حل تمارين السلسلة رقم 02

Solution of Series 02

Exercise 01:

a) $A \cup B = \{1, 2, 3, 4\}$ $\quad A \cap B = \{1, 2\}$

$A - B = \{3\}$ $\quad A \times B = \{(1,1), (1,2), (1,4), (2,1), (2,2), (2,4), (3,1), (3,2), (3,4)\}$

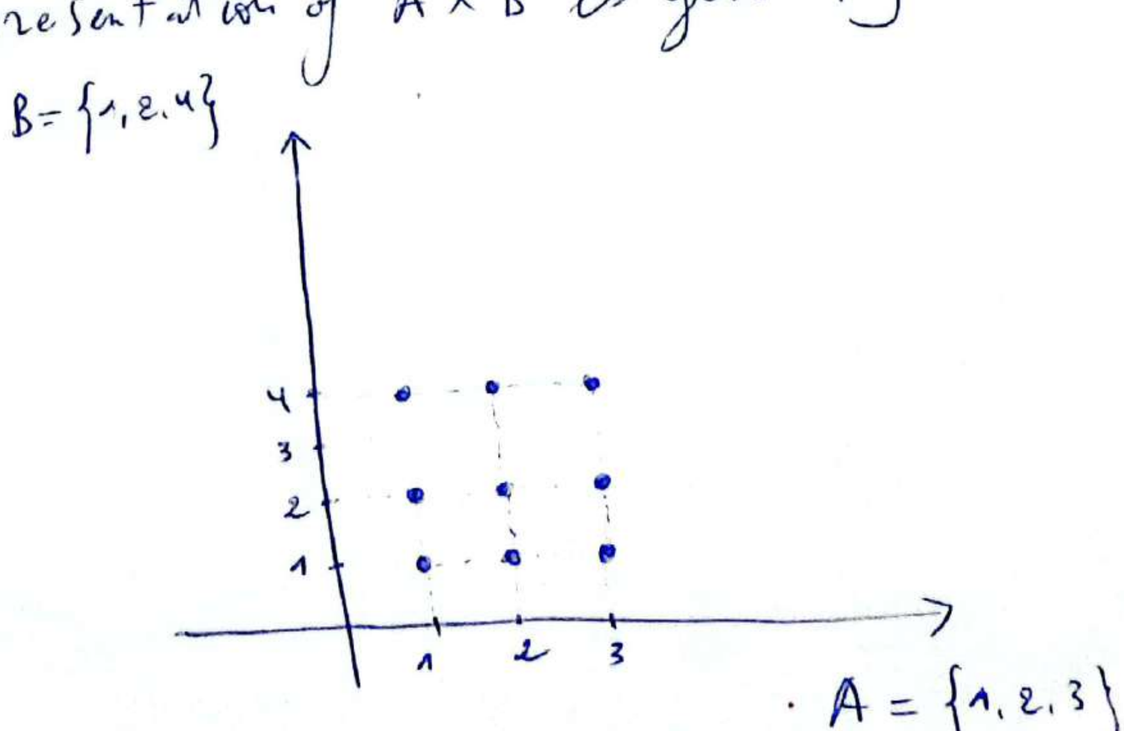
$P(A) = \{\{\phi\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

$A \cap \phi = \{\phi\}$, $\quad (A \cup B) - (A \cap B) = \{3, 4\}$

$\mathrm{Card}(P(A)) = 2^{\mathrm{Card}(A)} = 2^3 = 8$

$\mathrm{Card}(A \times B) = \mathrm{card}(A) \times \mathrm{Card}(B) = 3 \times 3 = 9$

- The representation of $A \times B$ is given by



2/ We know that: 1) $E=F \Rightarrow card(E)=card(F)$

2) $E=F \Rightarrow$ E and F have the same elements

1. $card(\{c\}) \neq card(\{a,b\}) \Rightarrow \{c\} \neq \{a,b\} \Rightarrow \{c\}=\{a\} \Rightarrow c=a$

2. $card(\{c,e\}) \neq card(\{a\}) \Rightarrow \{c,e\}=\{a,b\} \Rightarrow e=b$

Exercise 02

$\mathbb{Z}-(A\cup B)=\mathbb{Z}^{-}=\{-1,-2,-3,\cdots\}$

$(A\cup B)-\mathbb{Z}=\emptyset$ $\qquad (A\cup B)-\{0\}=\mathbb{N}$

$(A\cup B)-\mathbb{N}=\{0\}$ $\qquad (A\cup B)\cap\{0\}=\{0\}$

$\{0\}\cup\mathbb{N}=\{A\cup B\}$ $\qquad \mathbb{Z}\cap\mathbb{N}=\mathbb{N}$

Exercise 03

$D=(A\cup C_u^A)\cap B = u\cap B = B$

$E=C_u^A\cup(C_U^B\cap B)=C_u^A\cup\emptyset=C_u^A$

$F=C_U[(C_U^A\cap B)\cup(A\cap B)]^a = C_u^D = C_u^B$

Exercise 04

The statement if $(A - B) \cup B = A$, Then $B \subseteq A$

it can be shown as $\left((x \in A) \wedge (x \notin B) \vee (x \in A)\right) \Rightarrow \left((x \in B) \Rightarrow (x \in A)\right)$

the truth table of this statement gives

the statement $= A$ leads to that the statement

$(x \in A)$ is always true:

$x \in A$	$x \in B$	$x \notin B$	$(x \in A) \wedge (x \notin B)$	$(x \in A) \wedge (x \notin B) \vee (x \in B)$	$x \in B \Rightarrow x \in A$
T	T	F	F	T	T
T	F	T	T	T	T

The statement $(x \in A) \wedge (x \notin B) \vee (x \in B)$ and $x \in B \Rightarrow x \in A$ have the same truth value means that the statement is true.

Exercise 05

1- the pair $(1, 1)$ an element of S' if it is satisfying the relation S'. $a = 1$ and $b = 1$

$1 \times 1 \overset{?}{=} 2 \times 1 - 1 \Rightarrow$ the equality is correct, then

the pair $(1, 1)$ is an element of S'

2- the relation S' is symmetric if

$$\underset{\substack{\downarrow \\ ab = 2a - 1}}{a S' b} \Rightarrow \underset{\substack{\downarrow \\ ab = 2b - 1}}{b S' a}$$

the condition is that $2a - 1 = 2b - 1 \Rightarrow (b = a)$

4- transition property means

$$(a \mathcal{S}' b) \wedge (b \mathcal{S}' c) \Rightarrow a \mathcal{S}' c$$

$$\underset{①}{(ab = 2a-1)} \wedge \underset{②}{(bc = 2b-1)} \Rightarrow ac = 2a-1$$

$$① \times ② \Rightarrow ab^2c = (2a-1)(2b-1)$$

$$b = 1 \Rightarrow ac = 2a-1$$

$\Rightarrow$ the condition is that $\boxed{b=1}$ and $ab = 2a-1 \Rightarrow \boxed{a=1}$
and then $\boxed{c=1}$

Exercise 06:

$$R \text{ reflexive} \Rightarrow xRx \Rightarrow \frac{a e^{x-2}}{x} - \frac{e^x}{x} = 0$$

$$\Rightarrow a e^{x-2} = e^x \Rightarrow a e^{-2} = 1$$

$$\Rightarrow \boxed{a = e^2}$$

By replacing a in R we get:

$$\forall (x,y) \in E \times F \mid \frac{e^x}{y} - \frac{e^y}{x} = 0$$

R is an equivalence relation $\Rightarrow$ R is reflexive symmetric and transitive

1a is R reflexive?

$$xRx \Rightarrow \frac{e^x}{x} - \frac{e^x}{x} = 0$$ $\Rightarrow$ the relation is hold true $\Rightarrow$ R is reflexive

2b is R ~~symmetric~~ symmetric?

$$xRy \Rightarrow \underline{yRx} \Rightarrow \frac{e^y}{x} - \frac{e^x}{y} = 0$$

$$\frac{e^x}{y} - \frac{e^y}{x} = 0 \quad \text{--- (b1)}$$

$$(-1) \times (b1) \Rightarrow \frac{e^y}{x} - \frac{e^x}{y} = 0.$$

Thus, R is symmetric

2c is R transitive?

$$\underline{(xRy)} \wedge \underline{(yRz)} \Rightarrow \underline{(xRz)}$$

$$\left(\frac{e^x}{y} - \frac{e^y}{x} = 0\right)_{①} \wedge \left(\frac{e^y}{z} - \frac{e^z}{y} = 0\right)_{②} \Rightarrow \left(\frac{e^x}{z} - \frac{e^z}{x} = 0\right)$$

From ② $\Rightarrow e^y = z\frac{e^z}{y}$ we replace In ①

$$\frac{e^x}{y} - \frac{ze^z}{xy} = 0 \Rightarrow \times(xy) \Rightarrow xe^x - ze^z = 0$$

$$\Rightarrow \frac{e^x}{z} - \frac{e^z}{x} = 0$$

The statement is true and R is transitive and We can conclude that R is an equivalence relation

Exercise 01

1.a is R reflexive?

$$xRx \Rightarrow \cos^2 x + \sin^2 x = 1$$

the equality is true and thus R is reflexive

1.b is R symmetric?

$$\underbrace{xRy}_{} \Rightarrow \underbrace{yRx}_{}$$

$\cos^2 x + \sin^2 y = 1$ $\Rightarrow$ $\cos^2 y + \sin^2 x = 1$

$$\cos^2 x + \sin^2 y = (1 - \sin^2 x) + (1 - \cos^2 y) = 1$$

$$\Rightarrow \cos^2 y + \sin^2 x = 1$$

It is obvous tha R is symmetric.

1.c is R transitive?

$$(xRy) \wedge (yRz) \Rightarrow (xRz)$$

equal

$$\Big(\underbrace{\cos^2 x + \sin^2 y = 1}_{①}\Big) \wedge \Big(\underbrace{\cos^2 y + \sin^2 z = 1}_{②}\Big) \Rightarrow \Big(\underbrace{\cos^2 x + \sin^2 z = 1}_{③}\Big)$$

From ② we get $1 - \sin^2 y + \sin^2 z = 1 \Rightarrow \sin^2 y = \sin^2 z$

by replacing in ① we can get

$\cos^2 x + \sin^2 z = 1$ which equivalent to ③

and Thus R is transitive

R is reflexive + symmetric + transitive = R is an equivalence relation

2- the equivalence class of x

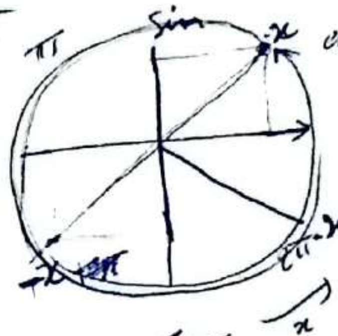
$$[x] = \{ y \in \mathbb{R} \mid yRx \}$$

$$yRx \Rightarrow \cos^2 y + \sin^2 x = 1$$

$$\cos^2 y = 1 - \sin^2 x$$

$$\cos^2 y = \cos^2 x \Rightarrow \cos y = \pm \cos x$$

Case 1: $\cos y = \cos x \Rightarrow \begin{cases} y_1 = x + 2n\pi \\ y_2 = -x + 2n\pi \end{cases}$ where $n \in \mathbb{Z}$



$\cancel{[x] = \{ \forall n \in \mathbb{Z} \}}$

$$[x] = \{ x + 2n\pi, -x + 2n\pi \}$$

Case 2: $\cos y = -\cos x \Rightarrow \begin{cases} y_3 = x + (2n+1)\pi \\ y_4 = -x + (2n+1)\pi \end{cases}$ $n \in \mathbb{Z}$

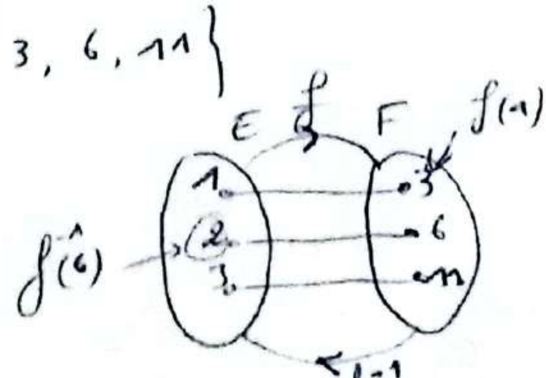
$$[x] = \{ x + (2n+1)\pi, -x + (2n+1)\pi \}.$$

Exercise 08:

1/ We have $E = \{1, 2, 3\}$ and $f : E \to F \mid f(x) = x^2 + 2$

$\rightarrow F \{ 1^2 + 2, 2^2 + 2, 3^2 + 2 \} \Rightarrow F = \{ 3, 6, 11 \}$

$f(1) = 3$ and $f^{-1}(6) = 2$



2/ the image of 3 $\Rightarrow f(3) = 3^2 + 2 = 11$

$y = f(x) = x^2 + 2 \Rightarrow y - 2 = x^2 \Rightarrow x = \sqrt{y-2}$

the preimage of 5 is $x = \sqrt{5-2} = \sqrt{3}$

$f^{-1}(5) = \sqrt{3}$

3/ $y = 1 = x^2 + 2 \Rightarrow x^2 = -2 + 1 \Rightarrow x^2 = -1 \Leftarrow$ impossible case

ther is no preimage of 1 under f becaus ~~that~~ there is no solution of $x^2 = -1$ in $\mathbb{R}$

Exercise 09

1- f injective $\Rightarrow f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

$x_1^2 + 2 = x_2^2 + 2 \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2$

f is not injective

- f Surjective!!?

$y = x^2 + 2 \Rightarrow x = \sqrt{y-2}$

x is defined Only if $y - 2 \geqslant 0 \Rightarrow y \geqslant 2$

there is no x in the case of $y < 2$.

$\Rightarrow f$ is not surjective

2. We can deduce that f is injective if $x \geqslant 0$ and it is surjective if $f(x) \geq 2$

and this means that: the function f is bijective in the case where

$$f : \mathbb{R}^+ \longrightarrow [2, \infty)$$
$$x \longrightarrow f(x) = x^2 + 2$$

3- $f(x) = x^2 + 2 \Rightarrow x = \sqrt{y-2} \Rightarrow f^{-1}(x) = \sqrt{x-2}$.

$$f^{-1}([2,6]) = [f^{-1}(2), f^{-1}(6)]$$
$$= [0, 2]$$

Exercise 10

1- $(g \circ f) = g(f(x)) = \frac{2x+1}{(2x+1)-2} = \frac{2x+1}{2x-1}$

2- is $(g \circ f)$ bijective? bijective = injective + surjective

is $(g \circ f)$ injective?

$g \circ f(x_1) = g \circ f(x_2) \Rightarrow \frac{2x_1+1}{2x_1-1} = \frac{2x_2+1}{2x_2-1} \Rightarrow \frac{2x_1+1-1+1}{2x_1-1} = \frac{2x_2+1-1+1}{2x_2-1}$

$\Rightarrow \cancel{2x} \; 1 + \frac{2}{2x_1-1} = 1 + \frac{2}{2x_2-1}$

$2x_2 - 1 = 2x_1 - 1$

$\Rightarrow x_1 = x_2 \Rightarrow g \circ f$ is injective

is $g \circ f$ surjective?

$\frac{2x+1}{2x-1} = y \Rightarrow 2x + 1 = 2xy - y$

$2x(1-y) = -y - 1$

$x = \frac{-y-1}{2(1-y)}$

$\Rightarrow x = \dfrac{1+y}{2(y-1)}$ is defined only if $y \neq 1$

We have $g \circ f : \mathbb{R} - \{\frac{1}{2}\} \longrightarrow \mathbb{R} - \{1\}$

the codomain of $g \circ f$ is defined ~~only if~~ on $\mathbb{R} - \{1\}$

and thus we can deduce that $g \circ f$ is ~~bijective~~ surjective $\Leftrightarrow$ for all $y \in \mathbb{R} - \{1\}$ there is a preimage x.

$g \circ f$ is injective + surjective $\Rightarrow g \circ f$ is bijective

and we have $x = (g \circ f)^{-1}(y) = \dfrac{1+y}{2(y-1)}$

Exercise 11:

1- $g(2+a) = (2+a)^2 - 4(2+a) + 4 = 4 + a^2 + 4a - 8 - 4a + 4$
$= a^2$

$g(2-a) = (2-a)^2 - 4(2-a) + 4 = 4 + a^2 - 4a - 8 + 4a + 4$
$= a^2$

injective $\Rightarrow \forall x_1, x_2 \in \mathbb{R} \mid (f(x_1) = f(x_2)) \Rightarrow (x_1 = x_2)$

We know that $p \Rightarrow q$ is false only if p: true and q: false

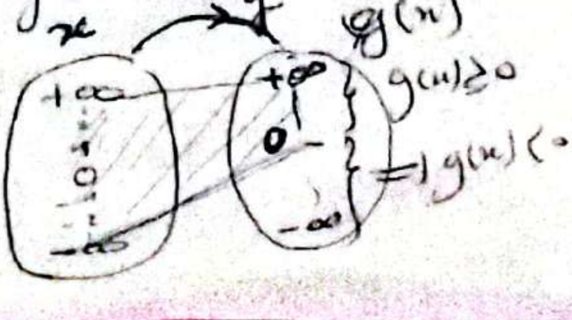
we have $g(2+a) = g(2-a) \Rightarrow 2+a \neq 2-a$

which means that g is not injective.

2- $g(x) = x^2 - 4x + 4 = (x-2)^2$ and we know that

$(x-2)^2 \geq 0 \Rightarrow g(x) \geq 0 \Rightarrow g(x) = y \geq 0$

$\Rightarrow$ ~~for g(x)<0~~ for all image $g(x) < 0$ we don't have a preimage in $\mathbb{R}$



$\Rightarrow g(x)$ is not surjective.