CHAPTER 2: Free Linear Systems with One Degree of Freedom.

II.1 Undamped free systems (Free oscillators)

A system oscillating without any excitation force (external forces) is called free (Undamped free oscillator). The number of independent variables describing the system is called the degree of freedom (DOF).

II.2 Harmonic oscillator

A harmonic oscillator is an oscillator that is returned to its equilibrium position when it is moved a certain distance due to a restoring force opposing the motion.

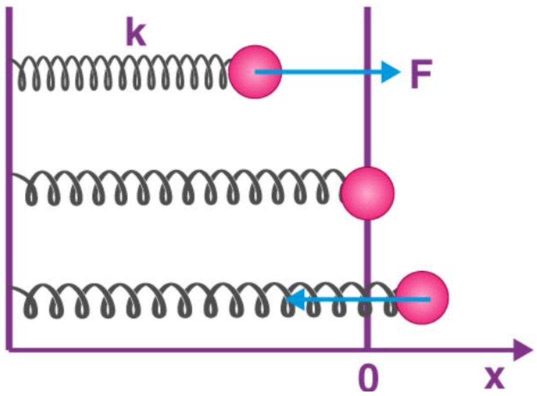
$$F = -c \times x$$

With:

C is a constant and x is the distance

Example: Mass-spring system

The mass is moved slightly away from its equilibrium position and released without initial speed; the experimenter observes that it begins to oscillate around this equilibrium position. the restoring force brings it back to its equilibrium position.



II.3 Equation of motion

The equation of motion for a harmonic oscillator has the form

$$\ddot{q} + \omega_0^2 q = 0$$

Where: q are the generalized coordinates,

for mechanics $q\ (x, y, z, \theta, \varphi\ldots\ldots)$ and $q(i, u, Q, \ldots\ldots)$ for electricity

The solution of the previous equation is written as:

$$q(t) = A\cos(\omega_0 t + \varphi)$$

A: Amplitude of the oscillations, φ: Initial phase.

ω_0: The pulsation of the system, depends on the constituent elements (mass, spring, wires, etc.).

The constants A and φ are calculated from the initial conditions:

$$\begin{cases} q(t=0) = q_0 \\ \dot{q}(t=0) = \dot{q}_0 \end{cases}$$

Note: The oscillation amplitude of a free harmonic oscillator does not depend on time. Such oscillations are said to be undamped.

II.4 Energy of a harmonic oscillator (E)

The energy of a harmonic oscillator is the sum of the kinetic (T) and potential (U) energy.

E=T+U

II.4.1 Kinetic energy T :

The kinetic energy of a mass m which performs a motion of:

- **Translation of a distance x**

$$T = \frac{1}{2} mV^2 = \frac{1}{2} m\dot{x}^2$$

- **Rotation of an angle θ**

$$T = \frac{1}{2} J\dot{\theta}^2$$

J is the moment of inertia relative to the point of the axis of rotation

- Translation x and Rotation θ at the same time:

$$T_{tot} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J\dot{\theta}^2$$

II.4.2 Potential energy U

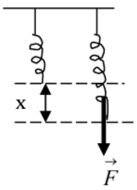
- The change in the gravitational potential energy (g.p.e.) of an object, Up, depends on the change in its height, h. We can calculate Up using this equation:

$$\text{Up} = \text{mgh}$$

With: m in kilograms (kg), g in newtons per kilogram ($N.kg^{-1}$) and h in meters (m)

- **The potential energy** of a spring with stiffness k during deformation x is:

$$U_k = \frac{1}{2} kx^2$$



- **A torsion spring** with a stiffness constant (k), a deformation (θ)

$$U_k = \frac{1}{2}k\theta^2$$

- **Electric potential energy:**

A charged particle placed in an electric field has an electric potential energy, denoted Epe. This energy depends on the charge q of the particle and the electric potential V of the point where the particle is located:

$$\text{Ue} = q.V$$

with q in coulombs (C) and V in volts (V).

II.5: Huygens' theorem

The moment of inertia I about any axis of rotation which is in parallel to the axis passing through the center of gravity is given by:

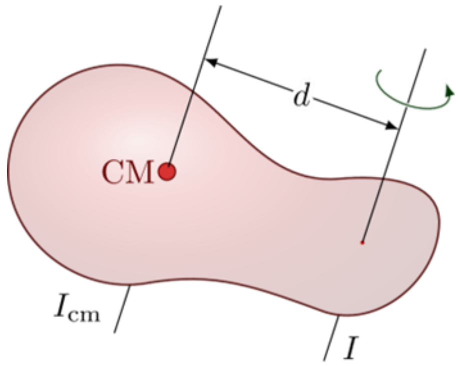
$$I = I_0 + Md^2$$

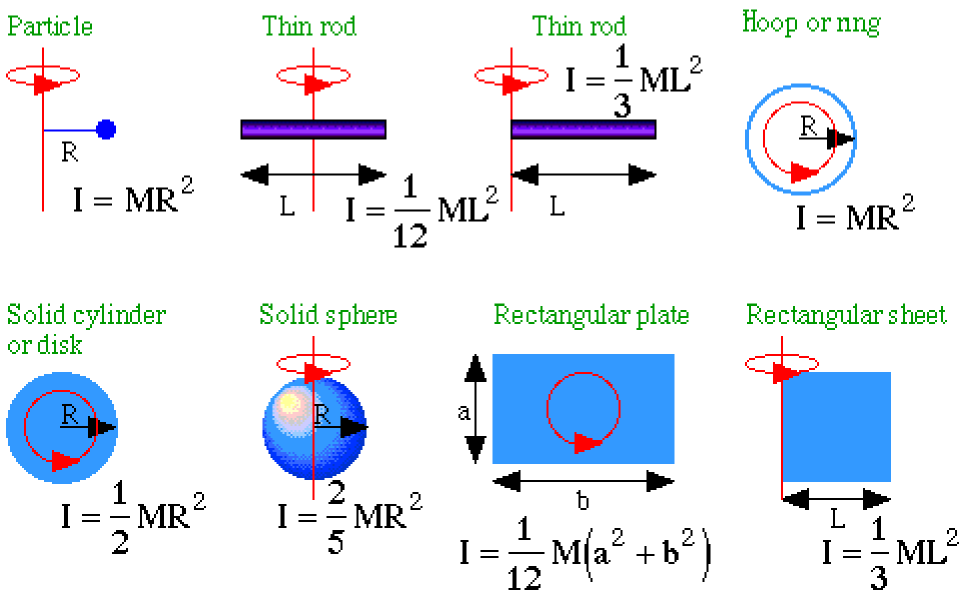
Where:

I_0 is the moment of inertia of the object

M the mass of the object

D is the distance between the two axis





Moment of inertia of some object

II.6: Study of the mechanical system

II.6.1: Obtaining the differential equation of motion

The choice of the calculation method to use to arrive at the equation of motion (EDM), we distinguish:

-Newton's equation.

-Lagrange's equation.

II.6.1.1 Newton's law:

This formalism is based on the fundamental principle of dynamics and is applied according to the case of the movement, namely: translation or rotation.

II.6.1.1.1 Translational motion:

If a system of mass m is subjected to external forces, the fundamental law of

dynamics (F.L.D.) gives us:

$$\sum_i \vec{F}_i = m\vec{\gamma}_i = m\vec{a}_i$$

II.6.1.1.2 Rotational motion:

The fundamental law of the dynamics of a rotational motion is written:

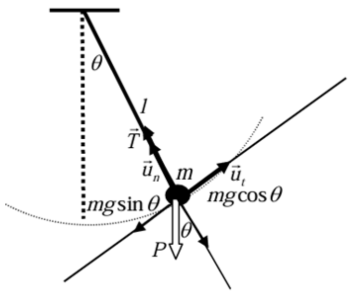
$$\sum_i M_\Delta\left(\vec{F}_i\right) = J\,\ddot{\theta} = J\frac{d^2\theta}{dt^2}$$

Example: simple pendulum

According to Newton's second law $\sum \vec{F} = m\vec{\gamma}$

We find

$$\vec{p} + \vec{T} = m\vec{\gamma}$$



By the projection on the axis $\vec{U}_t$

$$-mg\sin(\theta) = m\gamma_t$$

γ_t: Tangential acceleration and g: gravity.

By projection on the axis $\overrightarrow{U_n}$

$$-mg\cos(\theta) + T = m\gamma_n = 0$$

We have

$$v_t = l\dot{\theta} \quad et \quad \gamma_t = l\ddot{\theta}$$

By replacing in equation, we find that

$$-mg\sin(\theta) = ml\ddot{\theta}$$

Finally, we obtain:

$$\ddot{\theta} + \frac{g}{l}\sin(\theta) = 0$$

The pendulum is assumed to swing at small angles, which allows us to use the approximation sin θ=θ. The previous equation is therefore written as follows:

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

This is the differential equation of vibrational motion.

With $\omega_0^2 = \frac{g}{l}$

Note: ω_0 called proper pulsation because it only depends on the proper quantities of the oscillator (k and m)

II.6.1.2 Lagrange equation

The Lagrange equation is used to determine the equation of motion of mechanical systems. It is described by the following equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = F_i$$

Where (L) is the Lagrangian which is an explicit function of the generalized coordinates and the generalized velocities

$$L = T - U$$

T : est l’énergie cinétique totale du système

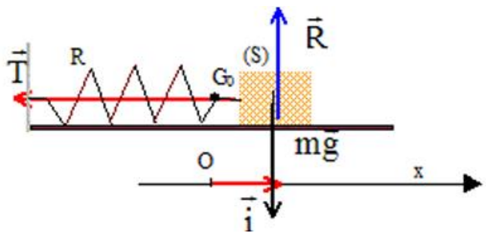
U: est l’énergie potentielle totale du système q_i et . $\dot{q}_i$ sont les coordonnées et les vitesses généralisées F_i sont les forces généralisée associées à q_i.

For the case of the conservative system with one degree of freedom, Lagrange equation reduces to:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)-\frac{\partial L}{\partial q}=0$$

Example1: Mass-spring

Consider the following system,



using the Lagrange method, write the equation of motion and deduce the proper pulsation.

- **Lagrangien** L=T-U

- **Kinetic energy**

$$T=\frac{1}{2}m\dot{x}^{2}$$

$$U=\underbrace{U_p}_{0}+U_k=U_k=\frac{1}{2}kx^2$$

$$L=\frac{1}{2}m\dot{x}^{2}-\frac{1}{2}kx^2$$

- **Lagrange equation**

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=0$$

After the derivations we get this equation

$$m\ddot{x}+kx=0$$

If we divide this equation into (m) we obtain the equation of motion

$$\ddot{x} + \frac{k}{m}x = 0$$

By transposition with the general equation

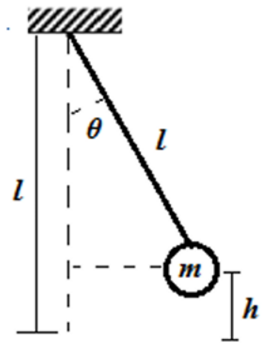
$$\ddot{x} + \omega_0^2 x = 0$$

So the pulsation is

$$\omega_0 = \sqrt{\frac{k}{m}}$$

Example 2 simple pendulum

Consider the pendulum in the figure opposite; calculate the pulsation of the small pulsations using the Lagrange formalism



The system is free (no external force applied) and conservative (undamped).

L: The Lagrangian of the system is given by: L=T-U

q: The generalized coordinate, in this case q= θ

T: The kinetic energy of the system

$$T = \frac{1}{2}\, mV^2 = \frac{1}{2} m\dot{x}^2$$

$x = l \times \theta \rightarrow$

$$T = \frac{1}{2} ml^2\dot{\theta}^2$$

U: The potential energy of the system U= mgl (1-cosθ)

$$L = \frac{1}{2} ml^2\dot{\theta}^2 - mgl(1 - \cos\theta)$$

- $$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{d\dot{\theta}}\right) = ml^2\ddot{\theta}$$

- $$\frac{\partial L}{\partial \theta} = -mgl\sin\theta\ ,$$

The pendulum is considered to oscillate (vibrate) with small angles, which made it possible to write $\sin\theta \approx \theta$

$$\frac{\partial L}{\partial \theta} = -mgl\theta$$

$$ml^2\ddot{\theta} + mgl\theta = 0$$

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

While: $\omega_0^2 = \frac{g}{l}$

II.7: Equilibrium conditions

According to the previous notions, the force is defined as deriving from a potential energy

$$F = -\frac{\partial U}{\partial q}$$

Our system is at equilibrium so F=0

So the equilibrium condition will be as follows

$$\frac{\partial U}{\partial q} = 0$$

The equilibrium of a system is stable if, once moved away from its equilibrium position, it returns to it. The system returns to its equilibrium if F is a restoring force. Since $F = -k\,x$, we will have a restoring force if $k > 0$.

$$\left.\frac{\partial^2 U}{\partial q^2}\right]_{q=0} > 0$$

The equilibrium of a system is unstable if the system does not regain its equilibrium during a separation, i.e. if k < 0. The unstable equilibrium condition is therefore written:

$$\left.\frac{\partial^2 U}{\partial q^2}\right]_{q=0} < 0$$

II.7: Variation in the energy of a vibrational system

Energy can neither be created nor destroyed, but only transferred from one system to another and transformed from one form to another. So if the total energy (kinetic and potential) of a system is invariable during its motion we will conclude that:

$$\frac{dE}{dt} = 0$$

For the harmonic oscillator,

$$E = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2$$

$$x = A\sin(w_n t + \varphi) \Rightarrow \dot{x} = Aw_n \cos(w_n t + \varphi)$$

$$E = \frac{1}{2} mA^2 w_n^2 \cos^2(w_n t + \varphi) + \frac{1}{2} kA^2 \sin^2(w_n t + \varphi)$$

$$E = \frac{1}{2} mA^2 w_n^2 (\cos^2(w_n t + \varphi) + \sin^2(w_n t + \varphi))$$

$$E = \frac{1}{2} mA^2 w_n^2$$

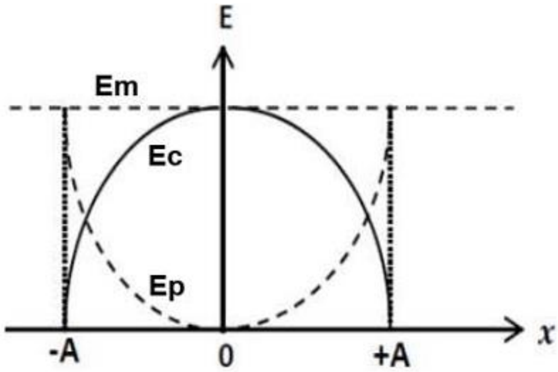
Knowing that:

$$\omega_0 = \sqrt{\frac{k}{m}} \Rightarrow k = m\omega_0^2$$

By substation in the previous equation of energy, we find that:

$$E=\frac{1}{2}kA^2\cos^2(\omega_0 t+\varphi)+\frac{1}{2}kA^2\sin^2(\omega_0 t+\varphi)$$

$$E=\frac{1}{2}kA^2$$



Variation of the kinetic and potential energy of an undamped vibrational system