Chapter 03: Real Functions of a Real Variable **(Lecture 1)**

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Plan

- References
- Definitions
- Bounded functions
- Definitions of Monotonic Functions
- Introduction to Maxima and Minima
- Limits of a Function
- Continuity

References

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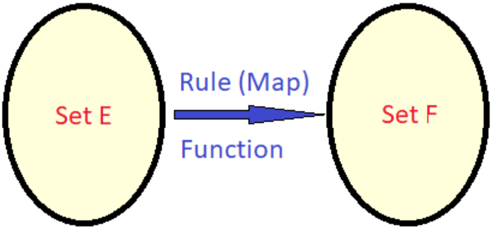
Definition:

- A real function of a real variable is a mapping f from a set $E \subset \mathbb{R}$ to a set $F \subset \mathbb{R}$, written as:

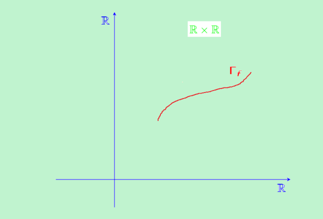
$$f : E \to F, \quad x \mapsto f(x).$$

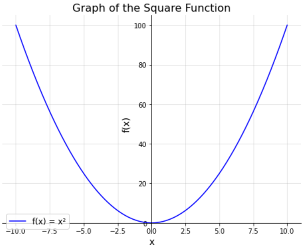
- Here, x is called the real variable, and $f(x)$ is called the image of x under f.
- The domain of definition of f is the set of values $x \in E$ for which $f(x) \in F$, denoted D_f.
- The set of all functions from E to F is denoted as $F(E, F)$.

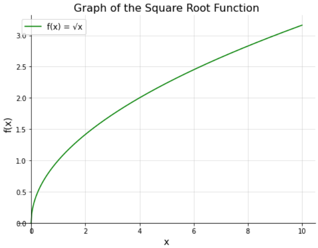


The graph of f is the subset Γ_f of the Cartesian product $\mathbb{R} \times \mathbb{R}$ defined as:

$$\Gamma_f = \{(x, f(x)) \mid x \in E\}.$$







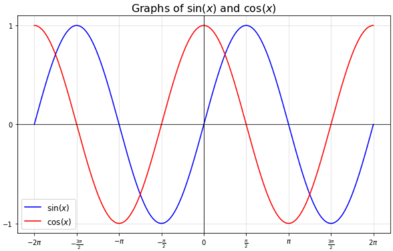
Definition: (Parity of a Function)
Let f be a function from $\mathbb{R}$ to $\mathbb{R}$.
- f is called even if $\forall x \in D_f,\ f(-x) = f(x)$, meaning the graph of f is symmetric with respect to the y-axis.
- f is called odd if $\forall x \in D_f,\ f(-x) = -f(x)$, meaning the graph of f is symmetric with respect to the origin.

Definition: (Periodicity of a Function) A function f is said to be periodic if there exists a strictly positive real number T such that:

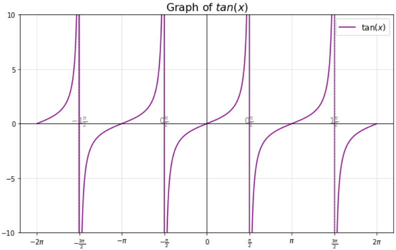
$$\forall x \in D_f,\ f(x+T) = f(x).$$

Examples:

- For $f(x) = \sin x$ or $f(x) = \cos x$, the period is $T = 2\pi$.



- For $f(x) = \tan x$, the period is $T = \pi$.



- For $f(x) = x - \lfloor x \rfloor$, the period is $T = 1$.

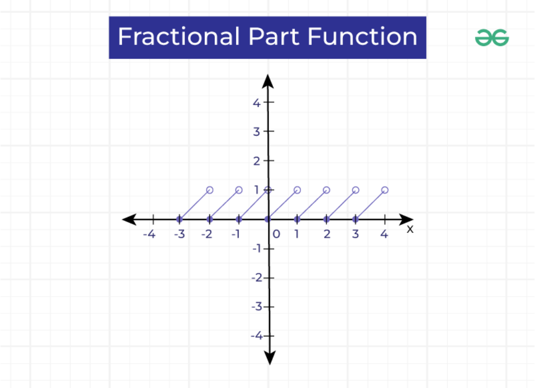


Figure: Source: https://www.geeksforgeeks.org/fractional-part-function/

- For $f(x) = \cos\left(\frac{5x}{2}\right)$, the period is $T = \frac{4\pi}{5}$.

Remarks:

1. If f is even or odd, it suffices to study it over half its domain.
2. There exist functions that are neither even nor odd.
3. If f is periodic with period T, it suffices to study it over a single period.

Bounded Functions

1. A function $f(x)$ is said to be **bounded above** in an interval (or set) if there exists a constant M such that:

$$f(x) \leq M \quad \text{for all } x \text{ in the interval.}$$

Here, M is called an **upper bound** for $f(x)$.
2. Similarly, $f(x)$ is **bounded below** if there exists a constant m such that:

$$f(x) \geq m \quad \text{for all } x \text{ in the interval.}$$

In this case, m is called a **lower bound**.
3. If both conditions are satisfied—i.e., there exist constants m and M such that:

$$m \leq f(x) \leq M \quad \text{for all } x \text{ in the interval,}$$

then $f(x)$ is called a **bounded function**. To denote that $f(x)$ is bounded, we can write:

$$|f(x)| \leq P, \quad P > 0.$$

Examples

1) For $f(x) = \cos(x)$ in the interval $-\infty < x < \infty$:

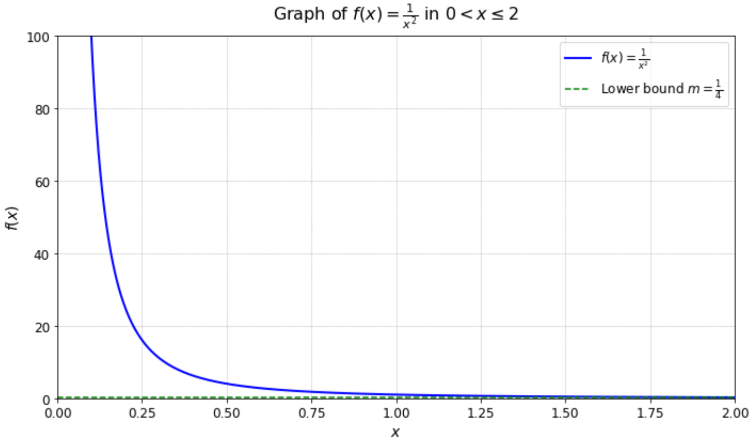
- The function is bounded since $-1 \leq f(x) \leq 1$ for all x.
- $M = 1$ is an **upper bound**, and $m = -1$ is a **lower bound**.

2) For $f(x) = x^3$ in the interval $-2 \leq x \leq 2$:

- The function is bounded because $-8 \leq f(x) \leq 8$ in this interval.
- $M = 8$ is an **upper bound**, and $m = -8$ is a **lower bound**.

3) For $f(x) = \frac{1}{x^2}$ in the interval $0 < x \leq 2$:

- The function is **not bounded above**, as $f(x) \to \infty$ as $x \to 0^+$.
- However, it is bounded below with $f(x) \geq \frac{1}{4}$.


Graph of $f(x) = \frac{1}{x^2}$ in $0 < x \leq 2$
$f(x) = \frac{1}{x^2}$
Lower bound $m = \frac{1}{4}$
100
80
60
40
20
0
$f(x)$
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
x

Definition of Monotonic Functions

Monotonically Increasing:

- A function $f(x)$ is monotonically increasing on an interval if, for any x_1, x_2 such that $x_1 < x_2$, we have:

$$f(x_1) \leq f(x_2).$$

- If $f(x_1) < f(x_2)$ strictly for all $x_1 < x_2$, then $f(x)$ is **strictly increasing**.

Monotonically Decreasing:

- A function $f(x)$ is monotonically decreasing on an interval if, for any x_1, x_2 such that $x_1 < x_2$, we have:

$$f(x_1) \geq f(x_2).$$

- If $f(x_1) > f(x_2)$ strictly for all $x_1 < x_2$, then $f(x)$ is **strictly decreasing**.

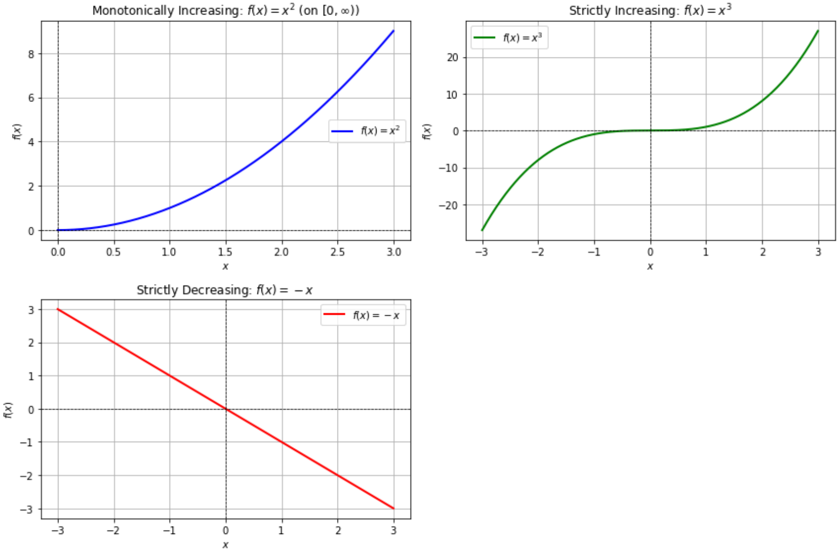
Examples:

Monotonically Increasing:

- The function $f(x) = x^2$ is monotonically increasing on $[0, \infty)$.
- The function $f(x) = x^3$ is strictly increasing on $\mathbb{R}$.

Monotonically Decreasing:

- The function $f(x) = -x$ is strictly decreasing on $\mathbb{R}$.


Monotonically Increasing: $f(x) = x^2$ (on $[0, \infty)$)
$f(x) = x^2$
$f(x)$
x
Strictly Increasing: $f(x) = x^3$
$f(x) = x^3$
$f(x)$
x
Strictly Decreasing: $f(x) = -x$
$f(x) = -x$
$f(x)$
x

Introduction to Maxima and Minima

- The study of maxima and minima, or extreme values of functions, was a key motivation for the development of calculus in the 17th century.
- Extreme values are important in both mathematical theory and real-world applications.
- They are classified into:
  - **Relative extrema** (local maxima and minima).
  - **Absolute extrema** (global maxima and minima).

Definition:

- A function $f(x)$ has a **relative maximum** at c if there exists an interval (a, b) containing c such that:

$$f(x) < f(c) \quad \text{for all } x \neq c \text{ in } (a, b).$$

- Similarly, $f(x)$ has a **relative minimum** at c if:

$$f(x) > f(c) \quad \text{for all } x \neq c \text{ in } (a, b).$$

Key Points:

- Relative extrema are the "high points" (maximum) or "low points" (minimum) within a local neighborhood.
- Functions may have multiple relative extrema or none at all.

Absolute Extrema

Definition:

- A function $f(x)$ has an **absolute maximum** at c if:

$$f(x) \leq f(c) \quad \text{for all } x \text{ in the domain of } f.$$

- A function $f(x)$ has an **absolute minimum** at c if:

$$f(x) \geq f(c) \quad \text{for all } x \text{ in the domain of } f.$$

Key Points:

- Absolute extrema consider the entire domain of the function.
- Strictly increasing or decreasing functions may have absolute extrema at the endpoints of a closed interval.
- Absolute extrema are not always unique, as in the case of constant functions.

Examples:

- Relative extrema:
  - The highest point on a specific hill is a **relative maximum**.
  - The lowest point in a valley is a **relative minimum**.
- Absolute extrema:
  - The tallest hill overall is the **absolute maximum**.
  - The deepest valley is the **absolute minimum**.

Special Cases:

- Strictly increasing or decreasing functions have no relative extrema but may have absolute extrema on closed intervals.
- Constant functions have infinite absolute extrema, as all points are maxima and minima.

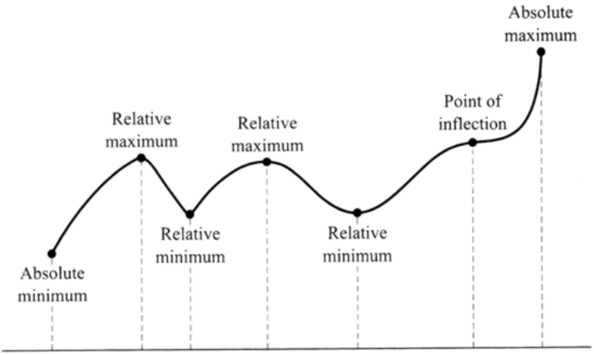


Figure 3.3

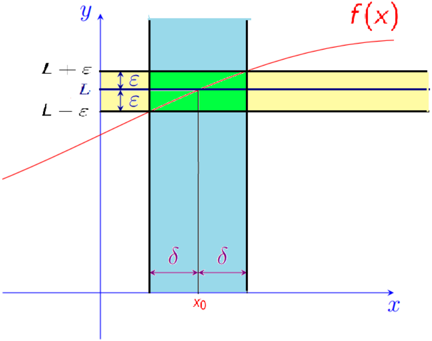
Figure: Source:Schaum's Outlines: Advanced Calculus By Robert C. Wrede, Murray R. Spiegel · 2011

Definition: Let f be a real-valued function defined on a set $E \subset \mathbb{R}$. We say that $f(x)$ tends toward a limit L as x approaches x_0 if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$\forall x \in E,\ 0 < |x - x_0| < \delta \implies |f(x) - L| < \epsilon.$$

This is written as:

$$\lim_{x \to x_0} f(x) = L.$$



Theorems on limits:

1. If $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = M$, then:

$$\lim_{x\to x_0}[f(x) + g(x)] = L + M.$$

2. If $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = M$, then:

$$\lim_{x\to x_0}[f(x) \cdot g(x)] = L \cdot M.$$

3. If $\lim_{x\to x_0} f(x) = L$, $\lim_{x\to x_0} g(x) = M$, and $M \neq 0$, then:

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

Remark: If $f(x)$ is continuous at x_0, then:

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Definition: A function $f(x)$ is said to have a limit L as x approaches infinity if, for every $\epsilon > 0$, there exists $N > 0$ such that:

$$\forall x > N, \ |f(x) - L| < \epsilon.$$

This is written as:

$$\lim_{x \to \infty} f(x) = L.$$

Theorem If f admits a limit at the point x_0, then this limit is unique.

Proof

Suppose, for contradiction, that f admits two different limits, l_1 and l_2 ($l_1 \neq l_2$). When x tends to x_0, we have:

$$\lim_{x \to x_0} f(x) = l_1 \iff$$

$$\forall \epsilon > 0, \exists \alpha_1 > 0, \forall x \in I, \ |x - x_0| < \alpha_1 \implies |f(x) - l_1| < \frac{\epsilon}{2}$$

$$\lim_{x \to x_0} f(x) = l_2 \iff$$

$$\forall \epsilon > 0, \exists \alpha_2 > 0, \forall x \in I, \ |x - x_0| < \alpha_2 \implies |f(x) - l_2| < \frac{\epsilon}{2}$$

Let $\epsilon > 0$, then:

$$|l_1 - l_2| = |(l_1 - f(x)) + (f(x) - l_2)|$$

For $\alpha = \min(\alpha_1, \alpha_2)$, we have:

$$|l_1 - l_2| \leq |f(x) - l_1| + |f(x) - l_2| < \epsilon$$

Thus, for all $\epsilon > 0$ (no matter how small), it follows that $l_1 = l_2$.

One-Sided Limits

Definition:

- f has a **left-hand limit** l_l as $x \to x_0^-$:

$$\lim_{x \to x_0^-} f(x) = l_l \iff$$

$$\forall \epsilon > 0, \exists \alpha > 0, \forall x \in I,\ x_0 - \alpha < x < x_0 \implies |f(x) - l_l| < \epsilon$$

- f has a **right-hand limit** l_r as $x \to x_0^+$:

$$\lim_{x \to x_0^+} f(x) = l_r \iff$$

$$\forall \epsilon > 0, \exists \alpha > 0, \forall x \in I,\ x_0 < x < x_0 + \alpha \implies |f(x) - l_r| < \epsilon$$

Remarks

1. If f has a limit l as $x \to x_0$, then:

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = l$$

2. If f has a left-hand limit l_l and a right-hand limit l_r at x_0, and $l_l = l_r$, then:

$$\lim_{x \to x_0} f(x) = l_l = l_r$$

3. If $l_l \neq l_r$, then f does not have a limit as $x \to x_0$.

Infinity as a Limit

1.
$$\lim_{x \to x_0} f(x) = +\infty \iff$$
$$\forall A > 0, \exists \alpha > 0, \forall x \in I, \; |x - x_0| < \alpha \implies f(x) > A$$

2.
$$\lim_{x \to x_0} f(x) = -\infty \iff$$
$$\forall A < 0, \exists \alpha > 0, \forall x \in I, \; |x - x_0| < \alpha \implies f(x) < A$$

Limits at Infinity

1. $$\lim_{x \to +\infty} f(x) = l \iff \forall \epsilon > 0, \exists \alpha > 0, \forall x > \alpha \implies |f(x) - l| < \epsilon$$

2. $$\lim_{x \to -\infty} f(x) = l \iff \forall \epsilon > 0, \exists \alpha < 0, \forall x < \alpha \implies |f(x) - l| < \epsilon$$

Definition: A function $f(x)$ is said to be continuous at a point $x_0 \in D_f$ if:

$$\lim_{x \to x_0} f(x) = f(x_0).$$

If $f(x)$ is continuous at every point in D_f, it is said to be continuous on D_f.

A function $f(x)$ is said to be **continuous** at a point $x = x_0$ if the following conditions hold:

1. The limit $\lim_{x\to x_0} f(x)$ exists.
2. The value $f(x_0)$ exists (i.e., f is defined at x_0).
3. The limit matches the function's value:

$$\lim_{x\to x_0} f(x) = f(x_0).$$

These conditions ensure the function behaves smoothly at x_0, with no jumps, gaps, or breaks.

Alternative Definition

A function $f(x)$ is continuous at x_0 if:

$\forall \epsilon > 0, \exists \delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$.

This means that we can make $f(x)$ arbitrarily close to $f(x_0)$ by taking x sufficiently close to x_0.

Intuitive Understanding of Continuity

- If $f(x)$ is continuous at x_0, the graph of $f(x)$ near x_0 can be drawn without lifting the pencil from the paper.
- If there is a gap or jump in the graph at x_0, the function is **not continuous** there.

In simple terms, $f(x)$ is continuous at x_0 if:

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Example 1: A Discontinuous Function

Let:

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 2, \\ 0 & \text{if } x = 2. \end{cases}$$

- For $x \neq 2$, $f(x) = x^2$, so $\lim_{x \to 2} f(x) = 4$.
- At $x = 2$, $f(2) = 0$.
- Since $\lim_{x \to 2} f(x) \neq f(2)$, $f(x)$ is **not continuous** at $x = 2$.

Example 2: A Continuous Function

Let $f(x) = x^2$ for all x:

- The limit $\lim_{x \to 2} f(x) = 4$.
- The value of the function at $x = 2$ is $f(2) = 4$.
- Since $\lim_{x \to 2} f(x) = f(2)$, $f(x)$ is **continuous** at $x = 2$.

Points of Discontinuity

Discontinuities are points where $f(x)$ fails to be continuous. These can occur if:

- $f(x)$ is undefined at the point (a gap),
- $f(x)$ jumps to a different value,
- The left-hand and right-hand limits do not match.

For example: - In Example 1, $f(x)$ is discontinuous at $x = 2$ because the limit and function value differ.

Property:

The sum, product, and quotient (where the denominator is nonzero) of continuous functions are continuous.

Examples:

1. The function $f(x) = x^2 + 3x + 2$ is continuous on $\mathbb{R}$.
2. The function $g(x) = \frac{1}{x}$ is continuous on $\mathbb{R} \setminus \{0\}$.