Chapter 3

Dynamics of material point

3.1 Introduction

- The previous chapter dealt with kinematics, we have discussed the elements that enter into the "description" of the motion of a particle (time equations, path, velocity and acceleration).
- In this chapter, we will investigate the reasons why bodies move.
- From daily experiences, we know that the motion of a body is a direct result of its interactions with the other bodies around it.
- Interactions are conveniently described by a mathematical concept called force.
- Dynamic studies the relation between force and the changes in the state of the body (rest or motion).

3.2 Definitions

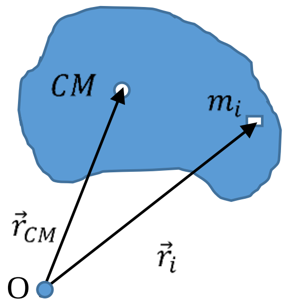
- **Mass**:
  - The mass of a system characterizes the quantity of matter it contains.
  - In physics, the mass is a quantitative measure a fundamental property of all matter called **inertia**.
- **Inertia** :
  Inertia is the property of an object that resisted change in its motion. An object at rest tends to stay at rest. An object on motion tends to keep moving at constant speed in a straight line.
- **Center of mass (center of inertia)**: The Center of mass of a body or a system is defined as the point at which the whole mass of the body or all the masses of the system appear to be concentrated.

The center of mass is defined as,

$$\sum m_i\vec{r}_i = M_{tot}\vec{r}_{CM}$$

Or we can use

$$\sum m_i\vec{r}_{CM} = \vec{0}$$



Example

Find the center of mass of the system represented by the front figure ($m_1 = 40g$, $m_2 = 10g$, and $l = 30cm$.

To clarify the idea of the vector $\vec{r}_{CM}$ we will find the center of mass by considering two cases:

- **The origin is at the middle of the barre.**

We have,

$\sum m_i\vec{r}_i = M_{tot}\vec{r}_{CM}$.

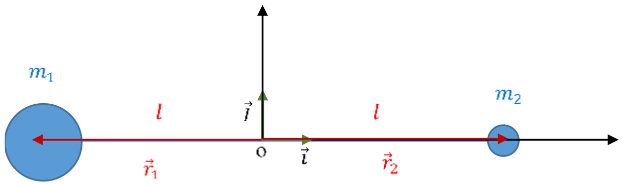
$\vec{r}_{CM} = \frac{\sum m_i\vec{r}_i}{M_{tot}}$

$\vec{r}_1 = -l\vec{\imath}, \quad \vec{r}_1 = l\vec{\imath}$

$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} = \frac{-m_1l\vec{\imath} + m_2l\vec{\imath}}{m_1 + m_2}$

$\vec{r}_{CM} = \frac{(-m_1 + m_2)l}{m_1 + m_2}\vec{\imath}$

$\vec{r}_{CM} = -18\vec{\imath}$



- **The origin is at the limit of the barre.**

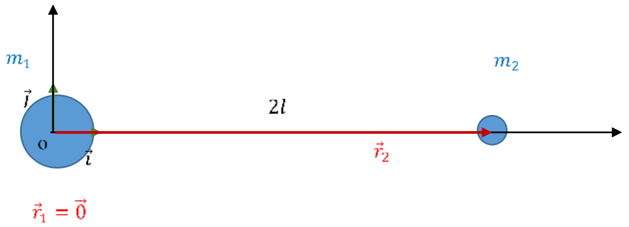
We have,

$\vec{r}_{CM} = \frac{\sum m_i\vec{r}_i}{M_{tot}}$

$\vec{r}_1 = 0\vec{\imath}, \quad \vec{r}_1 = 2l\vec{\imath}$

$\vec{r}_{CM} = \frac{2m_2\vec{r}_2}{m_1 + m_2} = \frac{2m_2l\vec{\imath}}{m_1 + m_2}$

$\vec{r}_{CM} = 12\vec{\imath}$



The center of mass is the same but it was calculated form different observers.

- **Centre of gravity:** The center of gravity can be taken as the point through which the force of gravity acts on an object or system. It is basically the point around which the resultant torque due to gravity forces disappears. In cases where the gravitational field is assumed to be uniform, the center of gravity and center of mass will be the same.
- **Free particle**: A free particle is one that is not subject to any interaction with the environment surrounding it.However, it would be impossible to observe it because, in the process of observation, there is always an interaction between the observer and the particle.
- **Momentum**

  The product of mass and velocity of a particle called momentum. The momentum is a vector quantity which can written as,

$$\vec{p} = m\vec{v}$$

In the case where the body is constituted of n particle, the total momentum is,

$$\vec{p} = \sum_{i=1}^{i=n} m_i \vec{v}_i$$

We know that $\vec{v}_i = \frac{d\vec{r}_i}{dt}$,

$$\vec{p} = \sum_{i=1}^{i=n} m_i \frac{d\vec{r}_i}{dt}$$

For the case where the element masses are constant, we can write

$$\vec{p} = \frac{d}{dt}\left(\sum_{i=1}^{i=n} m_i \vec{r}_i\right)$$

From the definition of the center of mass

$$\vec{p} = \frac{d}{dt} M_{tot}\vec{r}_{CM}$$

$M_{tot}$is constant, so

$$\vec{p} = M_{tot}\frac{d\vec{r}_{CM}}{dt}$$

$\frac{d\vec{r}_{CM}}{dt}$is the velocity of center of mass, we conclude that the total momentums of masses which constitute the whole body is equals to the total mass (mass of the body) times the velocity of center of mass. Then,

$$\vec{p} = M_{tot}\vec{v}_{CM}$$

The momentum of a system is equal to the momentum of a material point corresponding to its center of mass, where the total mass is concentrated.

Principle of Conservation of Momentum

To simplify the idea, let us consider that a system is constituted of two particles whichare subject only to their mutual interaction and are isolated from the rest of the world.

$$\vec{p} = \vec{p}_1 + \vec{p}_2 = m_1\vec{v}_1 + m_2\vec{v}_2$$

The two particles are in mutual interaction, their velocities change with time.

a. At time t_1,

$$\vec{p}(t_1) = m_1\vec{v}_1^1 + m_2\vec{v}_2^1$$

b. At time t_1,

$$\vec{p}(t_2) = m_1\vec{v}_1^2 + m_2\vec{v}_2^2$$

The momentum of this system still the same with respect to time because these two particles are isolated form the rest of word then,

$$\vec{p}(t_1) = \vec{p}(t_2)$$

This is what called the principle of conservation of momentum.

3.4 Newton's Law

3. 4.1 First Law(Law of Inertia)

- A free particle always moves with constant velocitywithout acceleration.That is, a free particle either moves in a straight line with constant speed or is at rest (zero velocity).
- Several easy experiments show that momentum is a more informative dynamical measure than velocity alone. For instance, even if both trucks are moving at the same speed, it is harder to stop or accelerate a loaded truckthan an empty one because of its higher momentum

We may now restate the law of inertia by saying that a free particle always moves with constant momentum.

$$\frac{d\vec{p}}{dt} = \vec{0}$$

Example

A gun with a mass of $0.80\,kg$ fires a bullet whose mass is $0.016\,kg$ with a velocity of $700\,ms^{-1}$, Compute the velocity of the gun's recoil.

Solution

Initially both the gun and the bullet are at rest and their total momentum is zero.

c. After the explosion, the bullet is moving forward with a momentum

$$\|\vec{p}_1\| = m_1\|\vec{v}_1\|$$

d. The gun must then recoil with an equal but opposite momentum. Therefore we must have also,

$$\|\vec{p}_2\| = m_2\|\vec{v}_2\|$$

$$m_1 v_1 = m_2 v_2$$

$$v_2 = \frac{m_1}{m_2} v_1$$

$$v_2 = \frac{0.016}{0.8} 700 = 14\ ms^{-1}$$

3. 4. 2 Newton's second law (Law of motion)

From the previous law, "the conservation of momentum", the free particle still have a constant momentum until interact with itsexternal surrounding (interact with other articles).We introduce another concept called **force,** which refers to the change in momentum of the particle with respect to time.Mathematically, the second law of Newton formulated by,

$$\vec{f} = \frac{d\vec{p}}{dt}.$$

This law is also called the **law of motion**.

We can rewrite this law as,

$$\vec{f} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt}$$

$$\vec{f} = \frac{dm}{dt}\vec{v} + m\frac{d\vec{v}}{dt}$$

In the case where the mass is constant, the second law will be written as,

$$\vec{f} = m\frac{d\vec{v}}{dt} = m\vec{a}$$

Note that the $\vec{f}$ represent the sum of forces act on the particle, $\vec{f} = \sum \vec{f_i}$

$$\sum \vec{f_i} = m\vec{a}$$

Example :A particle of mass 10 kg, subject to a force $\vec{f} = (120t + 40)N$, moves in a straight line. At time $t_0 = 0s$, the particle is at $x_0 = 5m$, with a velocity $v_0 = 6\, ms^{-1}$. Find its velocity and position at any later time.

Solution:

$$\vec{f} = m\vec{a}$$

The motion is linear then,

$$f = ma \rightarrow a = \frac{f}{m}$$

$$a = 12t + 4$$

$$a = \frac{dv}{dt} \rightarrow \int dv = \int (12t + 4)dt$$

$$v = 6t^2 + 4t + 6$$

$$v = \frac{dx}{dt} \rightarrow \int dx = \int (6t^2 + 4t + 6)dt$$

$$x = 2t^3 + 2t^2 + 6t + 5$$

3. 4. 3 Newton's third law (action and reaction)

Consider a system only constituted from two particles $m_1\ and\ m_2$ move with constant velocities $\vec{v}_1\ and\ \vec{v}_2$. The whole system is isolated,then asresult, the total momentum is conserved.

$$\frac{d\vec{p}}{dt} = \frac{d(\vec{p}_1 + \vec{p}_2)}{dt} = \vec{0}$$

We can write,

$$\frac{d\vec{p}_1}{dt} = -\frac{d\vec{p}_2}{dt}$$

If there is a change in the momentum of one particle, this indicates that there is a change in the momentum of the other particle of the same magnitude and in the opposite direction.

we can write, for the case where the masses are constant,

$$\begin{cases} \frac{d\vec{p}_1}{dt} = \vec{f}_1 \\ \frac{d\vec{p}_2}{dt} = \vec{f}_2 \end{cases} \rightarrow \quad then \quad \vec{f}_1 = -\vec{f}_2$$

When an object applies a force (action) to another object, the latter will apply a force (reaction) back to the first one. The action and reaction have the same magnitude, and they are in opposite directions.

3. 5. Force

As we said before the force refers to the change in momentum of the particle with respect to time.The change of momentum is due to the effect of interaction of the free particle with other particle (or particles).Otherwise, we can say that the force can just be defined as a push or pull on an objectwith another.

The unit of force can be directly definedin terms of the three basic units of mass, length, and time. The Newton(abbreviated N) is the unit of force in the InternationalSystem and is the force needed to acceleratea mass equals to 1kg at the rate of $1\text{m}s^{-2}$

$$1N = 1kg \times ms^{-2}$$

There is two type of force:

Forces interacted at a distance (Non-contact forces) such as:

- gravitational forces,
- electromagnetic forces, and
- nuclear cohesion forces.

Contact forces like,

- Action and reaction of two solids.
- Friction between two solids.
- Viscous friction with a fluid (gas or liquid).
- Spring return
- Thread tension

3. 5. 1.Examples of Non-contact forces (field forces)

a. Gravitational force

Gravitational force is the force of attraction experienced by two objects. The force of gravity is determined by the formula found by Newton, and is known as the gravitational force formula.

$$\vec{f}_{12} = -G\frac{m_1 \times m_2}{r^2}\vec{u}_{12}$$

From the action and reaction law,

$$\vec{f}_{21} = -G\frac{m_2 \times m_1}{r^2}\vec{u}_{21}$$

where G is the proportionality constant known as **Gravitational Constant**

$$\boldsymbol{G = 6.67 \times 10^{-11} Nm^2 kg^{-2}}$$

Law of Gravitation states that everything, including you, pulls every other object in the universe

b. Constant of Gravity of earth

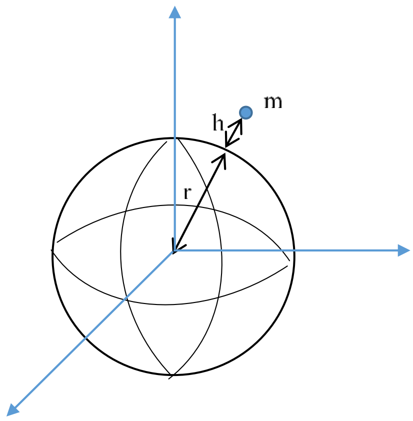
Due to the spherical symmetry of the earth, objects are attracted radially toward the center of the earth.

Consider an object of mass m is near to surface of the earth, then for force applies by earth on the object is,

$$\vec{f}_{12} = -G\frac{m \times m_{earth}}{(r_{earth} + h)^2}\vec{u}_{12}$$

$m_{earth} = 6 \times 10^{24} kg$: the mass of the earth.

$r_{earth} = 4 \times 10^{13} m$: the radius of the earth.



We can ignore the distance between the surface of the earth and the object, because it is very small comparing it by the radius of the earth.

$$f = G\frac{m \times m_{earth}}{(r^2)_{earth}}$$

$$f = m \times \frac{6.67 \times 10^{-11} \times 6 \times 10^{24}}{(6400000)^2}\ (N)$$

$$f = m \times 10\ (N)$$

We know that the force applied by the earth on an object near to its surface given by,

$$f = m \times g\ (N)$$

Then,$g = 10\ ms^{-2}$ which represents the acceleration the earth gravitation.

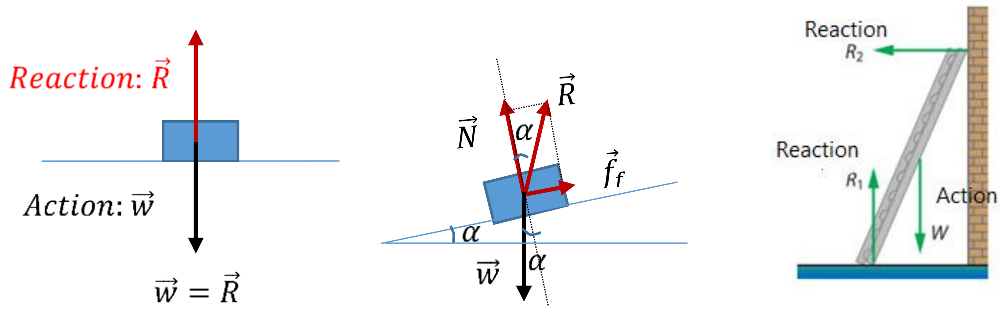
Note: there a slight difference between this constant when it calculated experimentally, which approximately equals to$g = 9.81\ ms^{-2}$.

3. 5. 2. Examples of contact forces

This type of force involves physical contact between two objects

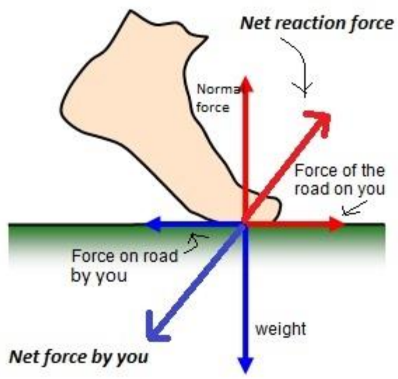
a. Actions and reactions forces

When a force affects the state (movement or rest) of an object, this object reacts with a force of the same magnitude and opposite direction to that exerted on it. A reaction may appears into more than one force, but the result of these forces are equal to the action force.



Reaction splits into two components

When a man walks on the ground, he exerts a force backward on the ground. At a result, the ground exerts a reaction force forward on the man that causes him to move forward.



Similarly, a car accelerates because the ground pushes forward on the drive wheels in reaction to the drive wheels pushing backward on the ground. We can see evidence of the wheels pushing backward when tires spin on a gravel road and throw rocks backward.

b. Friction force

Friction forces are forces that appear either during the movement of an object or if this object is subjected to a force which tends to move it. In all cases, the force of friction opposes the displacement that we seek to generate. It is important to distinguish two types of friction: viscous friction (solid-fluid contact) and solid friction (solid-solid contact).

- **Solid-solid contact**

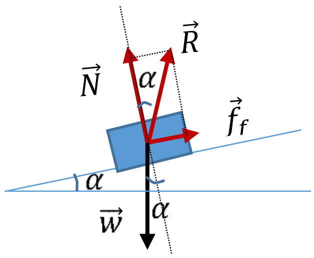
The direction of the force of static friction between any two surfaces in contact with each other is opposite the direction of relative motion and can have values.

$$\vec{F}_f = -\tan(\alpha)\, N\vec{u}_v$$

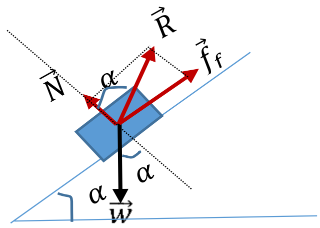
- $\alpha$is the angle between the normal and the reaction.
- $\vec{u}_v$indicates the direction of the motion.
- The negative sign in the equation indicates that the friction always be opposite to the motion.

The constant of proportionality $\mu = \tan(\alpha)$is called the coefficient of friction.

$$F_f = \mu N$$



$f_f = \tan(\alpha)N$



$f_f = \tan(\alpha)N$

Increasing α means increasing the force that moves the body downward (wsin(α)). As a result, the frictional forcef_f increases until it is no longer able to prevent movement. At this point, the friction force is at its maximum value. The coefficient of friction is called static frictionμ_s. Just after this angle, the movement begins and the resistance of the movement decreases. From this point, the coefficient of friction is called kinetic frictionμ_k.

We recapitulate,

- When multiplied the normal forceby the static coefficient of friction$\mu_s$gives the minimum force required to let the two bodies at relative rest.

$$F_f = \mu_s N$$

- When multiplied the normal forceby the kinetic coefficient of frictionμ_k gives the force required to maintain the two bodies in uniform relative motion.

$$F_f + \mu N = ma$$

Moment of force (Torque)

Moment of force measures the tendency of a force to rotate a body about a point or about an axis.Often, the moment of force is mathematically defined by the vector cross-product.

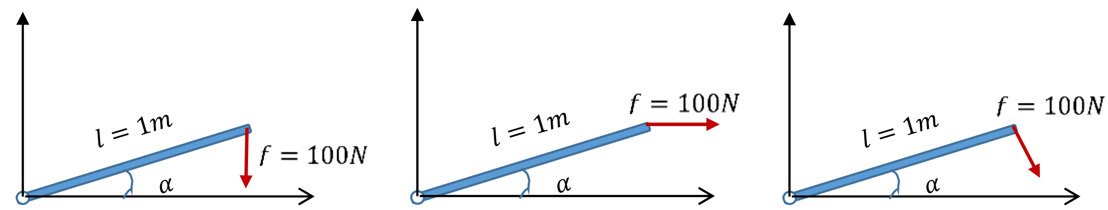
$$\vec{\mathcal{M}}(\vec{f})_o = \vec{r} \wedge \vec{f}$$

$\vec{r}$is the position vector from the rotating point "o" to any point on the line of action of $\vec{f}$.

It is clear that the moment of the force is a vector perpendicular to both $\vec{r}$and $\vec{f}$. Its magnitude indicates the ability of the force to make this object rotate.

Example:

Calculate the moment of the force $\vec{f}$ in three cases represented by the figure below.



Solution

- **first figure**

$$\vec{r} = l\cos(\alpha)\vec{\imath} + l\sin(\alpha)\vec{\jmath}$$

$$\vec{f} = -f\vec{\jmath}$$

$$\vec{\mathcal{M}}(\vec{f})_o = \vec{r} \wedge \vec{f} = \begin{vmatrix} \vec{\imath} & -\vec{\jmath} & \vec{k} \\ l\cos(\alpha) & l\sin(\alpha) & 0 \\ 0 & -f & 0 \end{vmatrix}$$

$$\vec{\mathcal{M}}(\vec{f})_o = -l\cos(\alpha) f\vec{k}$$

The negative sing indicates that the resulting vector and then,

$$\left\|\vec{\mathcal{M}}(\vec{f})_o\right\| = l\cos(\alpha) f$$

The moment equals tothe product of the magnitude of the force (F) and the perpendicular distance d (moment arm) from the line of action of the force to the rotation point.

- **Second figure**

$$\vec{r} = l\cos(\alpha)\vec{\imath} + l\sin(\alpha)\vec{\jmath}$$

$$\vec{f} = f\vec{\imath}$$

$$\vec{\mathcal{M}}(\vec{f})_o = \vec{r} \wedge \vec{f} = \begin{vmatrix} \vec{\imath} & -\vec{\jmath} & \vec{k} \\ l\cos(\alpha) & l\sin(\alpha) & 0 \\ f & 0 & 0 \end{vmatrix}$$

$$\vec{\mathcal{M}}(\vec{f})_o = -l\sin(\alpha) f\vec{k}$$

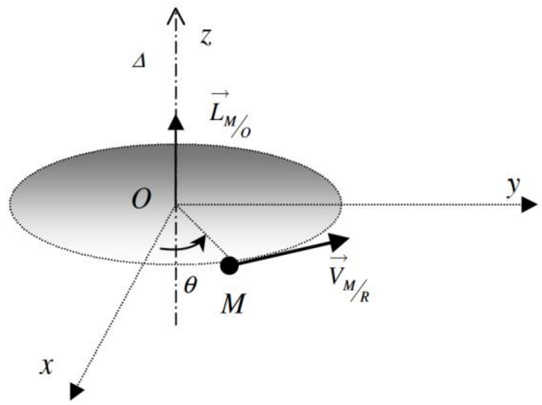
The negative sing indicates that the resulting vector and then,

$$\left\|\vec{\mathcal{M}}(\vec{f})_o\right\| = l\cos(\alpha)f$$

Angular Momentum

The angular momentum around point O of a particle of mass m moving with velocity v (and therefore having momentum p = mv) is defined by the vector product

$$\vec{L} = \vec{r} \wedge m\vec{v}$$



The angular momentum is therefore a vector perpendicular to the plane determined by $\vec{r}$and $\vec{v}$. The angular momentum of the particle in general changes in magnitude and direction while the particle moves (there are a change in $\vec{r}$and $\vec{v}$). We look for the change in the angular momentum with respect to the time,

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \wedge m\vec{v})$$

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \wedge (m\vec{v}) + \vec{r} \wedge \left(m\frac{d\vec{v}}{dt}\right)$$

$$\begin{cases} \frac{d\vec{r}}{dt} \wedge (m\vec{v}) = \vec{v} \wedge \left(m\frac{d\vec{v}}{dt}\right) = \vec{0} \\ \vec{r} \wedge (m\vec{v}) = \vec{r} \wedge \left(m\frac{d\vec{v}}{dt}\right) = \vec{r} \wedge \vec{f} \end{cases}$$ then,

$$\frac{d\vec{L}}{dt} = \vec{r} \wedge \vec{f}$$

The variation of angular momentum with respect to time is equals to the moment of force

$$\frac{d\vec{L}}{dt} = \vec{r} \wedge \vec{f}$$

$$\frac{d\vec{L}}{dt} = \vec{\mathcal{M}}(\vec{f})$$

The time rate of change of the angular momentum of a particle is equal to the moment of force (torque) of the force applied to it.

Note1: For a material point rotating around a fixed axis (or point), we can apply the fundamental principle of dynamics or the angular momentum theorem indifferently.

$$\frac{d\vec{L}}{dt} = \overrightarrow{\mathcal{M}}_{tot}(\vec{f})$$

Note 2:this equation is correct only if $\vec{L}$ and $\vec{r}$are measured relative to the same point (rotating point).

- For linear motion ----------------------- $\boldsymbol{\frac{d\vec{p}}{dt} = \vec{f}_{tot}}$
- For rotational motion ------------------- $\boldsymbol{\frac{d\vec{L}}{dt} = \overrightarrow{\mathcal{M}}_{tot}(\vec{f})}$

Example: