Tutorial Worksheet No.3

Exercise 1.

Let $\star$ be an operation on $\mathbb{R}^*$defined by

$$\star : \mathbb{R}^* \times \mathbb{R}^* \longrightarrow \mathbb{R}^*$$
$$(a, b) \longrightarrow a \star b = \frac{1}{a} + \frac{1}{b}$$

1. Is $\star$ satisfies the following properties : Closure, Identity, Inverse and Associativity.
2. Is $\star$ a binary operation ?
3. Does every elelment in $\mathbb{R}^{-\{-1,0,1\}}$ have its inverse under $\star$?

Exercise 2.

Let $A = \{$f,g,h,j$\}$and let $\sharp$ be an operation on A defined by the following

$\sharp$	f	g	h	j
f	f	g	h	j
g	g	f	j	h
h	h	j	g	f
j	j	h	f	g

1. Is $\sharp$ a binary operation ?
2. Determine the identity element and the inverse of each element
3. Is $\sharp$ satsfies the following : j$\sharp$(g$\sharp$h) = (j$\sharp$g)$\sharp$h

Exercise 3.

The binary operation $*$ is defined on $\mathbb{R}$ by : $\forall (x, y) \in \mathbb{R}^2 \mid x * y = x + y - axy,\ (a \in \mathbb{Z}^*)$

1. Is $(\mathbb{R}, *)$ a group ?
2. Solve $x * x = 2$,(This question is left for the students to solve on their own)
3. Determine G so that the combination $(G, *)$ forms a group
4. Is $(\mathbb{Q}, *)$ a subgroup of the group $(G, *)$?
5. Is $*$ distributive over multiplication ?

Exercise 4. (Homework)

Let $G = \mathbb{R}^* \times \mathbb{R}^*$and let $\diamond$ be an operation on G defined by
$\forall (a,b) \in G, \forall (c,d) \in G|\ (a,b) \diamond (c,d) = \left(ac, \dfrac{d}{a} + b\right)$

- Show that $(G, \diamond)$ is a non-commutative group.

Exercise 5.

Let $(G, *)$ and $(H, \triangleleft)$ be two groups and let $\phi(x)$ be a group homomorphism defined as follows

$$\begin{aligned} \phi : G &\longrightarrow H \\ x &\longrightarrow \phi(x) = 2x^2 + 1 \end{aligned}$$

1. Knowing that $e_G = 2\alpha$, determine e_H.
2. Prove that $\forall b \in G, \phi(b^{-1}) = (\phi(b))^{-1}$. Deduce the inverse element of β under $*$, by knowing that γ is the inverse element of $\phi(\beta)$ under $\triangleleft$.
3. Deduce that $\alpha = \dfrac{1}{2}\left(\beta * \sqrt{\dfrac{\gamma - 1}{2}}\right)$
4. Determine $Ker(\phi)$.

Exercise 6.

Let $\oplus$ and $\otimes$ two binary operations on $\mathbb{R}$ defined by :
$\forall a, b \in \mathbb{R} :\ a \oplus b = a + b + 1,\ a \otimes b = a + b + ab$

- Prove that the combination $(\mathbb{R}, \oplus, \otimes)$ forms an Identity ring ?

Exercise 7.

Let $\oplus$ and $\otimes$ two binary operations on $\mathbb{R}^2$ defined by :
$\forall (a, b), (c, d) \in \mathbb{R}^2 :\ (a, b) \oplus (c, d) = (a + c, b + d)\ ,\ (a, b) \otimes (c, d) = (ac - bd, ad + bc)$

- Does the combination $(\mathbb{R}^2, \oplus, \otimes)$, forms a field ?