system absorbs energy from the external force more efficiently. This can be seen in the formula for A, where the denominator becomes very small if $\Omega \approx \omega_0$, leading to large oscillations.

4.3.2 Summary

1. Set up the differential equation with the driving force.

2. Solve the homogeneous equation for the transient response.
3. Find the particular solution for the steady-state response using the form of the driving force.
4. Combine the solutions to get the general response.
5. Apply initial conditions if given.
6. Check for resonance conditions, as large amplitudes may result when the driving frequency matches the system's natural frequency.

4.3.3 Steady-State Response

This solution has a form similar to the driving force and solves the complete equation of motion:

$$x_p(t) = A\sin(\Omega t + \phi)$$

This represents the long-term behavior of the system as it oscillates at the driving frequency Ω.

4.4 Study of the Steady-State Regime

In the steady-state regime, only the particular solution remains:

$$x(t) = x_p(t)$$

Using complex representations, the amplitude A of the oscillation can be derived as:

$$A = \frac{F_0}{\sqrt{(k - m\Omega^2)^2 + (\alpha\Omega)^2}}$$

The phase ϕ, representing the phase difference between the displacement and the driving force, is given by:

$$\phi = \arctan\left(\frac{\alpha\Omega}{k - m\Omega^2}\right)$$

4.4.1 Study of the Steady-State Response

In the steady-state regime, only the particular solution exists, so:

$$x(t) \approx x_p = A\sin(\Omega t + \phi)$$

Substituting into the differential equation:

$$\ddot{x}_p + 2\lambda\dot{x}_p + \omega_0^2 x_p = \frac{F(t)}{m}$$

Using complex representations:

1. Assume

$$x_p = \tilde{A}e^{i(\Omega t+\phi)} = \tilde{A}e^{i\Omega t}, where \tilde{A}$$

is the complex amplitude.
2. Then:

$$-\dot{x}_p = i\Omega\tilde{A}e^{i\Omega t}$$

$$-\ddot{x}_p = -\Omega^2\tilde{A}e^{i\Omega t}$$

3. The driving force $F(t) = F_0 e^{i\Omega t}$

Substitute these into the equation to get:

$$-\Omega^2\tilde{A}e^{i\Omega t} + 2i\lambda\Omega\tilde{A}e^{i\Omega t} + \omega_0^2\tilde{A}e^{i\Omega t} = \frac{F_0}{m}e^{i\Omega t}$$

Simplify to find the complex amplitude $\tilde{A}$:

$$\tilde{A} = \frac{\frac{F_0}{m}}{\omega_0^2 - \Omega^2 + 2i\lambda\Omega}$$

To express this in terms of magnitude, we separate real and imaginary parts:

$$|\tilde{A}| = \frac{\frac{F_0}{m}}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\lambda\Omega)^2}}$$

Thus, the solution for this forced oscillator in the steady-state regime is:

$$x(t) = A\sin(\Omega t + \phi)$$

where the amplitude A is:

$$A = \left|\tilde{A}\right| = \frac{\frac{F_0}{m}}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\lambda\Omega)^2}}$$

This describes the amplitude of the oscillation at the driving frequency Ω, taking into account the damping in the system. The expression for the complex amplitude $\tilde{A}$ is given by:

$$\tilde{A} = \frac{\frac{F_0}{m}}{(\omega_0^2 - \Omega^2) + 2i\lambda\Omega} = \frac{\frac{F_0}{m}\left(\omega_0^2 - \Omega^2\right)}{\left(\omega_0^2 - \Omega^2\right)^2 + 4\lambda^2\Omega^2} - i\frac{\frac{F_0}{m} \cdot 2\lambda\Omega}{\left(\omega_0^2 - \Omega^2\right)^2 + 4\lambda^2\Omega^2}$$

The steady-state solution for this forced oscillator can then be written as:

$$x(t) = A\sin(\Omega t + \phi)$$

where the amplitude A of the oscillation is:

$$A = |\tilde{A}| = \sqrt{\left(\frac{\frac{F_0}{m} \cdot (\omega_0^2 - \Omega^2)}{(\omega_0^2 - \Omega^2)^2 + 4\lambda^2\Omega^2}\right)^2 + \left(\frac{\frac{F_0}{m} \cdot 2\lambda\Omega}{(\omega_0^2 - \Omega^2)^2 + 4\lambda^2\Omega^2}\right)^2}$$

This amplitude A depends on the natural frequency ω_0, the damping coefficient λ, the driving frequency Ω, and the driving force F_0. This relationship illustrates how the amplitude of forced oscillations changes with different driving frequencies and damping factors.

Explanation of the Solution and Important Remarks

The **amplitude** A of the oscillation is given by:

$$A = \left|\tilde{A}\right| = \frac{\frac{F_0}{m}}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\lambda\Omega)^2}}$$

The **phase** ϕ of the motion, which represents the phase shift between $x(t)$ and the driving force $F(t)$, is given by:

$$\tan\phi = \frac{-2\lambda\Omega}{\omega_0^2 - \Omega^2}$$

Thus, the particular solution $x_p(t)$ can be written as:

$$x_p(t) = \frac{B}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\lambda\Omega)^2}} \sin\left(\Omega t + \arctan\left(\frac{2\lambda\Omega}{\omega_0^2 - \Omega^2}\right)\right)$$

Important Remarks

1. General Solution**:
   - The general solution is the sum of two parts: - A **homogeneous solution** with an amplitude and energy that decay over time until they become zero. This is known as the **transient solution**. - A **particular solution** with

a constant amplitude A and frequency matching the driving force, representing the **steady-state** or **permanent solution**. - The amplitude A of the particular solution is given by:

$$A = \frac{B}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\lambda\Omega)^2}}$$

2. Long-Term Behavior:
   - Over time, the transient (homogeneous) solution disappears, leaving only the steady-state oscillatory motion. - For times $t < t_0$, the total motion is the sum of the transient and permanent solutions:

$$x(t) = x_p(t) + x_h(t)$$

   - For times $t \geq t_0$, only the steady-state motion remains:

$$x(t) = x_p(t)$$

This analysis describes how the system initially has both transient and steady-state components, but eventually, only the steady-state component persists.

The **damping ratio** is an important concept in many real-world systems, as it helps to predict how systems respond to disturbances. Here are some practical examples illustrating different damping ratios:

1. Car Suspension System :
   - Underdamped ($\zeta < 1$): Most car suspensions are designed to be slightly underdamped. This allows the car to absorb shocks from bumps in the road while returning to a stable position quickly. A perfectly underdamped suspension provides a comfortable ride with controlled oscillations after hitting a bump.
   - Overdamped ($\zeta > 1$): If the suspension is overdamped, the car's ride becomes stiff and uncomfortable, as it takes longer to return to the neutral position. This can cause the car to feel heavy and unresponsive.
   - Critically Damped ($\zeta = 1$): Some high-performance or luxury car suspensions

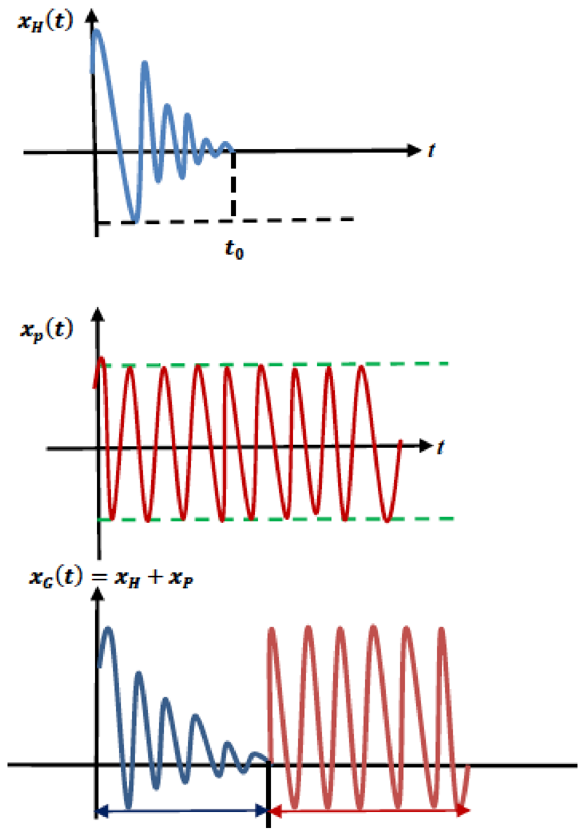

FIGURE 4.2: Transitional regime and permanent regime

aim for near-critical damping to minimize oscillations, providing a smooth, quick return to stability after a disturbance without noticeable bouncing.

2. Door Closers:
   - Critically Damped ($\zeta = 1$): Many automatic door closers, like those in office buildings, are designed to be critically damped. This means the door will close in the shortest possible time without bouncing or slamming shut. This provides a smooth, controlled closing motion.
   - Underdamped ($\zeta < 1$): If the damping is too low, the door might close quickly but could bounce slightly when it reaches the frame, which can be noisy and potentially damaging over time.
   - Overdamped ($\zeta > 1$): If the door closer is overdamped, the door closes very slowly, which can be inconvenient for people waiting for it to shut.

3. Building and Bridge Design:
   - Critically Damped or Overdamped ($\zeta \geq 1$): Tall buildings and bridges are designed with damping mechanisms to resist oscillations caused by wind or seismic activity. These structures often have **tuned mass dampers** or other damping systems that provide critical or overdamping, which prevents swaying or resonant oscillations.
   - Underdamped: In cases where buildings or bridges have insufficient damping, strong winds or earthquakes can cause prolonged oscillations. This was famously seen in the Tacoma Narrows Bridge collapse, where wind-induced vibrations led to oscillations that were not adequately damped.

4. Electronic Circuits (RLC Circuits) - In electronics, **RLC circuits** (circuits with resistors, inductors, and capacitors) often require specific damping characteristics:
   - Underdamped ($\zeta < 1$): Allows oscillations, which can be useful in signal processing and radio frequency applications. For example, radio receivers use underdamped circuits to tune into specific frequencies.
   - Critically Damped ($\zeta = 1$): Used when quick response without oscillations is desired, such as in some filter circuits.
   - Overdamped ($\zeta > 1$): In cases where a slow response is acceptable, such as in power supply circuits where stability is prioritized over speed.

5. Seismology and Earthquake Engineering:
   - Overdamped ($\zeta > 1$): Earthquake-resistant structures often incorporate heavy damping to avoid oscillations during seismic events. Engineers use base isolators and dampers to absorb energy from ground motion, allowing the structure to settle without extensive oscillations.
   - Critically Damped ($\zeta = 1$): Some advanced earthquake-proof designs aim for critical damping, allowing the building to return to rest as quickly as possible after ground shaking stops.

6. Aircraft Landing Gear:
   - Critically Damped or Slightly Underdamped:
   Aircraft landing gear is designed to absorb the impact energy upon landing and dissipate it quickly. A slightly underdamped or critically damped response allows the landing gear to compress and extend without causing the aircraft to bounce excessively on the runway, providing stability and passenger comfort.

7. Sports Equipment (Tennis Rackets, Golf Clubs):
   - Underdamped ($\zeta < 1$): Some sports equipment, like tennis rackets and golf clubs, use materials and designs that allow a slight rebound effect (underdamping) to transfer energy back to the ball effectively. However, if the damping is too low, vibrations can be uncomfortable for the player.
   - Overdamped: In some cases, extra damping materials (such as shock-absorbing grips) are added to reduce vibrations, particularly for beginners or people prone to joint pain. These materials increase the damping ratio to reduce the "buzzing" feel after hitting the ball.

8. Audio Systems and Loudspeakers:
   - Underdamped ($\zeta < 1$): A slightly underdamped loudspeaker design can create a fuller sound with strong bass, as it allows the speaker cone to oscillate more freely. However, too little damping can lead to a "boomy" sound with unwanted resonances.
   - Critically Damped or Overdamped: For a more accurate and clear sound, especially for high-fidelity audio, critical or slightly overdamped designs are preferred to avoid prolonged oscillations and provide precise sound reproduction.

- Underdamped ($\zeta < 1$): Allows some oscillation, which can be beneficial or desirable in systems that need to rebound or resonate (e.g., car suspensions, sports equipment). - **Critically Damped ($\zeta = 1$)**: Provides the quickest return to stability without oscillations, ideal for systems requiring precise, controlled motion (e.g., door closers, landing gear). - Overdamped ($\zeta > 1$): Slows the return to stability without oscillations, useful when stability is more important than response speed (e.g., seismic protection, heavy-duty machinery).

These examples demonstrate how engineers and designers use the damping ratio to optimize the performance and safety of various systems in response to oscillations or disturbances.

4.4.2 Resonance Phenomenon

Amplitude Resonance

We are going to study the variation of amplitude A as a function of the excitation frequency Ω:

$$A(\Omega) = \frac{\frac{F_0}{m}}{\sqrt{(\omega_0^2 - \Omega^2)^2 + 4\lambda^2\Omega^2}}$$

When: - $\Omega = 0$:

$$A(0) = \frac{\frac{F_0}{m}}{\omega_0^2} = \frac{F_0}{k}$$

- $\Omega \to \infty$:

$$A(\infty) = 0$$

The maximum amplitude is obtained when $\frac{dA}{d\Omega} = 0$:

$$\frac{dA}{d\Omega} = \frac{[2(-2\lambda)(\omega_0^2 - \Omega^2) + 8\lambda^2\Omega]\frac{F_0}{m}}{2((\omega_0^2 - \Omega^2)^2 + 4\lambda^2\Omega^2)^{\frac{3}{2}}}$$

$$\Rightarrow (-\omega_0^2 + \Omega^2) + 2\lambda\Omega = 0$$

$$\Rightarrow \Omega^2 = \omega_0^2 - 2\lambda^2$$

This frequency is called the **resonant frequency** and is denoted by:

$$\Omega_R = \sqrt{\omega_0^2 - 2\lambda^2}$$

At this frequency, the maximum amplitude exists only if:

$$\lambda < \frac{\omega_0}{2} \quad ; \quad (\Omega_R > 0)$$

The maximum amplitude is then:

$$A_{\max} = \frac{F_0}{m \cdot 2\lambda\sqrt{\omega_0^2 - \lambda^2}}$$

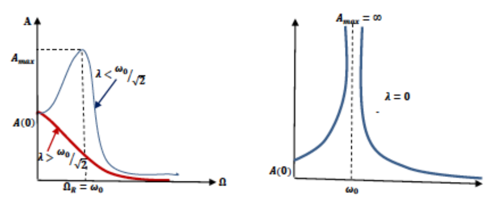

FIGURE 4.3: Variation of the amplitude a as a function of .

Variation of Phase as a Function of Ω

We have found:

$$\tan\phi = \frac{-2\lambda\Omega}{\omega_0^2 - \Omega^2}$$

$$\frac{d\phi}{d\Omega} = \cos^2\phi \cdot \frac{d(\tan\phi)}{d\Omega} \quad \text{and} \quad \cos\phi = \frac{1 - \frac{\Omega^2}{\omega_0^2}}{\sqrt{(\omega_0^2 - \Omega^2)^2 + 4\lambda^2\Omega^2}}$$

$$\frac{d\phi}{d\Omega} = \frac{-2\lambda\left(\omega_0^2 + \Omega^2\right)}{\left(\omega_0^2 - \Omega^2\right)^2 + 4\lambda^2\Omega^2}$$

Observations:

- If $\Omega \to 0$: - Then $\tan\phi \to 0$ and $\frac{d\phi}{d\Omega} \approx -\frac{2\lambda}{\omega_0}$.

- If $\Omega \to \omega_0$: - $\phi = \frac{\pi}{2}$, meaning the displacement is $\frac{\pi}{2}$ ahead of the excitation.

- When $\Omega \to \infty$: - $\tan\phi$ and $\frac{d\phi}{d\Omega}$ tend toward 0. - Since $\frac{d\phi}{d\Omega}$ is always negative, the phase shift is $-\pi$.

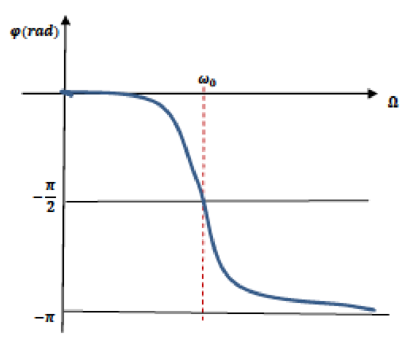

FIGURE 4.4: Phase variation as a function of

Resonance Phenomenon and Quality Factor

The resonance phenomenon appears when the excitation frequency approaches the natural frequency of the system.

In electrical systems, this phenomenon allows for the calculation of the quality factor Q, which increases as:

$$Q = \frac{A_{\max}}{A_0} = \frac{1}{2\zeta}$$

Another practical method to determine the quality factor is:

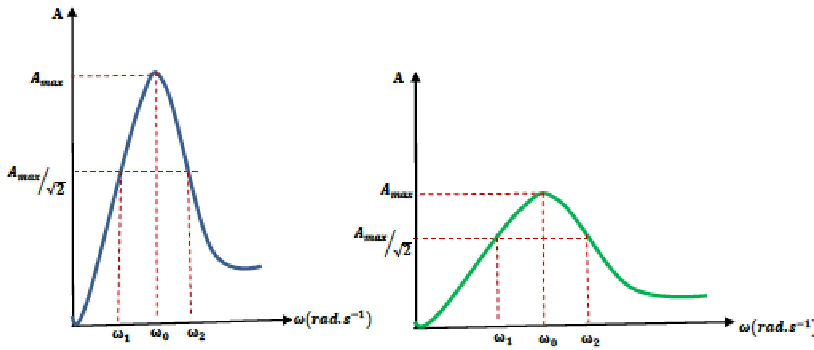

FIGURE 4.5: The resonance and the quality diminish

$$Q = \frac{\omega_0}{\omega_2 - \omega_1}$$

where $\omega_2 - \omega_1$ represents the bandwidth.

4.4.3 Conclusion

- When Q decreases $\Rightarrow \omega_2 - \omega_1$ increases $\Rightarrow$ the resonance curve becomes wider $\Rightarrow$ the amplitude at resonance and the quality diminish. - The ends of the bandwidth correspond to an amplitude that is $\sqrt{2}$ times smaller than at resonance.

Chapter 5

Coupled Oscillators

5.1 Introduction

5.1.1

Coupled Oscillators

Definition:

Coupled oscillators refer to systems with two or more oscillators that interact or influence each other through a coupling mechanism. In coupled systems, the oscillators are linked so that energy can transfer between them. This transfer of energy leads to complex oscillatory behavior that depends on the nature of the coupling and the characteristics of each oscillator.

Energy Exchange:

In coupled systems, the oscillators exchange energy. For example, in a mechanical system, two pendulums connected by a spring can transfer energy back and forth as they oscillate. The coupling allows each oscillator's motion to affect the other, resulting in a shared dynamic response.

Types of Coupling in Mechanical Oscillators:

1. Elastic Coupling:
   Here, the oscillators are connected by elastic elements, such as springs. The springs provide a restoring force that influences the movement of each oscillator, allowing them to interact. Elastic coupling can be set up with oscillators placed either horizontally or vertically. When the system is disturbed, the oscillators will start to oscillate, transferring energy through the spring.

2. Inertial Coupling:
   In this case, the oscillators are connected by a mass, which introduces inertia to the system. The mass acts as a bridge between the oscillators, causing them to influence each other's motion through the shared inertia. This type of coupling is commonly seen in pendulums connected by a shared mass at their center.

—

5.2 Example of Coupled Free Oscillators

5.2.1 Mechanical Systems

Free Mechanical Oscillators Coupled by Elasticity:

In this setup, two masses m_1 and m_2 are connected by two springs with spring constants k_1 and k_2. The springs create a force that acts on each mass, resulting in coupled motion when one mass is displaced. The restoring force from each spring causes energy to transfer between the two masses, leading to a synchronized oscillation Figure 5-1.

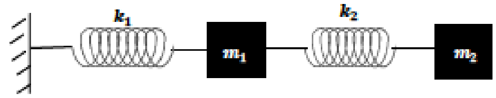

FIGURE 5.1: Free mechanical oscillators coupled by elasticity.

-Diagram Explanation:
The diagram (Figure 5-1) shows two masses m_1 and m_2 connected by springs with constants k_1 and k_2, representing an elastic coupling. When one of the masses is moved, the springs exert forces that affect the motion of both masses, creating a coupled oscillatory motion.

Free Mechanical Oscillators Coupled by Inertia:

Here, the coupling is achieved through a shared mass. For example, two pendulums, each with masses m_1 and m_2, are connected by a rod with a mass in the middle. The rod introduces inertia, linking the motion of the two pendulums so that their oscillations affect each other. When one pendulum moves, it causes the center mass to shift, influencing the motion of the other pendulum. -Diagram Explanation:
In Figure 5-2, two pendulums with masses m_1 and m_2 are shown, connected by a rod with a central mass. This central mass acts as the coupling mechanism, allowing the oscillations of each pendulum to influence each other through the inertia of the central mass. **-Elastic coupling (springs) for energy transfer through elastic forces.**
-Inertial coupling (central mass) for energy transfer through shared inertia.

Each type of coupling leads to different types of oscillatory behavior, influenced by factors like the stiffness of the springs in elastic coupling or the mass distribution in inertial coupling. These concepts are fundamental in understanding the dynamics of coupled systems, which are prevalent in fields such as physics, engineering, and biology. Here's the translation and explanation:

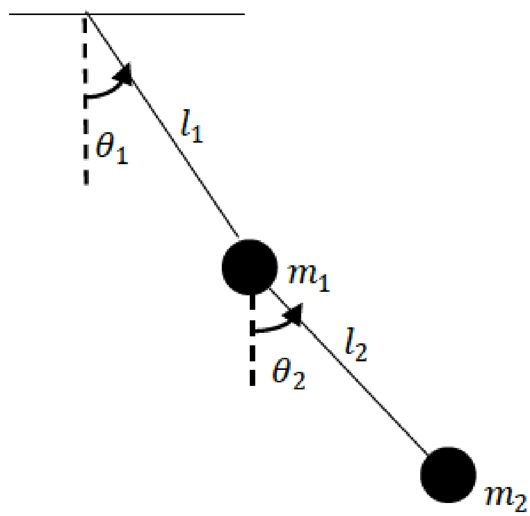

FIGURE 5.2: Free mechanical oscillators coupled by inertia

Viscous Coupling in Mechanical Oscillators:

In this configuration, the coupling is due to a "viscous friction" mechanism. This type of coupling is achieved through a damping element that introduces resistance to the motion, leading to energy dissipation between oscillators. -Diagram Explanation:

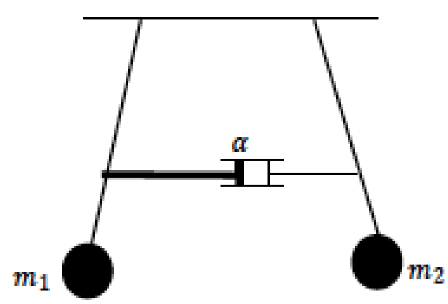

FIGURE 5.3: Viscous Coupling in Mechanical Oscillators

Figure 5-3 illustrates two masses m_1 and m_2 connected by a damping element with viscous friction coefficient α. The damping force between the masses allows for the transfer of energy in a way that gradually reduces the amplitude of oscillations due to the dissipative nature of the viscous coupling.

Inertial Coupling in Electrical Oscillators:

This coupling is represented by an inductive link in an electrical circuit. Here, inductors are used to couple two LC circuits. The inductors create a magnetic field that allows the oscillations of one circuit to influence the other.

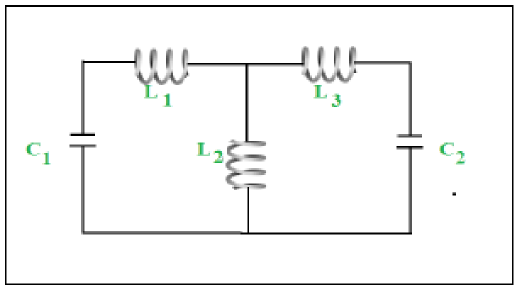

FIGURE 5.4: Inertial Coupling in Electrical Oscillators

- Diagram Explanation**:

Figure 5-4 shows two coupled LC circuits with inductors L_1, L_2, and L_3 and capacitors C_1 and C_2. The inductive coupling allows energy transfer through the mutual inductance, leading to synchronized oscillations between the circuits.

Viscous Coupling in Electrical Oscillators:

In electrical systems, viscous coupling is represented by a **resistive link**. A resistor is used to connect two oscillating circuits, which introduces a damping effect similar to mechanical viscous coupling.

- Diagram Explanation:

Figure 5-5 illustrates two LC circuits coupled by a resistor R between them. The resistor introduces a damping effect, allowing energy dissipation. This coupling method reduces the amplitude of oscillations over time, similar to mechanical viscous coupling.

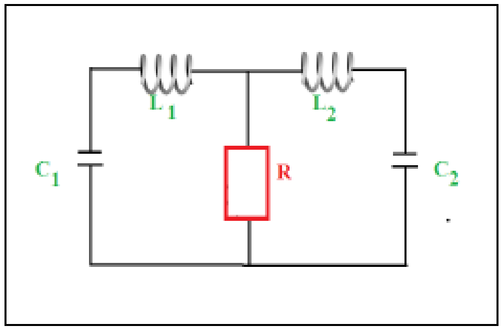

FIGURE 5.5: Viscous Coupling in Electrical Oscillators

These examples illustrate various ways of coupling oscillators, including: - Mechanical viscous coupling (damping force). - Electrical inertial coupling (inductive link). - Electrical viscous coupling (resistive link).

Each coupling type affects the system dynamics by controlling energy transfer and dissipation, impacting oscillation synchronization, amplitude, and frequency.

This diagram and accompanying text outline a system with two degrees of freedom, specifically a coupled mass-spring system. Here's a translation and explanation of each component:

5.3 Two Degrees of Freedom System

To analyze systems with two degrees of freedom, it's essential to write two differential equations of motion, which can be derived using Lagrange's equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

where q_1 and q_2 represent the generalized coordinates of the system.

For a system with two generalized coordinates, there will be two differential equations and two natural frequencies (denoted ω_1 and ω_2).

5.3.1 Example: Mass-Spring System

In this example, the system consists of two masses (m_1 and m_2) connected by springs with constants k_1, k_2, and k, as shown in the diagram.

Equations of Motion

To derive the equations of motion, we apply Newton's second law directly:

1. For Mass m_1:

The forces acting on m_1 include the force from the spring k_1 (connected to a fixed point), the force from the spring k (connected to m_2), and any external force F_{ext}.

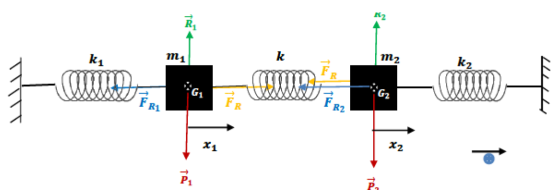

FIGURE 5.6: Mass-springs

The equation of motion for m_1 is:

$$m_1\ddot{x}_1 = F_{\text{ext}} - (k_1x_1) - k(x_1 - x_2)$$

2. For Mass m_2:

Similarly, the forces acting on m_2 include the force from the spring k_2 (connected to a fixed point), the force from the spring k (connected to m_1), and any external force F_{ext}.

The equation of motion for m_2 is:

$$m_2\ddot{x}_2 = F_{\text{ext}} - (k_2x_2) + k(x_1 - x_2)$$

System of Equations

The system of equations that describe the motion of the masses is as follows:

1. For m_1:

$$m_1\ddot{x}_1 + (k_1 + k)x_1 - kx_2 = 0$$

2. For m_2:

$$m_2\ddot{x}_2 + (k_2 + k)x_2 - kx_1 = 0$$

These coupled differential equations can be solved to find the natural frequencies ω_1 and ω_2 and the mode shapes of the system.

This analysis provides insights into the behavior of coupled oscillators, showing how the masses interact through the springs and how energy transfers between them in a coupled system.

1. Applying Newton's Second Law (PFD) Directly: - For mass m_1:

$$\sum F_{\text{ext}} = m_1\ddot{x}_1$$

- For mass m_2:

$$\sum F_{\text{ext}} = m_2\ddot{x}_2$$

2. Breaking Down the Forces Acting on Each Mass: - For m_1:

$$F_{k1} + F_k = m_1\ddot{x}_1$$

where F_{k1} is the force from spring k_1, and F_k is the force from the coupling spring k that connects m_1 to m_2. - For m_2:

$$F_{k2} + F_k = m_2\ddot{x}_2$$

where F_{k2} is the force from spring k_2, and F_k is the force from the coupling spring k.

3. Writing the Force Equations in Terms of Displacements: - Force on m_1 from springs:

$$-k_1x_1 + k(x_2 - x_1) = m_1\ddot{x}_1$$

- Force on m_2 from springs:

$$-k_2x_2 + k(x_1 - x_2) = m_2\ddot{x}_2$$

4. Simplifying the Equations: - Rearrange the equations to group terms with x_1 and x_2:

$$-(k + k_1)x_1 + kx_2 = m_1\ddot{x}_1$$

$$-(k + k_2)x_2 + kx_1 = m_2\ddot{x}_2$$

5. Final Form of the Coupled Differential Equations:
- For m_1:

$$m_1\ddot{x}_1 + (k + k_1)x_1 - kx_2 = 0$$

- For m_2:

$$m_2\ddot{x}_2 + (k + k_2)x_2 - kx_1 = 0$$

These two equations represent the coupled differential equations of motion for the two masses in the system. Solving these equations allows us to determine the natural frequencies and mode shapes of the system, which describe how the masses oscillate together under the influence The image shows the system of equations for coupled oscillators, where we treat the oscillators as decoupled for simplification. Here's a step-by-step translation and explanation of each part:

Equations of Motion

1. Starting System of Coupled Equations (V-1):** - The system of equations describes the motion of two masses, m_1 and m_2, connected by springs with constants k_1, k_2,

and a coupling spring k:

$$\ddot{x}_1 + \frac{k + k_1}{m_1} x_1 - \frac{k}{m_1} x_2 = 0$$

$$\ddot{x}_2 + \frac{k + k_2}{m_2} x_2 - \frac{k}{m_2} x_1 = 0$$

Decoupling the Oscillators

2. Using the Concept of Decoupled Oscillators: - To simplify, we consider each oscillator individually by fixing one mass at a time.

- Step 1: Fix m_2** (assuming m_2 does not move): - The natural frequency (pulsation) for the decoupled oscillator with m_1 free to move is:

$$\omega_{01} = \sqrt{\frac{k + k_1}{m_1}}$$

- Step 2: Fix m_1** (assuming m_1 does not move):
- The natural frequency for the decoupled oscillator with m_2 free to move is:

$$\omega_{02} = \sqrt{\frac{k + k_2}{m_2}}$$

Defining Natural Frequencies for Decoupled Oscillators

3. Defining the Natural Frequencies: - For oscillator m_1:

$$\omega_{01}^2 = \frac{k + k_1}{m_1}$$

- For oscillator m_2:

$$\omega_{02}^2 = \frac{k + k_2}{m_2}$$

4. Substituting into the System:
- The system of differential equations can now be rewritten in terms of the decoupled

natural frequencies:

$$\begin{cases} \ddot{x}_1 + \omega_{01}^2 x_1 - \frac{k}{m_1} x_2 &= 0 \\ \ddot{x}_2 + \omega_{02}^2 x_2 - \frac{k}{m_2} x_1 &= 0 \end{cases} \quad (V-1) \tag{5.1}$$

These equations describe the motion of each mass in terms of the modified (decoupled) natural frequencies, which simplifies further analysis.

5.4 Deriving the Equations of Motion using the Lagrange Method

We start by calculating the kinetic and potential energies for a system comprising two masses.

1. Kinetic Energy (T): The kinetic energy of the system is given by:

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

   where: - m_1 and m_2 are the masses, - $\dot{x}_1$ and $\dot{x}_2$ are the velocities of each mass.

2. Potential Energy (U): The potential energy of the system includes contributions from the springs connected to each mass and the coupling spring between the masses:

$$U = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 + \frac{1}{2} k (x_2 - x_1)^2$$

   where: - k_1 and k_2 are the spring constants of the individual springs, - k is the spring constant of the coupling spring.

Lagrangian (L):

The Lagrangian L is the difference between the kinetic and potential energies:

$$L = T - U$$

Applying Lagrange's Equations:

The equations of motion are derived by applying Lagrange's equations for each coordinate x_1 and x_2:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0$$

These equations allow us to describe the motion of each mass in terms of the system's kinetic and potential energies. Solving these differential equations will yield the motion of each mass as functions of time.

5.5 Finding the Natural Frequencies Using the Matrix Method

To find the natural frequencies (eigenfrequencies) of the system, we start by revisiting the equations (V.1) and looking for solutions of the form:

$$x_1 = A_1 \cos(\omega t + \varphi_1), \quad A_1 > 0$$

$$x_2 = A_2 \cos(\omega t + \varphi_2), \quad A_2 > 0$$

In complex notation, these can be represented as:

$$\bar{x}_1 = A_1 e^{i(\omega t - \varphi_1)}$$

$$\bar{x}_2 = A_2 e^{i(\omega t - \varphi_2)}$$

Substituting these expressions into system (V-1) gives:

$$\begin{cases} (\omega_{01}^2 - \omega^2)\,\bar{x}_1 - \frac{k}{m_2}\bar{x}_2 & = 0 \\ -\frac{k}{m_1}\bar{x}_1 + (\omega_{02}^2 - \omega^2)\,\bar{x}_2 & = 0 \end{cases} \quad (V-2) \tag{5.2}$$

These equations, now represented in matrix form as equation (V-2), allow us to solve for the eigenfrequencies of the system by setting the determinant to zero, leading to a characteristic equation. Solving this equation will yield the natural frequencies at which the system oscillates.

The homogeneous system admits non-zero solutions if, and only if, the determinant in equation (V.3) is zero. This gives:

$$\begin{vmatrix} \omega_{01}^2 - \omega^2 & -\frac{k}{m_2} \\ -\frac{k}{m_1} & \omega_{02}^2 - \omega^2 \end{vmatrix} = 0 \quad (V-3) \tag{5.3}$$

Expanding this determinant results in:

$$(\omega_{01}^2 - \omega^2)(\omega_{02}^2 - \omega^2) - \frac{k^2}{m_1 m_2} = 0$$

This can be written as:

$$(\omega_{01}^2 - \omega^2)(\omega_{02}^2 - \omega^2) = \frac{k^2}{m_1 m_2}$$

Further expanding this equation leads to a quartic (fourth-order) equation in ω:

$$\omega^4 - (\omega_{01}^2 + \omega_{02}^2)\omega^2 + \omega_{01}^2\omega_{02}^2 - \frac{k^2}{m_1 m_2} = 0 \quad (V-4) \tag{5.4}$$

This is denoted as equation (V-4).

The discriminant Δ of the quadratic equation in ω^2 is given by:

$$\Delta = (\omega_{01}^2 + \omega_{02}^2)^2 - 4\left(\omega_{01}^2\omega_{02}^2 - \frac{k^2}{m_1 m_2}\right)$$

Solving this equation for ω will provide the two natural frequencies of the coupled system, allowing us to understand the oscillatory modes of the system.
The discriminant Δ of the quadratic equation in ω^2 is given by:

$$\Delta = (\omega_{01}^2 + \omega_{02}^2)^2 - 4\left(\omega_{01}^2\omega_{02}^2 - \frac{k^2}{m_1 m_2}\right)$$

Expanding this further:

$$\Delta = (\omega_{01}^2 + \omega_{02}^2)^2 + \frac{k^2}{m_1 m_2}$$

Or in terms of the spring constants and masses:

$$\Delta = \left(\frac{k+k_1}{m_1} + \frac{k+k_2}{m_2}\right)^2 + \frac{4k^2}{m_1 m_2}$$

Since the determinant Δ is positive, we find that both solutions for the frequencies ω_1^2 and ω_2^2 are real and positive, given by:

$$\omega_1^2 = \frac{\omega_{01}^2 + \omega_{02}^2 - \sqrt{\Delta}}{2}$$
$$\omega_2^2 = \frac{\omega_{01}^2 + \omega_{02}^2 + \sqrt{\Delta}}{2}$$

These values represent the natural frequencies of the system. The polynomial $f(x)$ can be written as:

$$f(x) = x^2 - (\omega_{01}^2 + \omega_{02}^2)x + \omega_{01}^2\omega_{02}^2 - \frac{k^2}{m_1 m_2}$$

Assuming $\omega_{01} < \omega_{02}$, this results in two distinct solutions, which are the natural frequencies of the coupled oscillatory system. The polynomial $f(x)$ can be expressed as:

$$f(x) = (x^2 - \omega_1^2)(x^2 - \omega_2^2)$$

We observe that:

$$f(\omega_{01}^2) = f(\omega_{02}^2) = -\frac{k^2}{m_1 m_2} < 0$$

From this, we deduce that:

$$\omega_2 < \omega_{01} < \omega_{02} < \omega_1$$

Assuming $\omega_{01} < \omega_{02}$, we establish the relationship between the natural frequencies of the system, highlighting the range in which each frequency lies relative to the system's parameters.

5.6 Finding the Eigenmodes x_1 and x_2

By replacing x_1 and x_2 with their complex expressions:

$$x_1 = A_1 e^{i\varphi_1} e^{i\omega t} \quad \text{and} \quad x_2 = A_2 e^{i\varphi_2} e^{i\omega t}$$

In the system (V-2), we obtain for the first equation:

$$(\omega_{01}^2 - \omega^2) k x_1 e^{i\varphi_1} - \frac{k}{m_1} x_2 e^{i\varphi_2} = 0$$

So:

$$e^{i(\varphi_1 - \varphi_2)} = \frac{m_1(\omega_{01}^2 - \omega^2)}{k} x_1$$

Since this quantity is real, we conclude that this condition constrains the relative amplitudes and phases of x_1 and x_2, providing the solution for the normal modes of the coupled oscillator system. The equation presented here is focused on finding the phase difference between φ_1 and φ_2 for two coupled oscillators. Here's the interpretation of each case:

1. When $\sin(\varphi_1 - \varphi_2) = 0$:

This implies that $\varphi_1 - \varphi_2 = \pm\pi$ or 0, leading to two possible cases:

1. Case 1: $\omega = \omega_1$:
   (a) For this situation, we have $\omega_{01}^2 - \omega^2 < 0$.
   (b) Therefore, $\cos(\varphi_1 - \varphi_2) = -1$, implying $\varphi_1 - \varphi_2 = \pi$.
   (c) This result means that the two masses oscillate in opposite phase.
   (d) The amplitude relation between x_1 and x_2 becomes:

$$\frac{x_2(\omega)}{x_1(\omega)} = \frac{m_1(\omega_{01}^2 - \omega^2)}{k}$$

2. Case 2: $\omega = \omega_2$:
   (a) In this case, $\omega_{02}^2 - \omega^2 > 0$.
   (b) Here, $\cos(\varphi_1 - \varphi_2) = 1$, giving $\varphi_1 - \varphi_2 = 0$.
   (c) This result means that the two masses oscillate in phase.
   (d) The amplitude relation in this case is:

$$\frac{x_2(\omega)}{x_1(\omega)} = \frac{m_1(\omega_{02}^2 - \omega^2)}{k}$$

In summary:

- When $\omega = \omega_1$, the oscillators vibrate in opposite phase.
- When $\omega = \omega_2$, the oscillators vibrate in phase.

These two modes correspond to the two natural frequencies of the system, representing the fundamental modes of vibration for the coupled oscillators.
In this analysis of coupled oscillators, we see two specific configurations of vibration modes, describing how the masses m_1 and m_2 oscillate with respect to each other. The following describes each case:

1. Vibrations in Opposite Phase:

   (a) When the natural frequency of oscillation is $\omega = \omega_1$, the two masses vibrate in opposite phase. This means that as x_1 reaches a peak in one direction, x_2 reaches a peak in the opposite direction.

   (b) The phase relationship for this mode is $\varphi_1 - \varphi_2 = \pi$.

   (c) The amplitude relation between the displacements x_1 and x_2 is given by:

   $$\frac{x_2(\omega)}{x_1(\omega)} = \frac{m_1(\omega_{01}^2 - \omega^2)}{k}$$

   (d) For this mode, we have:

   $$x_1 = A_1(\omega_1)\cos(\omega_1 t + \varphi_1)$$

   $$x_2 = -A_1(\omega_1)\cos(\omega_1 t + \varphi_1)$$

2. Vibrations in Phase:

   (a) - For the natural frequency $\omega = \omega_2$, the two masses vibrate in phase, meaning x_1 and x_2 move in the same direction simultaneously.

   (b) The phase difference here is $\varphi_1 - \varphi_2 = 0$.

   (c) In this case, the amplitude relation is:

   $$\frac{x_2(\omega)}{x_1(\omega)} = \lambda_1 = \frac{m_1(\omega_1^2 - \omega^2)}{k}$$

These two cases represent the fundamental vibration modes of the system. The first mode corresponds to opposite-phase vibrations, while the second corresponds to in-phase vibrations, each with its characteristic frequency and amplitude relation. For the mode with frequency $\omega = \omega_2$, the two masses x_1 and x_2 oscillate in phase, meaning they move together in the same direction.

1. 1. Equations of Motion in Phase: - The positions of x_1 and x_2 are given by:

$$x_1 = A_1(\omega_2)\cos(\omega_2 t + \varphi_2)$$

$$x_2 = A_2(\omega_2)\cos(\omega_2 t + \varphi_2)$$

   - This relationship can also be expressed as:

$$x_2 = \lambda_2 A_2 \cos(\omega_2 t + \varphi_2)$$

   - Where $\lambda_2 = \frac{m_1(\omega_1^2 - \omega_2^2)}{k}$.

2. General Solution: - The complete solution that combines both vibration modes (mode 1 and mode 2) is written as:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = a_1 \begin{bmatrix} x_1(\omega_1, t) \\ x_2(\omega_1, t) \end{bmatrix} + a_2 \begin{bmatrix} x_1(\omega_2, t) \\ x_2(\omega_2, t) \end{bmatrix}$$

   - Here, a_1 and a_2 are coefficients that define the amplitudes of each mode, and the expressions $x_1(\omega_1, t), x_2(\omega_1, t)$ and $x_1(\omega_2, t), x_2(\omega_2, t)$ represent the motions of the system in each mode.

This general solution allows the system to be expressed as a combination of the two fundamental vibration modes, providing the real motion of the coupled oscillator system. The coefficients a_1 and a_2 will be determined by the initial conditions of the system. Here is the translation of the provided equations:

$$\begin{cases} x_1(t) = a_1 A_1(\omega_1)\cos(\omega_1 t + \varphi_1) + a_2 A_1(\omega_2)\cos(\omega_2 t + \varphi_2) \\ x_2(t) = -a_1 A_2(\omega_1)\cos(\omega_1 t + \varphi_1) + a_2 A_2(\omega_2)\cos(\omega_2 t + \varphi_2) \end{cases}$$

with

$$A_2(\omega_1) = \lambda_1 A_2(\omega_1)$$

and

$$A_2(\omega_2) = \lambda_2 A_2(\omega_2)$$

We obtain the system for the two modes given by (V.4)**

$$\begin{cases} x_1(t) = a_1 A_1(\omega_1)\cos(\omega_1 t + \varphi_1) + a_2 A_1(\omega_2)\cos(\omega_2 t + \varphi_2) \\ x_2(t) = -\lambda_1 a_1 A_1(\omega_1)\cos(\omega_1 t + \varphi_1) + \lambda_1 a_2 A_1(\omega_2)\cos(\omega_2 t + \varphi_2) \end{cases}$$

where

$$a_1 A_1(\omega_1) = a_1$$

and

$$a_2 A_1(\omega_2) = a_2$$

The four values $a_1, a_2, \varphi_1, \varphi_2$ are determined by four initial conditions on $x_1, x_2, \dot{x}_1, \dot{x}_2$.**

¿ For example, we take the conditions:

$$x_1(t=0) = 0, \quad x_2(t=0) = 0, \quad \dot{x}_1(t=0) = 0, \quad \dot{x}_2(t=2) = 0$$

From these conditions, we get:

1.

$$a_1 \cos\varphi_1 + a_2 \cos\varphi_2 = 0$$

2.

$$-\lambda_1 a_1 \cos\varphi_1 + \lambda_2 a_2 \cos\varphi_2 = 0$$

3.

$$\lambda_1 a_1 \sin\varphi_1 + \lambda_2 a_2 \sin\varphi_2 = 0$$

4.

$$\sin(\varphi_1 - \varphi_2) = 0$$

We deduce that:

$$a_1 + a_2 = 0$$

And:

$$-\lambda_1 a_1 + \lambda_2 a_2 = 0$$

$$a_1 = \frac{\lambda_2}{\lambda_1 + \lambda_2} a$$

$$a_1 = \frac{\omega_1^2 - \omega_2^2}{\omega_1^2 - \omega_2^2} a$$

$$a_2 = \frac{\lambda_1}{\lambda_1 + \lambda_2} a$$

This section solves for the coefficients a_1 and a_2 by setting up and solving a system of equations based on the initial conditions and parameters given.
The expression for a_2 is as follows:

$$a_2 = \frac{\omega_1^2 - \omega_{01}^2}{\omega_1^2 - \omega_2^2} a$$

This represents the solution for the coefficient a_2 in terms of the frequencies ω_1, ω_{01}, and ω_2, as well as a given constant a.

5.7 Finding the Natural Frequencies and eigenmodes Using the Method of Normal Coordinates

5.7.1 Finding the Natural Frequencies of Eigenmodes

We revisit the two-mass, three-spring system. To apply the method of normal coordinates, we assume:

$$m_1 = m_2 = m \quad \text{and} \quad k_1 = k_2$$

The differential equations of motion derived previously are rewritten as:

$$m\ddot{x}_1 = -k_1 x_1 + k(x_2 - x_1)$$

$$m\ddot{x}_2 = -k_2 x_2 - k(x_2 - x_1)$$

Which simplify to:

$$\begin{cases} m\ddot{x}_1 + (k_1 + k)x_1 - kx_2 & = 0 \\ m\ddot{x}_2 + (k_2 + k)x_2 - kx_1 & = 0 \end{cases} \quad (V-4)$$

This setup provides the foundation for analyzing the system's natural frequencies and corresponding eigenmodes using normal coordinates.

By subtracting and adding the two equations above, we obtain:

$$m(\ddot{x}_1 - \ddot{x}_2) + (k_1 + 2k)(x_1 - x_2) = 0$$

$$m(\ddot{x}_1 + \ddot{x}_2) + k_1(x_1 + x_2) = 0$$

These are labeled as equations (V-4).

We define new variables as follows:

$$X_1 = x_1 - x_2$$

$$\dot{X}_1 = \dot{x}_1 - \dot{x}_2$$

$$X_2 = x_1 + x_2$$

$$\dot{X}_2 = \dot{x}_1 + \dot{x}_2$$

The variables X_1 and X_2 are known as the normal coordinates of the system. Substituting these into the equations from system (V-3), we derive the new system (V-4):

$$\begin{cases} m\ddot{X}_1 + (k_1 + 2k)X_1 & = 0 \\ m\ddot{X}_2 + k_1 X_2 & = 0 \end{cases} \quad (V-5) \tag{5.5}$$

These are labeled as equations (V-5).

The equations of system (V-5) are decoupled, meaning that they represent two independent simple harmonic motions. This reformulation using normal coordinates X_1 and X_2 simplifies the problem by breaking it down into two separate harmonic oscillations, each with its natural frequency determined by the parameters m, k, and k_1. Let the solutions of the system (V-5) be:

$$X_1(t) = A_1 \sin(\omega_1 t + \varphi_1) \quad \text{with} \quad \omega_1 = \sqrt{\frac{k_1 + 2k}{m}}$$

$$X_2(t) = A_2 \sin(\omega_2 t + \varphi_2) \quad \text{with} \quad \omega_2 = \sqrt{\frac{k_1}{m}}$$

Therefore:

$$x_1 = \frac{X_1 + X_2}{2}$$

and

$$x_2 = \frac{X_1 - X_2}{2}$$

Thus:

$$\begin{cases} x_1(t) = \frac{A_1 \sin(\omega_1 t + \varphi_1) + A_2 \sin(\omega_2 t + \varphi_2)}{2} \\ x_2(t) = \frac{A_1 \sin(\omega_1 t + \varphi_1) - A_2 \sin(\omega_2 t + \varphi_2)}{2} \end{cases} \qquad (V-6) \qquad (5.6)$$

Regarding the order of natural frequencies, we can also list them in increasing or decreasing order (for systems with 2 degrees of freedom).

5.7.2 Superposition of Vibrations with Different Frequencies

In this section, the concept of *superposition of vibrations* is explained. Since harmonic oscillators follow linear equations, multiple harmonic oscillations can occur together without affecting each other. This allows us to analyze each oscillation separately and then combine their displacements to get the overall motion. When oscillations with slightly different frequencies are combined, they can create a *beating effect*, where the amplitude of the resulting wave fluctuates periodically. If the system is simultaneously subjected to several different oscillations, we can solve the equations corresponding to each oscillation separately. The displacement of the oscillator is then obtained by adding the displacement solutions of each equation. This results in the superposition of vibrations with the same frequency or with different, yet close, frequencies. In the case of different but close frequencies, as treated in (V-6), we encounter the phenomenon of beating.

Beating

A beating effect occurs when two vibrations with nearly identical frequencies are superimposed. This results in a new vibration with an average frequency $\frac{\omega_1+\omega_2}{2}$, whose amplitude varies periodically with the frequency $\frac{\omega_1-\omega_2}{2}$.

In this case, the displacement x_1 given by the system (equation V-6) is considered. The values A_1, A_2, φ_1, and φ_2 are determined by initial conditions. Here, it's assumed that $A_1 = A_2 = A$ and $\varphi_1 = \varphi_2 = 0$.

The resulting displacement becomes:

$$x_1 = \frac{A}{2}(\sin\omega_1 t + \sin\omega_2 t)$$

This expression represents the phenomenon of beats, where the amplitude oscillates at the beat frequency due to the interference between the two close frequencies. The expression for x_1 can be rewritten by applying the trigonometric identity:

$$\sin a + \sin b = 2\cos\left(\frac{a-b}{2}\right)\sin\left(\frac{a+b}{2}\right)$$

Thus,

$$x_1(t) = \frac{A}{2}\sin\omega_1 t + \frac{A}{2}\sin\omega_2 t = x_1(t) + x_2(t)$$

Using the identity, we get:

$$x_1(t) = A\cos\left(\frac{\omega_1-\omega_2}{2}t\right)\sin\left(\frac{\omega_1+\omega_2}{2}t\right)$$

This form shows the phenomenon of beating, where $A\cos\left(\frac{\omega_1-\omega_2}{2}t\right)$ represents the slowly varying amplitude, and $\sin\left(\frac{\omega_1+\omega_2}{2}t\right)$ represents the fast oscillation at the average frequency.

Mean Period and Beat Period

The mean period corresponds to the average frequency $\frac{\omega_1+\omega_2}{2}$, while the beat period is associated with the difference frequency $\frac{\omega_1-\omega_2}{2}$, giving the rate at which the amplitude oscillates. The following parameters are defined:

1. Mean Frequency ω_{moyenne}:

$$\omega_{\text{moyenne}} = \frac{\omega_1 + \omega_2}{2}$$

2. Mean Period T_{moyenne}:

$$T_{\text{moyenne}} = \frac{2\pi}{\omega_{\text{moyenne}}} = \frac{2\pi}{\frac{\omega_1+\omega_2}{2}}$$

   where:

$$T_1 = \frac{2\pi}{\omega_1}, \quad T_2 = \frac{2\pi}{\omega_2}$$

3. Modulation Frequency $\omega_{\text{modulation}}$:

$$\omega_{\text{modulation}} = \frac{\omega_1 - \omega_2}{2}$$

4. Beat Period T_b:

   The time interval between two consecutive zero crossings of the amplitude of the beats (represented by T_b) is given by:

$$T_b = \frac{2\pi}{\omega_{\text{modulation}}} = \frac{2\pi}{\frac{\omega_1-\omega_2}{2}}$$

The beat period T_b is given by the formula:

$$T_b = \frac{2\pi}{\omega_{\text{modulation}}}$$

Alternatively, it can be expressed in terms of the individual periods T_1 and T_2 of the two oscillations as:

$$T_b = \frac{T_1 T_2}{|T_1 - T_2|}$$

The figure (V-6) illustrates the superposition of two vibratory motions with close frequencies.

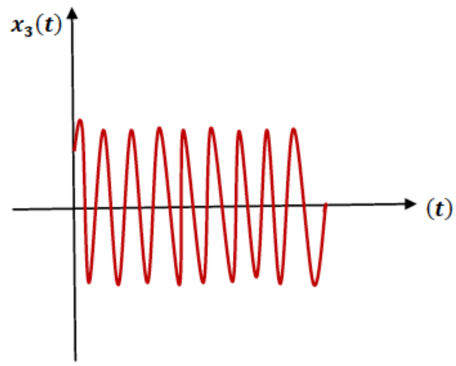

FIGURE 5.7: Representation of vibratory movement

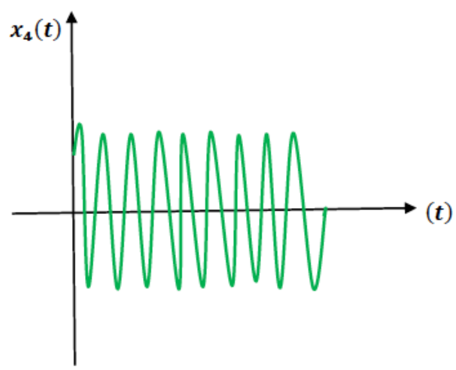

FIGURE 5.8: Representation of vibratory movement

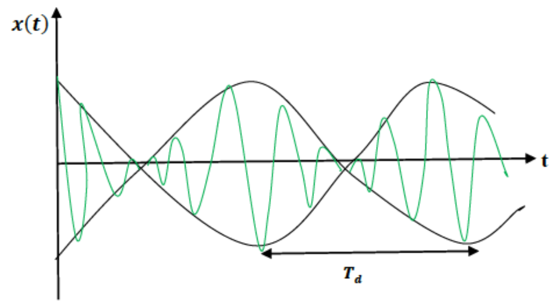

FIGURE 5.9: Beating of the two oscillations $x_3(t)$ and $x_4(t)$

5.8 Forced Oscillations with Degrees of Freedom

5.8.1 Forced Oscillations without Damping

We consider the system (two masses, three springs) shown in Figure (V.7). The

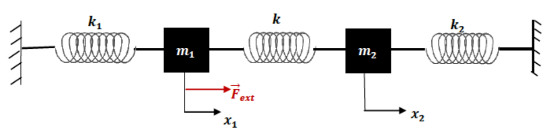

FIGURE 5.10: Elastically coupled mechanical oscillators are subjected to an external force

differential equations of motion are:

$$\begin{cases} m_1\ddot{x}_1 + (k_1 + k)x_1 - kx_2 = F_0 \cos \Omega t \\ m_2\ddot{x}_2 + (k_2 + k)x_2 + kx_1 = 0 \end{cases}$$

We are interested in the steady-state regime, meaning the solution that follows the particular solution of the system of differential equations. We seek sinusoidal solutions with angular frequency Ω. To do this, we use the expressions for the natural frequencies of the decoupled oscillators, defined by: The natural frequencies for the decoupled oscillators are defined as:

$$\omega_{01}^2 = \frac{k + k_1}{m_1}$$

and

$$\omega_{02}^2 = \frac{k + k_2}{m_2}$$

Using these, we obtain the following system of differential equations:

$$\begin{cases} \ddot{x}_1 + \omega_{01}^2 x_1 - \frac{k}{m_1} x_2 = \frac{F_0}{m_1} \cos \Omega t \\ \ddot{x}_2 + \omega_{02}^2 x_2 - \frac{k}{m_2} x_1 = 0 \end{cases}$$

If we use the complex notation, we replace x_1, x_2, and the external force with their complex expressions:

$$\bar{A}_1 = A_1 e^{i\varphi_1} e^{i\Omega t} = A_1 e^{i\Omega t}$$

$$\bar{A}_2 = A_2 e^{i\varphi_2} e^{i\Omega t} = A_2 e^{i\Omega t}$$

This complex notation simplifies the analysis of the system. With the complex amplitudes defined as:

$$\bar{A}_1 = A_1 e^{i\varphi_1} \quad \text{and} \quad \bar{A}_2 = A_2 e^{i\varphi_2}$$

we also define the external force as:

$$\bar{F}_0 = F_0 e^{i\Omega t}$$

The system of equations becomes:

$$\begin{cases} (\omega_{01}^2 - \Omega^2)\bar{A}_1 - \frac{k}{m_1}\bar{A}_2 = \frac{F_0}{m_1} \\ -\frac{k}{m_2}\bar{A}_1 + (\omega_{02}^2 - \Omega^2)\bar{A}_2 = 0 \end{cases} \quad (V-7) \tag{5.7}$$

For the system (V-7) to admit solutions, the determinant must be non-zero:

$$\Delta = \begin{vmatrix} \omega_{01}^2 - \Omega^2 & -\frac{k}{m_1} \\ -\frac{k}{m_2} & \omega_{02}^2 - \Omega^2 \end{vmatrix} \neq 0 \quad (V-8) \tag{5.8}$$

The solutions for the system (V-8) are:

$$\bar{A}_1 = \frac{F_0}{m_1(\omega_{02} - \Omega^2)}$$

and the complex solution involving cosine and sine is expressed as:

$$\bar{A}_1 = A_1 \cos \Omega t + iA_1 \sin \varphi_1$$

$$\bar{A}_1 = \begin{vmatrix} \frac{F_0}{m_1} & -\frac{k}{m_1} \\ 0 & \omega_{02}^2 - \Omega^2 \end{vmatrix} = A_1 \cos \varphi_1 + iA_1 \sin \varphi_1 \quad (V-9) \tag{5.9}$$

These expressions give us the complex amplitudes for the forced oscillation system. The expression for $\bar{A}_2$ in the forced oscillation system is given by:

$$\bar{A}_2 = \begin{vmatrix} \omega_{01}^2 - \Omega^2 & \frac{F_0}{m_1} \\ -\frac{k}{m_2} & 0 \end{vmatrix} = A_2 \cos \varphi_2 + iA_2 \sin \varphi_2 \quad (V-10) \tag{5.10}$$

In this example, the expressions found for $\bar{A}_1$ and $\bar{A}_2$ are purely real quantities. This implies that the resulting oscillatory motion, described by these amplitudes, does not involve any imaginary components, indicating that the solution is aligned with real physical displacements. The imaginary parts of expressions (V.9) and (V.10) must be zero. Since $\bar{A}_1$ and $\bar{A}_2$ are necessarily non-zero, we derive the following conditions:

$$\sin \varphi_1 = 0$$

Thus:

$$\varphi_1 = 0 \text{ or } \pi$$

And similarly:

$$\varphi_2 = 0 \text{ or } \pi$$

This ensures that the phases φ_1 and φ_2 are restricted to values that eliminate any imaginary components, preserving the real nature of the oscillatory solution. - The movement of the mass in question is either in phase or in opposite phase with the external force.

- If we set both phase shifts to zero, the displacements take the following forms:

$$x_1(t) = \frac{F_0(\omega_{02}^2 - \Omega^2)}{m_1\Delta} \cos \Omega t = A_1 \cos \Omega t$$

and

$$x_2(t) = \frac{-F_0 k}{m_1 m_2 \Delta} \cos \Omega t = A_2 \cos \Omega t$$

This gives the displacement expressions $x_1(t)$ and $x_2(t)$ in terms of cosine functions, reflecting oscillations with amplitude factors A_1 and A_2. The main discriminant Δ becomes zero for pulse values given by the square root of the expressions in (V-9) and (V-10). The amplitudes A_1 and A_2 tend toward infinity at this point, representing resonance for an external pulse equal to ω_{02}. At this resonance, oscillator m_1 is stationary, and $A_1 = \frac{F_0}{k}$, indicating an anti-resonance phenomenon.

Oscillator 2 (right) vibrates sinusoidally at its natural frequency and exerts a force on oscillator 1 (left) through the coupling spring. This coupling force is always equal and opposite to the external force applied to oscillator 1. The variations in amplitudes A_1 and A_2 are plotted as a function of the pulse of oscillator 1.

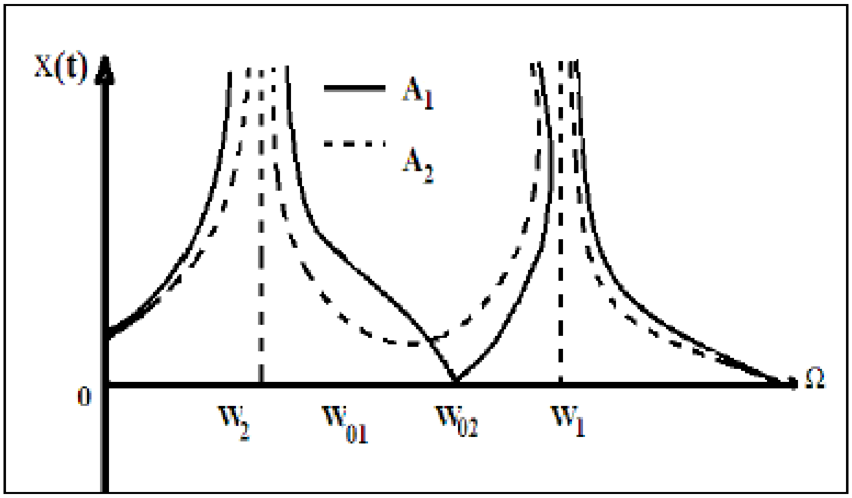

FIGURE 5.11: Variation of Amplitudes A_1 and A_2 as a Function of the Pulse

Damped Forced Oscillation

Consider the system below:

- An external force $\vec{F}_{\text{ext}} = F_0 \cos \Omega t$ is applied to mass m_1. - The system consists of two masses, m_1 and m_2, connected by springs with spring constants k_1, k_2, and k. - Each mass has its own damping coefficient, α_1 for m_1 and α_2 for m_2.

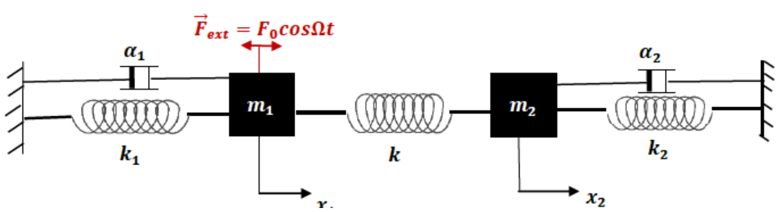

FIGURE 5.12: Mechanical oscillators by elasticity and subjected to an external force and a damper

Figure 5-12 shows mechanical oscillators connected by springs and subject to an external force and damping.

The differential equations of motion for the system are:

$$\begin{cases} m_1\ddot{x}_1 + (k_1 + k)x_1 + \alpha_1\dot{x}_1 - kx_2 = F_0 \cos \Omega t \\ m_2\ddot{x}_2 + (k_2 + k)x_2 + \alpha_2\dot{x}_2 - kx_1 = 0 \end{cases} \qquad (V - 11) \qquad (5.11)$$

These equations describe the motion of each mass under the influence of elastic (spring) forces, damping, and the external periodic force applied to m_1.

Resonance and Anti-resonance

(With $D = 0$ and $F \neq 0$: Forced but undamped system)

The steady-state solution from (V-10) is:

$$\begin{cases} x_1 = A_1 \cos(\Omega t + \varphi_1) \\ x_2 = A_2 \cos(\Omega t + \varphi_2) \end{cases}$$

This represents the displacement of each mass in the system under forced oscillation at a frequency Ω, where each mass has its own amplitude A_1, A_2 and phase φ_1, φ_2. In this context, resonance and anti-resonance behavior can be observed depending on the relationship between the driving frequency Ω and the natural frequencies of the system. Where:

A_1, A_2, φ_1, and φ_2 depend on the frequency Ω and the force amplitude F_0. To find A_1 and A_2, we use the complex representation.

$$\begin{cases} F(t) = F_0 \cos(\Omega t) \to \bar{F}(t) = F_0 e^{-i\Omega t} \\ x_1(t) = A_1 \cos(\Omega t + \varphi_1) \to \bar{x}_1(t) = A_1 e^{-i\Omega t} \\ x_2(t) = A_2 \cos(\Omega t + \varphi_2) \to \bar{x}_2(t) = A_2 e^{-i\Omega t} \end{cases}$$

This complex form simplifies the analysis of the forced oscillation by allowing us to work with complex exponentials instead of trigonometric functions. The system of equations (V-10) becomes:

When $\Delta = 0$:

$$\begin{cases} m_1\bar{x}_1 + (k_1+k)\bar{x}_1 + a_1\bar{x}_1 - kx_2 = F_0e^{-i\Omega t} \\ m_2\bar{x}_2 + (k_2+k)\bar{x}_2 + a_2\bar{x}_2 - kx_1 = 0 \end{cases}$$

Which simplifies to the following system (V-12):

$$\begin{cases} \left(-\Omega^2 + \frac{k_1+k}{m_1}\right)\bar{A}_1 - \frac{k}{m_1}\bar{A}_2 = \frac{F_0}{m_1} \\ \left(-\Omega^2 + \frac{k_2+k}{m_2}\right)\bar{A}_2 - \frac{k}{m_2}\bar{A}_1 = 0 \end{cases} \quad (V-12) \tag{5.12}$$

This system can now be solved for the complex amplitudes A_1 and A_2, representing the solutions for each mass in the forced oscillation system. In the case where $m_1 = m_2 = m$ and $k_1 = k_2 = k$:

Setting $\omega_0^2 = \frac{k}{m}$, equation (V-12) becomes:

$$\begin{cases} (-\Omega^2 + 2\omega_0^2)\bar{A}_1 - \omega_0^2\bar{A}_2 = \frac{F_0}{m} \\ (-\Omega^2 + 2\omega_0^2)\bar{A}_2 - \omega_0^2\bar{A}_1 = 0 \end{cases}$$

This leads to the following expressions for A_1 and A_2:

$$\begin{cases} A_1 = \frac{F_0}{m}\frac{|2\omega_0^2-\Omega^2|}{|(2\omega_0^2-\Omega^2)^2-\omega_0^4|} \\ A_2 = \frac{F_0}{m}\frac{|\omega_0|}{|(2\omega_0^2-\Omega^2)^2-\omega_0^4|} \end{cases} \quad (V-13) \tag{5.13}$$

From this (equation V-13), we observe:

- $A_1 = A_2 = \infty$, when $\Omega = \omega_0 = \Omega_{R1}$ (referred to as the first resonance frequency). - $\Omega = \sqrt{3}\omega_0 = \Omega_{R2}$ (referred to as the second resonance frequency). - $A_1 = 0$, when $\Omega = \sqrt{2}\omega_0 = \Omega_{R1}$ (referred to as the anti-resonance frequency). Anti-resonance is a phenomenon that occurs in certain coupled oscillating systems, where one part of the system remains nearly stationary while another part experiences a significant oscillation, despite being driven by an external force. In simpler terms, at the anti-resonance frequency, one of the oscillators remains almost completely unaffected by the driving force, effectively "canceling out" the oscillatory effect on that particular part of the system.

Key Aspects of Anti-Resonance:

1. Frequency Dependence: - In coupled systems, anti-resonance occurs at a specific frequency, known as the **anti-resonance frequency** Ω_{AR}, which is distinct from the resonance frequencies where both oscillators would normally respond strongly. - The anti-resonance frequency depends on the coupling properties of the system, including parameters like the masses, stiffness (spring constants), and damping (if any).

2. Destructive Interference: - At the anti-resonance frequency, the motion induced by the external force on one part of the system interferes destructively with the natural oscillations of the other part. - This destructive interference causes one of the oscillators to have zero or minimal amplitude, meaning it effectively does not respond to the driving force. - In practical terms, the oscillatory energy is redistributed such that one part "absorbs" the motion, leaving the other almost motionless.

3. Energy Flow and Isolation: - In anti-resonance, energy from the external driving force is not effectively transferred to the oscillating mode that includes the "silent" or "stationary" part. - This isolation of motion at the anti-resonance frequency is used in engineering applications to reduce vibrations in specific parts of a system while allowing other parts to oscillate.

4. Contrast with Resonance: - In resonance, the system absorbs energy very effectively at specific frequencies, leading to large amplitudes of oscillation. - In anti-resonance, the system behaves oppositely: it avoids absorbing energy in certain oscillating modes, resulting in minimal oscillation in one part of the system despite the presence of a driving force.

Mathematical Explanation (Based on Coupled Oscillators)

For a system of two coupled oscillators with masses m_1 and m_2 and spring constants k_1, k_2, and coupling k:

1. Equations of Motion: - The system's response to an external driving force $F(t) = F_0 \cos(\Omega t)$ applied to one of the masses can be described by differential equations that include terms for the coupling between the masses.

2. Anti-Resonance Condition: - Solving the equations of motion reveals frequencies where one oscillator's amplitude A_1 goes to zero. This occurs at the anti-resonance frequency Ω_{AR}. - Mathematically, the condition for anti-resonance is when the response amplitude A_1 or A_2 satisfies $A_1 = 0$ or $A_2 = 0$, leading to minimal oscillation in one part of the system.

3. Example Calculation: - For identical masses and springs in a symmetric system, the anti-resonance frequency Ω_{AR} might be calculated as $\Omega_{AR} = \sqrt{2}\,\omega_0$, where ω_0 is the natural frequency of one of the oscillators if it were isolated. This depends on the specific coupling configuration.

Practical Implications of Anti-Resonance:

Anti-resonance is useful in designing systems where vibration needs to be minimized in a specific component while allowing another part to respond to external forces. Examples include:

- Mechanical Systems: Reducing vibrations in sensitive parts of machinery, such as in precision instruments, where you want to isolate certain components from vibrations. - Structural Engineering: Designing buildings or bridges with tuned mass dampers that reduce oscillations in certain parts by exploiting anti-resonance. - Acoustic Engineering: In loudspeakers, where anti-resonance can prevent certain frequencies from causing unwanted resonance in parts of the speaker housing.

In summary, anti-resonance is a phenomenon in coupled oscillators where, at a particular frequency, one part of the system remains nearly stationary due to destructive interference in the oscillatory response. This concept is used in various applications to isolate parts of a system from vibrations and oscillations at specific frequencies.

Chapter 6

Lorem Ipsum Dolor Sit Amet

6.1 Introduction

6.2 Lorem Ipsum Dolor Sit Amet

6.3 Experiments

6.4 Conclusions

Chapter 7

Measurement

7.1 Introduction

7.2 Metric A

Chapter 8

Proposed Work and Timeline

8.1 Lorem Ipsum Dolor Sit Amet

8.1.1 Description

8.1.2 Time Estimate

Chapter 9

Expected Contributions

Bibliography