Directed Work Series No.1

29 novembre 2024

Solution of the Exercises

Exercise 1.

$$Case1 : (3\ is\ odd\ number) \Longrightarrow (3\ is\ a\ prime\ number)$$

— The statement $P \Rightarrow Q$ is true only if P is true and Q is false and then the statement (3 is odd number)$\Longrightarrow$(3 is a prime number) is true becaus (3 is odd number) is true and (3 is a prime number) is true

$$Case2 : ((3\ is\ odd\ number) \Longrightarrow (3\ is\ a\ prime\ number)) \Rightarrow (9\ is\ prime\ number)$$

— We show from the previous that the statement (3 is odd number)$\Longrightarrow$(3 is a prime number) is true but (9 is prime number) is false imply that the statement is false

$$Case3 : (4\ is\ odd\ number) \Longrightarrow (11\ is\ a\ prime\ number)$$

— The statement is true because the hypothesis $(4\ is\ odd\ number)$ is false and the conclusion $(11\ is\ a\ prime\ number)$ is true.

$$Case4 : ((\forall x \in]-5,-1[\ |\ \mid x+3 \mid < 2)) \wedge \left(\forall x \in\ [-5,1]\ \mid x^2+2x-8 \leq 0\right)$$

— The statement $P \wedge Q$ is tue only if both P and$Q$ are true.

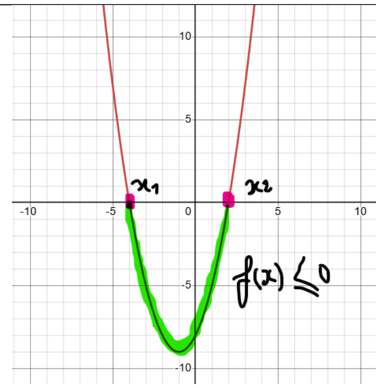
$$we\ have\ P : ((\forall x \in]-5,-1[\ |\ \mid x+3 \mid < 2))$$
$$P\ is\ true\ because\ -2 < x+3 < 2$$
$$-5 < x < -1$$

To show if the statement $Q : (\forall x \in [-5,1] \mid f(x) \leq 0)$is true we have to determine the interval where the inequality $f(x) = x^2+2x-8 \leq 0$ holds. The inequality $f(x) \leq 0$ is valid where the graph of $f(x)$ is below the x-axis. We start by studing the variations of The function

$$f(x) = 0 \Rightarrow x_1 = \frac{-2-\sqrt{36}}{2} = -4,\ x_2 = \frac{-2+\sqrt{36}}{2} = 2$$

$$f'(x) = 2x + 2 \Rightarrow x_{min} = -1 \; and \; f(-1) = -9$$

graph and the table of variation are shown in the following

<table>
<tr><th>The table of variations $f(x)$</th><th>The graph of the function $f(x)$</th></tr>
<tr><td>
<table>
<tr><td>x</td><td>$-\infty$ -1 $+\infty$</td></tr>
<tr><td>$f'(x)$</td><td>$-$ $|$ $+$</td></tr>
<tr><td>$f(x)$</td><td>$\searrow$ -9 $\nearrow$</td></tr>
</table>
</td><td>
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The inequality $f(x) \leq 0$ is valid if $x \in [-4, 1]$, and for that the function $f(x)$ is not valid for all $x \in [-5, 1]$. Thus the statement $Q : (\forall x \in [-5, 1] \mid f(x) \leq 0)$ is false. We have P is true and Q is false, then $P \wedge Q$ is false.

$$Case5 : \overline{(x^2 = 4 \Leftrightarrow x = 2)} \Leftrightarrow (x^2 = 4 \Leftrightarrow x \neq 2)$$

— We know that $\overline{P \Leftrightarrow Q}$ and $P \Leftrightarrow \overline{Q}$ are two equivalent statements, so fourth $\overline{(x^2 = 4 \Leftrightarrow x = 2)} \equiv (x^2 = 4 \Leftrightarrow x \neq 2)$. Therefor, the statement $\overline{(x^2 = 4 \Leftrightarrow x = 2)} \Leftrightarrow (x^2 = 4 \Leftrightarrow x \neq 2)$ is equivalent to $(x^2 = 4 \Leftrightarrow x \neq 2) \Leftrightarrow (x^2 = 4 \Leftrightarrow x \neq 2)$ which is always true because every statement is logically equivalent to itself.

$$Case6 : (\int_2^5 \sqrt{x+2}dx = \int_2^7 ydy, wherey^2 = x + 2) \quad \vee \quad (\forall k \in \mathbb{N} \; k^2 + 1 \; is \; odd)$$

— The statement $P \vee Q$ is false only if both P and Q are false. the statement P : $\int_2^5 \sqrt{x+2}dx = \int_2^7 ydy, wherey^2 = x + 2$ is false for the reason that $\sqrt{x+2} \neq 7$ for $x = 5$. The statement $Q : (\forall k \in \mathbb{N} \; k^2 + 1 \; is \; odd)$ is false because $1^2 + 1 = 2$. We deduce that P is false and Q is false, then $P \vee Q$ is false.

Exercise 2.

Two statements said to be equivalent if they have the same truth values.

ThetruthTableofthestatementa :$((P \Rightarrow Q) \Rightarrow R) \quad and \quad P$

P	Q	R	$(P \Rightarrow Q)$	$((P \Rightarrow Q) \Rightarrow R)$
T	**T**	**T**	**T**	**T**
T	**T**	**F**	**T**	**F**
T	**F**	**T**	**F**	**T**
T	**F**	**F**	**F**	**T**
F	**T**	**T**	**T**	**T**
F	**T**	**F**	**T**	**F**
F	**F**	**T**	**T**	**T**
F	**F**	**F**	**T**	**F**

From the truth table we can notice that the two statements are not equivalent.

ThetruthTableofthestatementb :$(P \Leftrightarrow Q) \quad and \quad ((P \Rightarrow Q) \wedge (Q \Rightarrow P))$

P	Q	$(P \Leftrightarrow Q)$	$(P \Rightarrow Q)$	$(Q \Rightarrow P)$	$((P \Rightarrow Q) \wedge (Q \Rightarrow P))$
T	**T**	**T**	**T**	**T**	**T**
T	**F**	**F**	**F**	**T**	**F**
F	**T**	**F**	**T**	**F**	**F**
F	**F**	**T**	**T**	**T**	**T**

From the truth table we can notice that the two statements $(P \Leftrightarrow Q) \quad and \quad ((P \Rightarrow Q) \wedge (Q \Rightarrow P))$ are not equivalent.

ThetruthTableofthestatementc : $(\overline{P} \vee Q) \quad and \quad (\overline{Q} \vee P)$

P	Q	$\overline{P}$	$\overline{Q}$	$(\overline{P} \vee Q)$	$(\overline{Q} \vee P)$
T	**T**	**F**	**F**	**T**	**T**
T	**F**	**F**	**T**	**F**	**T**
F	**T**	**T**	**F**	**T**	**F**
F	**F**	**T**	**T**	**T**	**T**

From the truth table we can notice that the two statements are not equivalent.

ThetruthTableofthestatementd :$\overline{(P \Leftrightarrow Q)} \quad and \quad (P \Leftrightarrow \overline{Q})$

P	Q	$\overline{Q}$	$P \Leftrightarrow Q$	$\overline{(P \Leftrightarrow Q)}$	$(P \Leftrightarrow \overline{Q})$
T	**T**	**F**	**T**	**F**	**F**
T	**F**	**T**	**F**	**T**	**T**
F	**T**	**F**	**F**	**T**	**T**
F	**F**	**T**	**T**	**F**	**F**

From the truth table we can notice that the two statements are equivalent.

ThetruthTableofthestatemente : $(P \vee (Q \Rightarrow R)) \quad and \quad ((P \Rightarrow Q) \Leftrightarrow (P \Rightarrow R))$

P	Q	R	$(Q \Rightarrow R)$	$(P \Rightarrow Q)$	$(P \Rightarrow R)$	$(P \vee (Q \Rightarrow R))$	$((P \Rightarrow Q) \Leftrightarrow (P \Rightarrow R))$
T	**T**	**T**	**T**	**T**	**T**	**T**	**T**
T	**T**	**F**	**F**	**T**	**F**	**T**	**F**
T	**F**	**T**	**T**	**F**	**T**	**T**	**F**
T	**F**	**F**	**T**	**F**	**F**	**T**	**T**
F	**T**	**T**	**T**	**T**	**T**	**T**	**T**
F	**T**	**F**	**F**	**T**	**T**	**F**	**T**
F	**F**	**T**	**T**	**T**	**F**	**T**	**F**
F	**F**	**F**	**T**	**T**	**T**	**T**	**T**

From the truth table we can notice that the two statements are equivalent.

Thestatemente,fitcanbe completed by student at home (as homework for example).

Exercise 3.

a) Proving the truthy or falsy of the statements :

1. i) if k is odd, then $k+1$is even.ii)if k is even, then $k+1$is odd. The multiplication of odd number by an odd number is always gives an even number. The statement $\forall k \in \mathbb{N} \mid \; k(k+1)$ is odd is false
2. It is known that $(x-2)^2 \geq 0 \Rightarrow x+4 \geq 2x$ and this leads to $x+5 > 2x$. Thus, the statement $\forall x \in \mathbb{R} \mid \; x^2+5 > 2x$ is true.
3. We can easly show that $x(x^2+1) < 0$ if $x < 0$ and therefor the $\forall x \in \mathbb{R} \mid x^3+x \geq 1$ is false. Also you can show that the statement is not valid if $x = 0$.
4. The statement $\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z} \mid \; x^3+y^2 \geq 1$ is true since there exist a least an $x = 1$ where $1+y^2 \geq 1$.
5. The statement $\exists x \in \mathbb{R}, \forall n \in \mathbb{N} \mid \; x+3n$ is multiple of 3 is true because it can be shown that the statement is vrified as $x = 3$.
6. The statement $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \mid \; y > x(2-x)+1$is true. One can take $y = x(2-x)+2$ to check that $y > x(2-x)+1$.

b) The negation of the statements :

— $\exists k \in \mathbb{N} \mid \; k(k+1)$ is not odd
— $\exists x \in \mathbb{R} \mid \; x^2+5 \leq 2x$
— $\exists x \in \mathbb{R} \mid x^3+x-1 < 0$
— $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z} \mid \; x^3+y^2 < 1$
— $\forall x \in \mathbb{R}, \exists n \in \mathbb{N} \mid \; x+3n$ is not a multiple of 3
— $\exists x \in \mathbb{R}, \forall y \in \mathbb{R} \mid \; y \leq x(2-x)+1$

Exercise 04

1) let $a, b, c \in \mathbb{N}$ - a divides b $\Rightarrow b = \ell a$ where $\ell \in \mathbb{N}$

- a divides c $\Rightarrow c = ra$ " $r \in \mathbb{N}$

$\Rightarrow b + c = \ell a + ra = (\ell + r) a$

Since $(\ell + r) \in \mathbb{N}$, then a divide $(b+c)$

2) n is odd $\Rightarrow n = 2k + 1 \,/\, k \in \mathbb{N}$

$$n^2 = (2k+1)^2 = 4k^2 + 1 + 2k$$
$$= 2k(2k+1) + 1$$

$2k$ is even and $(2k+1)$ is odd $\Rightarrow 2k(2k+1)$ is even

Thus, $2k(2k+1) + 1$ is odd

3) $(a = b) \Rightarrow \left(\frac{a}{b+1} = \frac{b}{b+1}\right)$

We assume that $a = b$ is to be true

and by multiplying by $\frac{1}{b+1}$, we can get

$$\frac{a}{b+1} = \frac{b}{b+1}$$

4) $\left(\frac{a}{b+1} = \frac{b}{a+1}\right) \Leftrightarrow (a = b)$

This equivalent to $\left(\frac{a}{b+1} = \frac{b}{a+1}\right) \Rightarrow (a = b)$

and $(a = b) \Rightarrow \left(\frac{a}{b+1} = \frac{b}{a+1}\right)$

4.a/ We start from $\frac{a}{b+1} = \frac{b}{a+1}$ which impley that

$$a(a+1) = b(b+1) \Rightarrow a^2 + a - b^2 - b = 0$$
$$\Rightarrow (a-b)(a+b) + a - b = 0$$
$$\Rightarrow (a-b)(a+b+1) = 0$$

We have two solution $\begin{cases} a = b \\ a = -b-1 = \end{cases}$

We know that $a, b \in \mathbb{R}_+^*$ lead us to cancel the second solution. $\Rightarrow \left(\frac{a}{b+1} = \frac{b}{a+1}\right) \Rightarrow a = b$

4.b/ We start from $(a = b)$

$$a = b \Rightarrow a\,\frac{b+1}{b+1} = b\,\frac{a+1}{a+1}$$
$$\frac{a}{b+1} = \frac{b}{a+1}\left(\frac{a+1}{b+1}\right)$$

the last equation is ~~equal to a~~ equivalent to $\frac{a}{b+1} = \frac{b}{a+1}$ if $\left(\frac{a+1}{b+1}\right) = 1 \Rightarrow (a = b)$

and we can deduce that $\left(\frac{a}{b+1} = \frac{b}{a+1}\right) \Rightarrow (a = b)$

$$\left(\frac{a}{b+1} = \frac{b}{a+1}\right) \Rightarrow (a = b) \text{ and } (a = b) \Rightarrow \left(\frac{a}{b+1} = \frac{b}{a+1}\right)$$

Which gives $\left(\frac{a}{b+1} = \frac{b}{a+1}\right) \Leftrightarrow (a = b)$

5) We know that $(a-b)^2 \geqslant 0$

$$a^2+b^2-2a \geqslant 0 \Rightarrow a^2+b^2+2ab-2ab \geqslant 2ab$$
$$\Rightarrow (a+b)^2 \geqslant 4ab$$
$$\Rightarrow a+b \geqslant 2\sqrt{ab}$$
$$\Rightarrow \frac{a+b}{2} \geqslant \sqrt{ab}$$

Exercise 05 :

proof by contrapositiv:

to prove $(p \Rightarrow \varphi)$ it is sufficent to prove $\overline{\varphi} \Rightarrow \overline{p}$

1- p: a divides b φ: a divides c R: a divides b+c

to prove $(p \wedge \varphi) \Longrightarrow R$ it is sufficent to prove equivalent to

$\overline{R} \Rightarrow (\overline{p \wedge \varphi})$ and this is equivalent to

$$\overline{R} \Longrightarrow \overline{p} \vee \overline{\varphi}$$

$\overline{R}$: a does not divide b+c $\Rightarrow b+c = la+\delta \quad 0 \leqslant \delta < a$

$\overline{p}$: a does not divide b $\Rightarrow b = sa+\varepsilon \quad 0 \leqslant \varepsilon < a$

$\overline{p}$: a does not divide b $\Rightarrow$ ... $0 \leqslant \gamma < a$

$\overline{\varphi}$: a does not divide $c \Rightarrow c = ra+\gamma$

a does not divide $b+c \Rightarrow b+c = la+\delta \Rightarrow b = la-c+\delta$

if we assume that a divides $c \Rightarrow c = ra$

$\Rightarrow b = (l-r)a+\delta \Rightarrow$ a does not divide b

$\overline{R} \Rightarrow \overline{p} \vee \overline{\varphi}$ is true $\Rightarrow$

$\bar{R} \Rightarrow \bar{P} \vee \bar{Q}$ is true meaning that

$P \wedge Q \Rightarrow R$ is true.

2) We supose that P : n is an odd integer.

Q : n^2 is also an odd integer

$P \Rightarrow Q$ is true if $\bar{Q} \Rightarrow \bar{P}$ is true

$\bar{Q}$: n^2 is not an odd integer $\Rightarrow$ $\left.\begin{array}{l} n^2 = 4K^2 \\ n = 2m \end{array}\right\} m, K \in \mathbb{N}$

$\bar{P}$: n is not an odd integer

n^2 is not odd $\Rightarrow$ n^2 is even $\Rightarrow$ $n^2 = 4K^2$

$\Rightarrow$ ~~[illegible]~~ $n = 2K$

3) $\left(\frac{a}{b+1} \neq \frac{b}{b+1}\right) \Rightarrow (a \neq b)$

$$\frac{a}{(b+1)} \neq \frac{b}{(b+1)} \Rightarrow a\left(\frac{b+1}{b+1}\right) \neq b$$

$$\Rightarrow a \neq b$$

Exercise 06

1) $p: \forall k \in \mathbb{Z} : \sqrt{k^2+1} \geq k$

we supose the statement $\sqrt{k^2+1} < k$ is true $\Rightarrow k^2+1 < k^2$

$\Rightarrow 1 < 0 \Rightarrow$ contradiction

the statement $\sqrt{k^2+1} < k$ leads to a contradiction and this is a good reason to show that $\sqrt{k^2+1} \geq k$ is true.

2) $(a+b)^n \neq a^n + b^n$

We assume that $(a+b)^n = a^n + b^n$

$\Rightarrow (a+b)^{n+1-1} = a^n + b^n$

$\Rightarrow (a+b)^{n+1} = (a^n + b^n)(a+b)$

$(a+b)^{n+1} = a^{n+1} + b^{n+1} + a^n b + b^n a$

$\Rightarrow (a+b)^{n+1} \neq a^{n+1} + b^{n+1}$ leading to a contradiction. (تناقض)

$(a+b)^n = a^n + b^n$ Bring us to a contradiction

$\Rightarrow (a+b)^n \neq a^n + b^n$ is true

3/ $p: 2^n > n \Rightarrow \bar{p}: 2^n \leq n$

We assume that $2^n \leq n$ is true, then

$$2 \leq n^{1/n}$$

For $n = 2 \Rightarrow 2 \leq \sqrt{2} \Leftrightarrow$ contradiction

assuming $\bar{p}$ to be true leads to a contradiction

and thus, $2^n > n$ is true.

4/ $\bar{p}: \frac{a+b}{2} < \sqrt{ab}$

$\Rightarrow a + b < 2\sqrt{ab}$

$\Rightarrow (a+b)^2 < 4ab$

$\Rightarrow (a+b)^2 - 2ab < 2ab$

$\Rightarrow a^2 + b^2 < 2ab$

$\Rightarrow a^2 + b^2 - 2ab < 0$

$\Rightarrow (a-b)^2 < 0$

$\frac{a+b}{2} < \sqrt{ab}$ leads to $(a-b)^2 < 0$ and that ~~which~~ is a contradiction $\Rightarrow \frac{a+b}{2} \geq \sqrt{ab}$

is true