People's Democratic Republic of Algeria

Ministry of Higher Education and scientific Research

University Center abdelhafid boussouf (Alger)

Institute of mathematics and computer sciences

Department of mathematics

Serie of exercise
Introduction To Dynamics Systems .

Master 1 (first year) fundamental and applied mathematics

The first semester

By: ROUIBAH KHAOULA

University year: 2023 – 2024

CHAPTER 1

LINEAR SYSTEMS

exercise 1

Find the general solution and draw the phase portrait for the following linear systems:

(a)

$$\dot{x}_1 = x_1$$
$$\dot{x}_2 = x_2$$

(b)

$$\dot{x}_1 = x_1$$
$$\dot{x}_2 = 2x_2$$

(c)

$$\dot{x}_1 = x_1$$
$$\dot{x}_2 = 3x_2$$

(d)

$$\dot{x}_1 = -x_2$$
$$\dot{x}_2 = x_1$$

(e)

$$\dot{x}_1 = -x_1 + x_2$$
$$\dot{x}_2 = -x_2$$

Hint: Write (d) as a second-order linear differential equation with constant coefficients, solve it by standard methods, and note that $x_1^2 + x_2^2$ = constant on the solution curves. In (e), find $x_2(t) = c_2 e^{-t}$ and then the x_1-equation becomes a first order linear differential equation.

solution

Let $\mathbf{x} = (x_1, x_2, x_3)^\top = (x, y, z)^\top$ and $\mathbf{x}(0) = (x_0, y_0, z_0)^\top$.

1. (a) $x(t) = x_0 e^t, y(t) = y_0 e^t$, and solution curves lie on the straight lines $y = (y_0/x_0)\, x$ or on the y-axis. The phase portrait is given in 1.1.

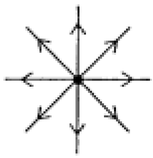

Figure 1.1: Phase portrait

(b) $x(t) = x_0e^t, y(t) = y_0e^{21}$, and solution curves, other than those on the x and y axes, lie on the parabolas $\mathrm{y} = \left(\mathrm{y}_0/\mathrm{x}_0^2\right)\mathrm{x}^2$. Cf. 1.2.

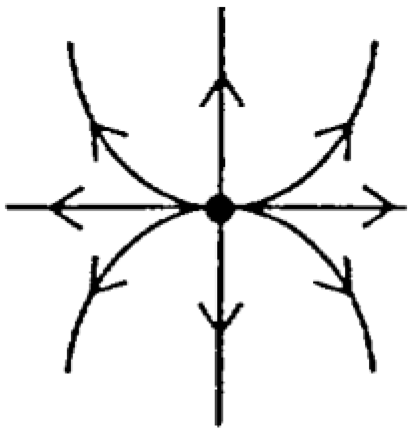

Figure 1.2: Phase portrait

(c) $\mathrm{x(t) = x_0e^t, y(t) = y_0e^{34}}$, and solution curves lie on the curves $\mathrm{y} = \left(\mathrm{y}_0/\mathrm{x}_0^3\right)\mathrm{x}^3$.

(d) $\dot{x} = -y, \dot{y} = x$ can be written as $\dot{y} = \dot{x} = -y$ or $\dot{y} + y = 0$ which has the general solution $y(t) = c_1 \cos t + c_2 \sin t$; thus, $x(t) = \dot{y}(t) = -c_1 \sin t + c_2 \cos t$; or in terms of the initial conditions $x(t) = x_0 \cos t - y_0 \sin t$ and $y(t) = x_0 \sin t + y_0 \cos t$. It follows that for all $t \in R$, $x^2(t) + y^2(t) = x_0^2 + y_0^2$ and solution curves lie on these circles. Cf. Figure 1.3.

(e) $\mathrm{y}(t) = \mathrm{c}_2\mathrm{e}^{-1}$ and then solving the first-order linear differential equation $\dot{x} + \mathrm{x} = c_2\mathrm{e}^{-t}$ leads to

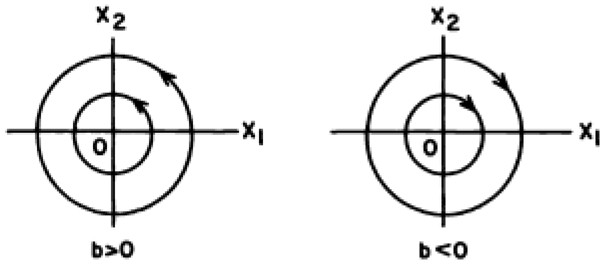

Figure 1.3: Phase portrait

$x(t) = c_1 e^{-1} + c_2 t e^{-1}$ with $c_1 = x_0$ and $c_2 = y_0$. Cf. Figure 1.4.

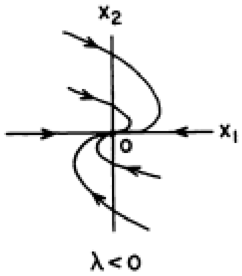

Figure 1.4: Phase portrait

exercise 2

Find the general solution and draw the phase portraits for the following three-dimensional linear systems:

(a)

$$\dot{x}_1 = x_1$$
$$\dot{x}_2 = x_2$$
$$\dot{x}_3 = x_3$$

(b)

$$\dot{x}_1 = -x_1$$
$$\dot{x}_2 = -x_2$$
$$\dot{x}_3 = x_3$$

(c)

$$\dot{x}_1 = -x_2$$
$$\dot{x}_2 = x_1$$
$$\dot{x}_3 = -x_3$$

Hint: In (c), show that the solution curves lie on right circular cylinders perpendicular to the x_1, x_2 plane. Identify the stable and unstable subspaces in (a) and (b). The x_3-axis is the stable subspace in (c) and the x_1, x_2 plane is called the center subspace in (c); cf. Section 1.9 in the course .

solution

2.

(a) $x(t) = x_0 e^t, y(t) = y_0 e^t, z(t) = z_0 e^t$, and $E^a = R^3$.

(b) $x(t) = x_0 e^{-1}, y(t) = y_0 e^{-1}, z(t) = z_0 e^t, E^s = Span\ \left\{(1,0,0)^T, (0,1,0)^T\right\}$, and $E^a = \text{Span}\ \{(0,0,1)\}$. Cf. 1.5 with the arrows reversed.

(c) $x(t) = x_0 \cos t - y_0 \sin t, y(t) = x_0 \sin t + y_0 \cos t, z(t) = z_0 e^{-t}$; solution curves lie on the cylinders $x^2 + y^2 = c^2$ and approach circular periodic orbits in the x, y plane as $t \to \infty$; $E^c = \text{Span}\left\{(1,0,0)^T, (0,1,0)^T\right\}, E^s = \text{Span}\left\{(0,0,1)^T\right\}$.

exersise 3

3. Find the general solution of the linear system

$$\dot{x}_1 = x_1$$
$$\dot{x}_2 = a x_2$$

where a is a constant. Sketch the phase portraits for $a = -1, a = 0$ and $a = 1$ and notice that the qualitative structure of the phase portrait is the same for all $a < 0$ as well as for all $a > 0$, but that it changes at the parameter value $a = 0$ called a bifurcation value.

solution

3. $x(t) = x_0 e^t, y(t) = y_0 e^x$.

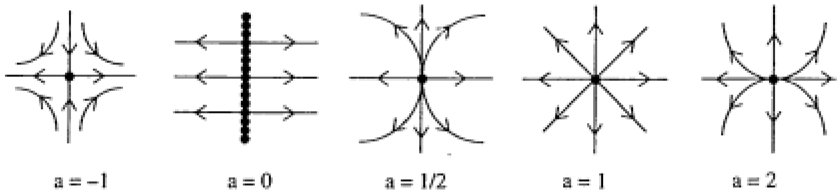

Figure 1.5: Phase portrait

exercise 4

Find the general solution of the linear system (1) when A is the $n \times n$ diagonal matrix $A = \text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]$. What condition on the eigenvalues $\lambda_1, \ldots, \lambda_n$ will guarantee that $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$ for all solutions $\mathbf{x}(t)$ of (1)

solution

$x_1(t) = x_{10} e^{\lambda_{11}}, x_2(t) = x_{20} e^{\lambda_2 t}, \cdots, x_n(t) = x_{20} e^{\lambda_3 t}$. Thus, $x(t) \to 0$ as $t \to \infty$ for all $\mathbf{x}_0 \in \mathbf{R}^{\mathrm{n}}$ if $\lambda_1 < 0, \cdots, \lambda_{\mathrm{n}} < 0$ (and also if $R_e\left(\lambda_j\right) < 0$ for j $= 1, 2, \cdots, n$).

exercise 5

What is the relationship between the vector fields defined by

$$\dot{\mathbf{x}} = A\mathbf{x}$$

and

$$\dot{\mathbf{x}} = kA\mathbf{x}$$

where k is a non-zero constant? (Describe this relationship both for k positive and k negative.)

solution

If $k > 0$, the vectors Ax and kAx point in the same direction and they are related by the scale factor k.

If k < 0, the vectors Ax and kAx point in opposite directions and are related by the scale factor $|k|$.

exercise 6

(a) If $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are solutions of the linear system (1), prove that for any constants a and $b, \mathbf{w}(t) = a\mathbf{u}(t) + b\mathbf{v}(t)$ is a solution.

(b) For

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

find solutions $\mathbf{u}(t)$ and $\mathbf{v}(t)$ of $\dot{\mathbf{x}} = A\mathbf{x}$ such that every solution is a linear combination of $\mathbf{u}(t)$ and $\mathbf{v}(t)$.

solution

6. (a) $\dot{w}(t) = a\dot{u}(t) + b\dot{v}(t) = aAu(t) + bAv(t) = A[au(t) + bv(t)] = Aw(t)$ for all $t \in R$.

(b) $\mathbf{u}(t) = (e^t, 0)^T$, $\mathbf{v}(t) = \left(0, e^{-22}\right)^T$ and the general solution of $\dot{\mathbf{x}} = A\mathbf{x}$ is given by $\mathbf{x}(t) = x_0\mathbf{u}(t) + y_0\mathbf{v}(t)$.

BIBLIOGRAPHY

[1] P. Auger, C. Lett et J. C. Poggiale. *Modélisation mathématique en écologie : Cours et exercices corrigés*. Dunod, Paris, 2010.

[2] G. C. Layek. *An introduction to dynamical systems and chaos*. Springer, India, 2015.

[3] J. L. Pac. *Systèmes dynamiques : Cours et exercices corrigés*. Dunod, Paris, 2016.

[4] H. O. Peitgen, H. Jürgens et D. Saupe. *Chaos and Fractals*. Springer-Verlag, New York,Inc , 2004.

[5] L. Perko. *Differential Equations and Dynamical Systems*. Springer, 2006.

[6] S. H. Strogatz, Nonlinear dynamics and chaos : *With Applications to Physics, Biology, Chemistry, and Engineering*. Perseus Books Publishing, L.L.C, 1994.

[7] T. Vialar et A. Goergen. *Complex and Chaotic Nonlinear Dynamics*. Springer-Verlag Berlin, Heidelberg, 2009.

[8] Morris W. Hirsch, Stephen Smale, and Robert L. Devaney. Differential equations, dynamical systems, and an introduction to chaos. Elsevier/Academic Press, Amsterdam, third edition, 2013