People's Democratic Republic of Algeria

Ministry of Higher Education and scientific Research

University Center abdelhafid boussouf (Alger)

Institute of mathematics and computer sciences

Department of mathematics

Course Of Introduction To Dynamics Systems.

Master 1 (first year) fundamental and applied mathematics

The first semester

By: ROUIBAH KHAOULA

University year: 2023 – 2024

CONTENTS

INTRODUCTION

This course covers those topics necessary for a clear understanding of the qualitative theory of ordinary differential equations and the concept of a dynamical system. It is written for ferst yers master students.

It begins with a study of linear systems of ordinary differential equations, a topic already familiar to the student who has completed a first course in differential equations. An efficient method for solving any linear system of ordinary differential equations is presented in Chapter 1. The major part of this course is devoted to a study of nonlinear systems of ordinary differential equations and dynamical systems. Since most nonlinear differential equations cannot be solved, this course focuses on the qualitative or geometrical theory of nonlinear systems of differential equations originated by Henri Poincarc in his work on differential equations at the end of the nineteenth century as well as on the functional properties inherent in the solution set of a system of nonlinear differential equations embodied in the more recent concept of a dynamical system. Our primary goal is to describe the qualitative behavior of the solution set of a given system of differential equations including the invariant sets and limiting behavior of the dynamical system or flow defined by the system of differential equations. In order to achieve this goal, it is first necessary to develop the local theory for nonlinear systems. This is done in Chapter 2 which includes the fundamental local existence-uniqueness theorem, the Hartman-Grobman Theorem and the Stable Manifold Theorem. These latter two theorems establish that the qualitative behavior of the solution set of a nonlinear system of ordinary differential equations near an equilibrium point is typically the same as the qualitative behavior of the solution set of the corresponding linearized system near the equilibrium point.

CHAPTER 1

CONTINUOUS DYNAMICAL SYSTEMS

1.1 Dynamical Systems

Dynamics is primarily the study of the time-evolutionary process and the corresponding system of equations is known as dynamical system. Generally, a system of n first-order differential equations in the space $\mathbb{R}^*$ is called a dynamical system of dimension n which determines the time behavior of evolutionary process. Evolutionary processes may possess the properties of determinacy/non-determinacy, finite/infinite dimensionality, and differentiability. A process is called deterministic if its entire future course and its entire past are uniquely determined by its state at the present time. Otherwise, the process is called nondeterministic. However, the process may be semi-deterministic (determined, but not uniquely). In classical mechanics the motion of a system whose future and past are uniquely determined by the initial positions and the initial velocities is an example of a deterministic dynamical system. The evolutionary process may describe, viz. (i) a continuous-time process and (ii) a discrete-time process. The continuous-time process is represented by differential equations, whereas the discrete-time process is by difference equations (or maps). The continuous-time dynamical systems may be described mathematically as follows:

Let $x = x(t) \in \mathbb{R}^n, t \in I \subseteq \mathbb{R}$ be the vector representing the dynamics of a continuous system (continuous-time system). The mathematical representation of the system may be written as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} = f(x, t) \tag{1.1}$$

where $f(x,t)$ is a sufficiently smooth function defined on some subset $U \subset \mathbb{R}^n \times \mathbb{R}$. Schematically, this can be shown as

$$\underset{\text{(state space)}}{\mathbb{R}^n} \times \underset{\text{(time)}}{\mathbb{R}} = \underset{\text{(space of motions)}}{\mathbb{R}^{n+1}}$$

The variable t is usually interpreted as time and the function $f(x,t)$ is generally nonlinear. The time interval may be finite, semi-finite or infinite. On the other hand, the discrete system is related to a discrete map (given only at equally spaced points of time) such that from a point x_0, one can obtain a point x_1 which in turn maps into x_2, and so on. In other words, $x_{n+1} = g(x_n) = g(g(x_{n-1}))$, etc. This is also written in the form $x_{n+1} = g(x_n) = g^2(x_{n-1}) = \cdots$. The discrete system will be discussed in the later course .

If the right-hand side of Eq.1.1 is explicitly time independent then the system is called **autonomous**. The trajectories of such a system do not change in time. On the other hand, if the right-hand side of Eq. 1.1 has explicit dependence on time then the system is called **nonautonomous**.
An n-dimensional nonautonomous system can be converted into autonomous form by introducing a new dependent variable x_{n+1} such that $x_{n+1} = t$. In general, the solution of Eq. 1.1 is difficult or sometimes impossible to obtain when the function $f(x,t)$ is nonlinear, except in some special cases. Examples of autonomous and nonautonomous systems are given below.

(i) Autonomous systems

(a) $\ddot{x} + \alpha\dot{x} + \beta x = 0, \alpha, \beta > 0$. This is a damped linear harmonic oscillator. The parameters α and β are, respectively, the strength of damping and the strength of linear restoring force.

(b) $\ddot{x} + \omega^2 \sin x = 0, \omega = \sqrt{g/L}$.g is the gravitational acceleration, L the string length. This is a simple undamped nonlinear oscillator (pendulum).

(c) $\left.\begin{aligned} \dot{x} &= \alpha x - \beta xy \\ \dot{y} &= -\gamma y + \delta xy \end{aligned}\right\}$. . This is the well-known Lotka-Volterra predator-prey model, where $\alpha, \beta, \gamma, \delta$ are all positive constants.

(d) $\ddot{x} - \mu\left(1 - x^2\right)\dot{x} + \beta x = 0, \mu > 0$. This is the well-known van der Pol oscillator.

(ii) Nonautonomous systems

(a) $\ddot{x} + \alpha\dot{x} + \beta x = f \cos \omega t, \alpha, \beta > 0$. This is an example of linear oscillator with external time-dependent force. f and ω are the amplitude and frequency of driving force, respectively.

(b) $\ddot{x} + \alpha\dot{x} + \omega_0^2 x + \beta x^3 = f \sin \omega t$. This is a Duffing nonlinear oscillator with cubic restoring force. α is the strength of damping, ω_0 is the natural frequency and β is the strength of the nonlinear restoring force.

(c) $\ddot{x} - \mu\left(1 - x^2\right)\dot{x} + \beta x = f \cos \omega t, \mu > 0$. This is a van der Pol nonlinear forced oscillator.

(d) $\ddot{x} - \mu\left(1 - x^2\right)\dot{x} + \omega_0^2 x + \beta x^3 = f \cos \omega x$. This is a Duffing-van der Pol nonlinear forced oscillator.

1.2 Flows

The time-evolutionary process may be described as a flow of a vector field.
Generally, flow is frequently used for discussing the dynamics as a whole rather than the evolution of a system at a particular point. The solution $x(t)$ of a system $\dot{x} = f(x)$ which satisfies $x(t_0) = x_0$ gives the past ($t < t_0$) and future ($t > t_0$) evolutions of the system. Mathematically, the flow is defined by $\phi_t(x) : U \to \mathbb{R}^n$ where $\phi_t(x) = \phi(t, x)$ is a smooth vector function of $x \in U \subseteq \mathbb{R}^n$ and $t \in I \subseteq \mathbb{R}$ satisfying the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_t(x) = f\left(\phi_t(x)\right)$$

for all t such that the solution through x exists and $\phi(0, x) = x$. The flow $\phi_t(x)$ satisfies the following properties:
(a) $\phi_o = I_d$,
(b) $\phi_{t+s} = \phi_t \circ \phi_s$.
Some flows may also satisfy the property (c)

$$\phi(t + s, x) = \phi(t, \phi(s, x)) = \phi(s, \phi(t, x)) = \phi(s + t, x).$$

Flows in $\mathbb{R}$: Consider a one-dimensional autonomous system represented by $\dot{x} = f(x), x \in \mathbb{R}$. We can imagine that a fluid is flowing along the real line with local velocity $f(x)$. This imaginary fluid is called **the phase fluid** and the real line is called the **phase line**.

For solution of the system $\dot{x} = f(x)$ starting from an arbitrary initial position x_0, we place an imaginary particle, called a **phase point**, at x_0 and watch how it moves along with the flow in phase line in varying time t. As time goes on, the phase point (x, t) in the one-dimensional system $\dot{x} = f(x)$ with $x(0) = x_0$ moves along the x-axis according to some function $\phi(t, x_0)$. The function $\phi(t, x_0)$ is called the **trajectory** for a given initial state x_0, and the set $\left\{\phi(t, x_0) \mid t \in I \subseteq \mathbb{R}\right\}$ is the orbit of $x_0 \in \mathbb{R}$. The set of all qualitative trajectories of the system is called phase portrait.
Flows in $\mathbb{R}^2$: Consider a two-dimensional system represented by the following equations $\dot{x} = f(x, y), \dot{y} = g(x, y), (x, y) \in \mathbb{R}^2$. An imaginary fluid particle flows in the plane $\mathbb{R}^2$, known as phase plane of the system. The succession of states given parametrically by $x = x(t), y = y(t)$ trace out a curve through some initial point $P(x(t_0), y(t_0))$ is called a **phase path**.
The set $\left\{\phi(t, x_0) \mid t \in I \subseteq \mathbb{R}\right\}$ is the orbit of x in $\mathbb{R}^2$. There are an infinite number of trajectories that would fill the phase plane when they are plotted. But the qualitative behavior can be determined by plotting a few trajectories with different initial conditions. The phase portrait displays how the qualitative behavior of a system is changing as x and y varies with time t. An orbit is called periodic if $x(t + p) = x(t)$ for some $p > 0$, for all t. The smallest integer p for which the relation is satisfied is called the prime period of the

orbit. Flows in $\mathbb{R}$ cannot have oscillatory or closed path.

Flows in $\mathbb{R}^n$: Let us now define an autonomous system representing n ordinary differential equations as

$$\left.\begin{aligned} \dot{x}_1 &= f_1\left(x_1, x_2, \ldots, x_n\right) \\ \dot{x}_2 &= f_2\left(x_1, x_2, \ldots, x_n\right) \\ &\vdots \\ \dot{x}_n &= f_n\left(x_1, x_2, \ldots, x_n\right) \end{aligned}\right\}$$

which can also be written in symbolic notation as $\dot{x} = f(x)$, where $x = (x_1, x_2, \ldots, x_n)$ and $f = (f_1, f_2, \ldots, f_n)$. The solution of this system with the initial condition $x(0) = x_0$ can be thought as a continuous curve in the phase space $\mathbb{R}^n$ parameterized by time $t \in I \subseteq \mathbb{R}$.

So the set of all states of the evolutionary process is represented by an n-valued vector field in $\mathbb{R}^n$. The solutions of the system with different initial conditions describe a family of phase curves in the phase space, called the phase portrait of the system. The vector field $f(x)$ is everywhere tangent to these curves and their orientation is directed by the direction of the tangent vector of $f(x)$.

1.3 Evolution

Consider a system $\dot{x} = f(x), x \in \mathbb{R}^n$ with initial conditions $x\left(t_0\right) = x_0$. Let $E \subset \mathbb{R}^n$ be an open set and $f \in C^1(E)$. For $x_0 \in E$, let $\phi\left(t, x_0\right)$ be a solution of the above system on the maximum interval of existence $I\left(x_0\right) \subset \mathbb{R}$. The mapping $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\phi_t(x_0) = \phi(t, x_0)$ is known as **evolution operator** of the system.

The linear flow for the system $\dot{x} = Ax$ with $x\left(t_0\right) = x_0$, is defined by $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ and $\phi_t = e^{At}$, the exponential matrix. The mappings ϕ_t for both linear and nonlinear systems satisfy the following properties:

(i) $\phi_0(x) = x$

(ii) $\phi_s\left(\phi_t(x)\right) = \phi_{s+t}(x), \forall s, t \in \mathbb{R}$

(iii) $\phi_t\left(\phi_{-t}(x)\right) = \phi_{-t}\left(\phi_t(x)\right) = x, \forall t \in \mathbb{R}$

In general a dynamical system may be viewed as group of nonlinear / linear operators evolving as $\left\{\phi_t(x), t \in \mathbb{R}, x \in \mathbb{R}^n\right\}$. The following dynamical group properties hold good:

(i) $\phi_t\phi_s \in \left\{\phi_t(x), t \in \mathbb{R}, x \in \mathbb{R}^n\right\}$ (Closure property)

(ii) $\phi_t\left(\phi_s\phi_r\right) = (\phi_t\phi_s)\phi_r$ (Associative property)

(iii) $\phi_0(x) = x, \phi_0$ being the Identity operator.

(iv) $\phi_t\phi_{-t} = \phi_{-t}\phi_t = \phi_0$, where ϕ_{-t} is the Inverse of ϕ_t.

For some cases the flow satisfies the commutative property $\phi_t\phi_s = \phi_s\phi_t$.

1.4 Fixed Points of a System

The notion of fixed point is important in analyzing the local behavior of a system. The fixed point is nothing but a constant or equilibrium or invariant solution of a system. A point is a fixed point of the flow generated by an autonomous system $\dot{x} = f(x), x \in \mathbb{R}^n$ if and only if $\phi(t, x) = x$ for all $t \in \mathbb{R}$. Consequently in continuous system, this gives $\dot{x} = 0 \Rightarrow f(x) = 0$. For nonautonomous systems fixed point can be defined for a fixed time interval. $\tilde{A}$ fixed point is also known as a **critical point** or an **equilibrium point** or a **stationary point**. This point is also called **stagnation point** with respect to the flow ϕ_t in $\mathbb{R}^n$. Flows on line may have no fixed points, only one fixed point, finite number of fixed points, and infinite number of fixed points. For example, the flow $\dot{x} = 5$ (no fixed points), $\dot{x} = x$ (only one fixed point), $\dot{x} = x^2 - 1$ (two fixed points), and $\dot{x} = \sin x$ (infinite number of fixed points).

1.5 Linear Stability Analysis

A fixed point, say x_0 is said to be stable if for a given $\varepsilon > 0$, there exists a $\delta > 0$ depending upon ε such that for all $t \geq t_0$, $\|x(t) - x_0(t)\| < \varepsilon$, whenever $\|x(t_0) - x_0(t_0)\| < \delta$, where $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ denotes the norm of a vector in $\mathbb{R}^n$. Otherwise, the fixed point is called unstable. In linear stability analysis the quadratic and higher order terms in the Taylor series expansion about a fixed point x^* of a system $\dot{x} = f(x), x \in \mathbb{R}$ are neglected due to the smallness of the terms. Consider a small perturbation quantity $\xi(t)$, away from the fixed point x^*, such that $x(t) = x^* + \xi(t)$. We see whether the perturbation grows or decays as time goes on. So we get the perturbation equation as

$$\dot{\xi} = \dot{x} = f(x) = f(x^* + \xi).$$

Taylor series expansion of $f(x^* + \xi)$ gives

$$\dot{\xi} = f(x^*) + \xi f'(x^*) + \frac{\xi^2}{2} f''(x^*) + \cdots$$

According to linear stability analysis, we get

$$\dot{\xi} = \xi f'(x^*) \, [\because f(x^*) = 0]$$

Assuming $f'(x^*) \neq 0$, the perturbation $\xi(t)$ grows exponentially if $f'(x^*) > 0$ and decays exponentially if $f'(x^*) < 0$. Linear theory fails if $f'(x^*) = 0$ and then higher order derivatives must be considered in the neighborhood of fixed point for stability analysis of the system.

Example 1.1 Find the evolution operator ϕ_t for the one-dimensional flow $\dot{x} = -x^2$. Show that ϕ_t forms a dynamical group. Is it a commutative group?

Solution The solutions of the given system are obtained as below:

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = -x^2 \Rightarrow \frac{1}{x} = t + A \Rightarrow x(t) = \frac{1}{t+A}$$

in any interval of $\mathbb{R}$ that does not contain the point $x = 0$, where A is a constant. If we take starting point $x(0) = x_0$, then $A = 1/x_0$ and so we get

$$x(t) = \frac{x_0}{1 + x_0 t}, \quad t \neq -1/x_0.$$

The point $x = 0$ is not included in this solution. But it is the fixed point of the given system, because $\dot{x} = 0 \Leftrightarrow x = 0$. Therefore, $\phi_t(0) = 0$ for all $t \in \mathbb{R}$. So the evolution operator of the system is given as $\phi_t(x) = \frac{x}{1+xt}$ for all $x \in \mathbb{R}$.

The evolution operator ϕ_t is not defined for all $t \in \mathbb{R}$. For example, if $t = -1/x, x \neq 0$, then ϕ_t is undefined. Thus we see that the interval in which ϕ_t is defined is completely dependent on x.

We shall now examine the group properties of the evolution operator ϕ_t below:

(i) $\phi_r\phi_s \in \left\{\phi_t(x), t \in \mathbb{R}, x \in \mathbb{R}\right\} \forall r, s \in \mathbb{R}$ (Closure property)

Now,

$$\begin{aligned} \phi_r(y) &= \frac{y}{1+yr}. \text{ Take } y = \frac{x}{1+xs} \\ &= \frac{x/1+sx}{1+\frac{x}{1+x}} = \frac{x}{1+xs+xr} = \frac{x}{1+x(s+r)} \\ &= \phi_{s+r} \in \left\{\phi_t(x), t \in \mathbb{R}, x \in \mathbb{R}\right\} \end{aligned}$$

(ii) $\phi_t\left(\phi_s\phi_r\right) = \left(\phi_t\phi_s\right)\phi_r$ (Associative property)

$$\text{L.H.S.} = \phi_t\left(\left(\phi_s\phi_r\right)(x)\right) = \phi_t(y) = \frac{y}{1+yt} = \frac{z}{1+zs} = \frac{x}{1+x(r+s)}, y = \phi_s\left(\phi_r(x)\right)$$

$$\left(\text{where } y = \phi_s(z), z = \phi_r(x) = \frac{x}{1+rx}\right)$$

$$\therefore \text{ L.H.S} = \frac{x}{1+x(t+r+s)} = \phi_{r+r+s}(x)$$

$$\text{R.H.S.} = \left(\left(\phi_t\phi_s\right)\phi_r(x)\right)$$

Now,

$$\begin{aligned} \phi_t(y) &= \frac{y}{1+yt}, y = \phi_s(x) = \frac{x}{1+sx} \\ &= \frac{x}{1+x(t+s)} = \phi_{t+s}(x) \\ \phi_{t+s}\left(\phi_r\right)(x) &= \phi_{t+s}(z) = \frac{z}{1+z(t+s)}, z = \phi_r(x) = \frac{x}{1+rx} \\ \phi_{t+s}\left(\phi_r\right)(x) &= \frac{x}{1+x(t+s+r)} = \phi_{t+s+r}(x) \end{aligned}$$

Hence, $\phi_t\left(\phi_s\phi_r\right)(x) = \left(\phi_s\phi_r\right)\phi_t(x), \forall x \in \mathbb{R}$.

(iii) $\phi_0(x) = \frac{x}{1+x.0} = x, \phi_0$ is the identity operator.

$$\begin{aligned}\phi_t\phi_{-t}(x) = \phi_t(y) &= \frac{y}{1+ty}, y = \phi_{-t}(x) = \frac{x}{1-tx}\\ &= \frac{x}{1-tx+tx} = x = \phi_0(x) \quad \left(\phi_{-t} \text{ is the inverse of } \phi_t\right)\end{aligned}$$

Hence the flow evolution operator forms a dynamical group.

(v) $\phi_t\phi_s = \phi_s\phi_t$

Now,

$$\begin{aligned}\left(\phi_t\phi_s\right)(x) = \phi_t(y) &= \frac{y}{1+ty}, y = \phi_s(x) = \frac{x}{1+xs}\\ &= \frac{x}{1+x(t+s)} = \phi_{t+s}(x)\\ \phi_s\phi_t(x) = \phi_s(z) &= \frac{z}{1+sz}, z = \phi_t(x) = \frac{x}{1+tx}\\ &= \frac{x}{1+tx+sx} = \frac{x}{1+(s+t)x} = \phi_{s+t}(x)\end{aligned}$$

So, $\phi_t\phi_s = \phi_s\phi_t$ (Commutative property).

Thus, the evolution operator ϕ_t forms a commutative group.

Example 1.2 Using linear stability analysis determine the stability of the critical points for the following systems

$$\text{(i) } \dot{x} = \sin x, \quad \text{(ii) } \dot{x} = x^2$$

Solution (i) The given system has infinite numbers of critical points. The critical points are $x_n^* = n\pi, n = 0, \pm 1, \pm 2, \ldots$. When n is even, $f'\left(x_n^*\right) = \cos\left(x_n^*\right) = \cos(n\pi) = (-1)^n = 1 > 0$. So, these critical points are unstable. When n is odd, $f'\left(x_n^*\right) = -1 < 0$, and so these critical points are stable.

(ii) The critical point of the system is at $x^* = 0$. Now, $f'\left(x^*\right) = 0$ and $f''\left(x^*\right) = 2 > 0$. Hence, x^* is attracting when $x < 0$ and repelling when $x > 0$. Actually, the critical point is semi-stable in nature.

1.6 Analysis of One-Dimensional Flows

As we know qualitative approach is the combination of analysis and geometry and is a powerful tool for analyzing solution behaviors of a system qualitatively. By drawing trajectories in phase line/plane/space, the behaviors of phase points may be found easily. In qualitative analysis we mainly look for the following solution behaviors:

(i) Local stabilities of fixed points for a system;

(ii) Analyzing the existence of periodic/quasi-periodic solutions, limit cycle, relaxation oscillation, hysteresis, etc.;

(iii) Local and asymptotic solution behaviors of a system;

(iv) Topological features of flows such as bifurcations, catastrophe, topological equivalence, transitiveness, etc.

We shall now analyze a simple one-dimensional system as follows.

Consider a one-dimensional system represented as $\dot{x}(t) = \sin x$ with the initial condition $x(t = 0) = x(0) = x_0$. The characteristic features of the system are (i) it is a one-dimensional system, (ii) nonlinear system (iii) autonomous system, and (iv) its closed-form solution (analytical solution) exists. This is a one-dimensional flow and we analyze the system on the basis of flow. The analytical solution of the system is obtained easily

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sin x \Rightarrow \mathrm{d}t = \mathrm{cosec}(x)\mathrm{d}x$$

Integrating, we get

$$\begin{aligned} t &= \int \mathrm{cosec}(x)\mathrm{d}x \\ &= -\log|\mathrm{cosec}(x) + \cot(x)| + c \end{aligned}$$

where c is an integrating constant. Using the initial condition $x(0) = x_0$, we get the integrating constant c as

$$c = \log|\mathrm{cosec}\,(x_0) + \cot\,(x_0)|\,.$$

Thus the solution of the system is given as

$$t = \log\left|\frac{\mathrm{cosec}\,(x_0) + \cot\,(x_0)}{\mathrm{cosec}(x) + \cot(x)}\right|$$

From this closed-form solution, the behaviors of solutions for any initial conditions are difficult to analyse. Moreover, the asymptotic values of the system are
also difficult to obtain. The qualitative approach can give better dynamical behavior about this simple system.

We consider t as time, x as the position of an imaginary particle moving along the flow in real line and $\dot{x}$ as the velocity of that particle. The differential equation $\dot{x} = \sin x$ represents a vector field on the line. It gives the velocity vector $\dot{x}$ at each position x. The arrows point to the right when $\dot{x} > 0$ and to the left when $\dot{x} < 0$. We shall draw the graph of $\sin x$ versus x in $x\dot{x}$ - plane which gives the flow in the x-axis (see Fig. 1.1).

We may imagine that fluid is flowing steadily along the x-axis with a velocity $\dot{x}$ which varies from place to place, according to equation $\dot{x} = \sin x$. At points $\dot{x} = 0$, there is no flow and such points are called equilibrium points (fixed points). According to the definition of fixed point, the equilibrium points of this system are obtained as $\sin x = 0 \Rightarrow x = n\pi (n = 0, \pm 1, \pm 2, \ldots)$. This simple looking autonomous system has infinite numbers of equilibrium points in $\mathbb{R}$. We can see that there are two kinds of equilibrium points. The equilibrium point where the flow is toward the point is called sink or attractor (neighboring trajectories approach asymptotically to the point as $t \to \infty$). On the other hand, when the flow is away from the point, the point is called source or repellor (neighboring trajectories move away from the point as $t \to \infty$). From the above figure the solid circles represent the sinks that are stable equilibrium points and the open circles are the sources, which are unstable equilibrium points. The names are given because the sinks and sources are common in fluid flow problems. From the geometric approach one can get local stability behavior of the equilibrium points of the system easily and is valid for all time. We shall now re-look the analytical solution of the system. The analytical solution can be expressed as

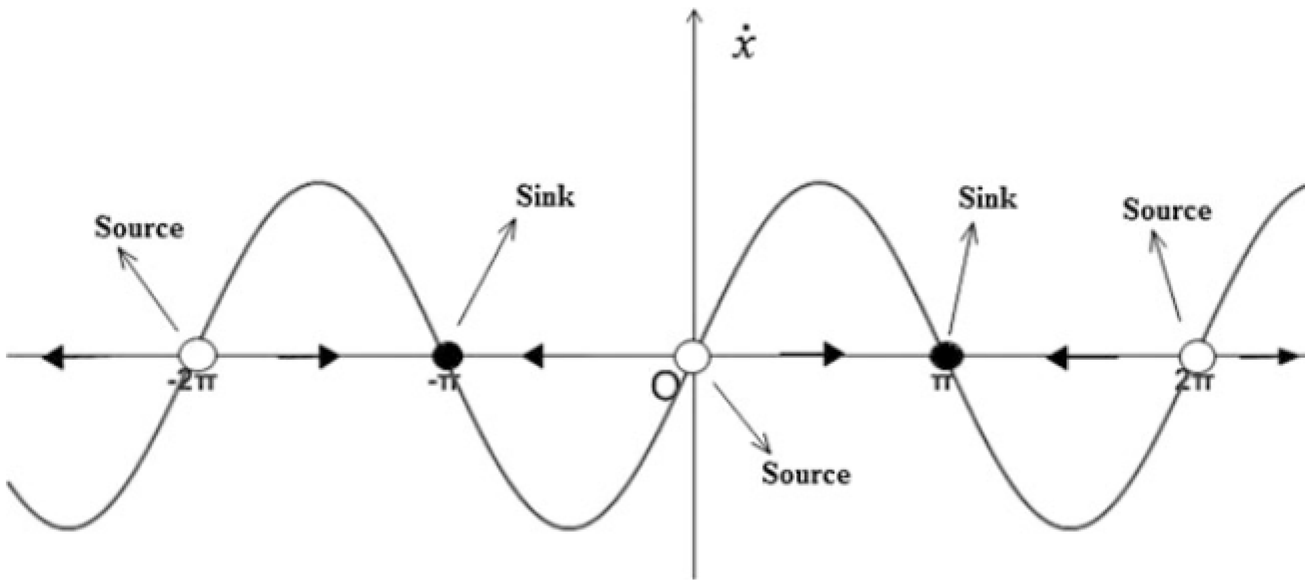

Figure 1.1: Graphical representation of flow generated by $\sin(x)$

Let the initial condition be $x_0 = x(0) = \pi/4$. Then from the above solution we obtain

$$A = \tan(\pi/8) = -1 + \sqrt{2} = 1/(1 + \sqrt{2})$$

So the solution is expressed as

$$x(t) = 2\tan^{-1}\left(\frac{e^t}{1 + \sqrt{2}}\right)$$

We see that the solution $x(t) \to \pi$ and $t \to \infty$.

Without using analytical solution for this particular initial condition the same result can be found by drawing the graph of x versus t. So the solution's behavior at any initial condition can be obtained easily by geometric approach. This simple one-dimensional system also has an interesting application. For a slow motion of a spring immersed in a highly viscous fluid such as grease or viscoelastic fluid (the combined effects of fluid viscosity and elasticity for example, synovial fluid in the joints of human bones), the viscous damping force is very strong compared to the inertia of motion. So one can neglect acceleration term (that is, inertia) and the spring-mass system may be governed by the equation $\alpha\dot{x} = \sin x$, where $\alpha > 0$ (string constant) is a real number and the dynamics can be obtained using this approach for different values of α (see the book Strogatz [5] for more physical examples and explanations).

We shall discuss a few worked out examples presented below.

Example 1.5 With the help of flow concept discuss the local stability of the fixed points of $\dot{x} = f(x) = \left(x^2 - 1\right)$.

Solution The fixed points of the given autonomous system are given by setting $f(x) = 0$. This gives $x = \pm 1$. So the fixed points of the system are 1 and -1 . For the local stability of the system about these fixed points we plot the graph of the function $f(x)$ and then sketch the vector field. The flow is to the right direction, indicated by the symbol $' \rightarrow '$, where the velocity $\dot{x} > 0$, that is, where $\left(x^2 - 1\right) > 0$ and to the left direction, indicated by the symbol $' \leftarrow '$, where $\dot{x} < 0$, that is, $\left(x^2 - 1\right) < 0$. We also use solid circles to represent stable fixed points and open circles for unstable fixed points.

In Fig.1.2 the arrows indicate the flow of the system. From the figure, we see that the fixed point $x = 1$ is unstable, since it acts as a source point and the fixed point $x = -1$ is stable, since it acts as a sink point.

Example 1.6 Discuss the stability character of the fixed points for the system $\dot{x} = x(1 - x)$ using the concept of flow.

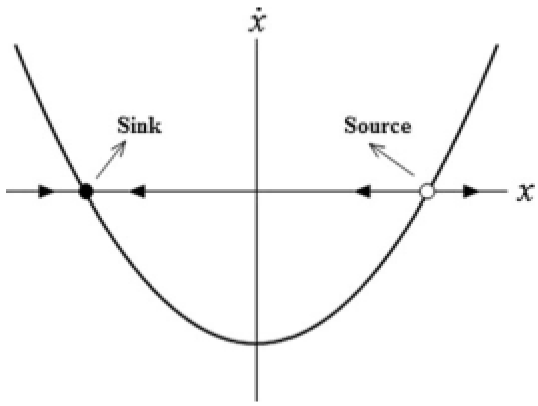

Figure 1.2: Graphical representation of $f(x) = \left(x^2 - 1\right)$

Solution Here $f(x) = x(1 - x)$. Then for the fixed points, we have

$$f(x) = 0 \Rightarrow x(1-x) = 0 \Rightarrow x = 0, 1$$

Thus the fixed points are 0 and 1 . To discuss the stability of these fixed points we plot the system (x versus $\dot{x}$) and then sketch the vector field. The flow is to the right direction, indicated by the symbol $' \rightarrow '$, when the velocity $\dot{x} > 0$, and to the left direction, indicated by the symbol $' \leftarrow '$, when $\dot{x} < 0$. We also use solid circle to represent stable fixed point and open circle to represent unstable fixed point.

From Fig. 1.3 we see that the fixed point $x = 1$ is stable whereas the fixed point $x = 0$ is unstable.

Example Find the fixed points and analyze the local stability of the following systems (i) $\dot{x} = x + x^3$ (ii) $\dot{x} = x - x^3$ (iii) $\dot{x} = -x - x^3$

Solution (i) Here $f(x) = x + x^3$. Then for fixed points $f(x) = 0 \Rightarrow x + x^3 = 0 \Rightarrow x = 0$, as $x \in \mathbb{R}$. So, 0 is the only fixed point of the system. We now see that when $x > 0, \dot{x} > 0$ and when $x < 0, \dot{x} < 0$. Hence the fixed point $x = 0$ is unstable. The graphical representation of the flow generated by the system is displayed in Fig.1.4.

(ii) Here $f(x) = x - x^3$. Then $f(x) = 0 \Rightarrow x - x^3 = 0 \Rightarrow x = 0, 1, -1$. Therefore, the fixed points of the system are $0, 1, -1$. We now see that

(a) when $x < -1$, then $\dot{x} > 0$

(b) when $-1 < x < 0, \dot{x} < 0$

(c) when $0 < x < 1, \dot{x} > 0$

(d) when $x > 1$, then $\dot{x} < 0$.

This shows that the fixed points 1 and -1 are stable whereas the fixed point 0 is unstable (Fig. 1.5).

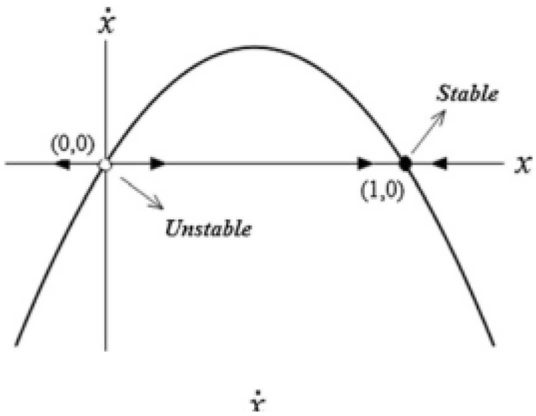

Figure 1.3: Pictorial representation of $f(x) = x(1-x)$

(iii) Here $f(x) = -x - x^3$. Then $f(x) = 0 \Rightarrow -x - x^3 = 0 \Rightarrow x = 0$, as $x \in \mathbb{R}$. So $x = 0$ is the only fixed point of the system. We now see that $\dot{x} > 0$ when $x < 0$ and $\dot{x} < 0$ when $x > 0$. This shows that the fixed point $x = 0$ is stable. The graphical representation of the flow generated by the system is displayed in

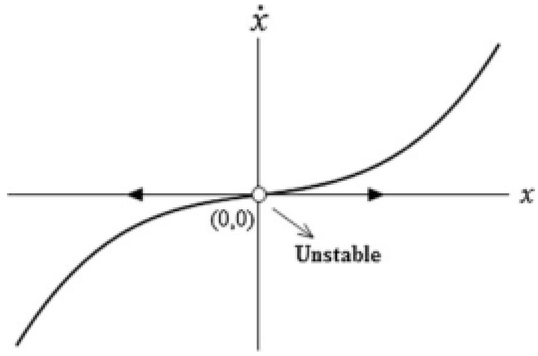

Figure 1.4: Graphical representation of $f(x) = \left(x + x^3\right)$

Fig. 1.6.

Example 1.8 Determine the equilibrium points and sketch the phase diagram in the neighborhood of the equilibrium points for the system represented as $\dot{x} + x\,\mathrm{sgn}(x) = 0$.

Solution Given system is $\dot{x} + x\,\mathrm{sgn}(x) = 0$, that is, $\dot{x} = -x\,\mathrm{sgn}(x)$, where the function $\mathrm{sgn}(x)$ is defined as

$$\mathrm{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

For equilibrium points, we have

$$\dot{x} = 0 \Rightarrow x\,\mathrm{sgn}\,x = 0 \Rightarrow x = 0$$

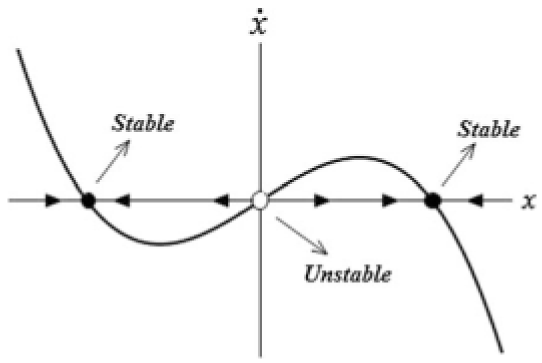

Figure 1.5: raphical representation of the flow generated by $\left(x - x^3\right)$

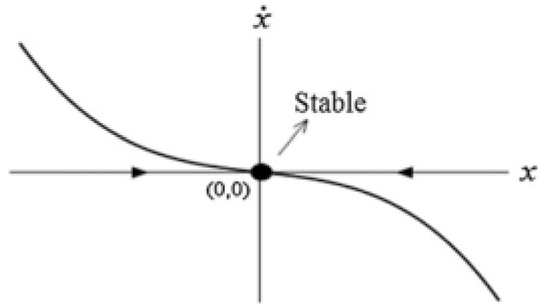

Figure 1.6: Graphical representation of $f(x) = \left(-x - x^3\right)$ versus x

This shows that the system has only one equilibrium point at $x = 0$. In flow analysis we see that the velocity $\dot{x} < 0$ for all $x \neq 0$. The flow is to the right direction, when $\dot{x} > 0$, in the negative x-axis and to the left direction, when $\dot{x} < 0$, in the positive x-axis. This is shown in the phase diagram depicted in Fig. 1.7, which shows that the fixed point origin is semi-stable.

1.7 Conservative and Dissipative Dynamical Systems

The dichotomy of dynamical systems in conservative versus dissipative is very important. They have some fundamental properties. Particularly, conservative systems are the integral part of Hamiltonian mechanics. We give here only the formal definitions of conservative and dissipative systems. Consider an autonomous system represented as

$$\dot{x} = f(x), x \in \mathbb{R}^n. \tag{1.2}$$

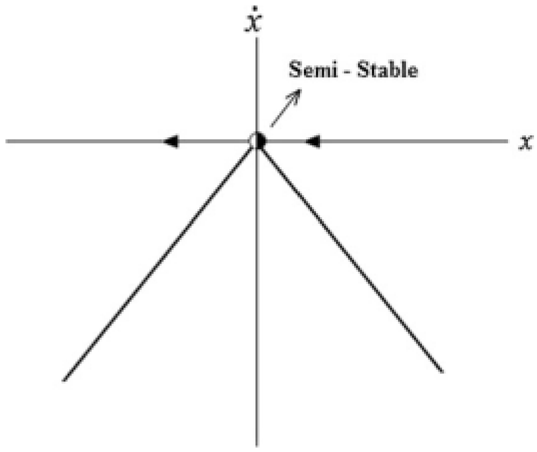

Figure 1.7: Graphical representation of the flow $\dot{x} = -x \operatorname{sgn} x$

The conservative and dissipative systems are defined with respect to the divergence of the corresponding vector field, which in turn refers to the conservation of volume or area in their state space or phase plane, respectively as follows:

A system is said to be **conservative** if the divergence of its vector field is zero. On the other hand, it is said to be **dissipative** if its vector field has negative divergence. The phase volume in a conservative system is constant under the flow while for a dissipative system the phase volume occupied by the system is gradually decreased as the time t increases and shrinks to zero as $t \to \infty$. When divergence of vector field is positive, the phase volume is gradually expanding. We shall discuss it in a later chapter. We state a lemma below which gives the change of volume in a phase space for an autonomous system.

Sometimes, it is useful to find the evolution of volume in the phase space of a system $\dot{x} = f(x), x \in \mathbb{R}^n$. The system generates a flow $\phi(t, x)$. We give Liouville's theorem which describes the time evolution of volume under the flow $\phi(t, x)$. Before this we now give the following lemma.

Lemma Consider an autonomous vector field $\dot{x} = f(x), x \in \mathbb{R}^n$ and generates a flow $\phi_t(x)$. Let D_0 be a domain in $\mathbb{R}^n$ and $\phi_t\,(D_0)$ be its evolution under the flow. If $V(t)$ is the volume of D_t, then the time rate of change of volume is given as $\left.\frac{\mathrm{d}v}{\mathrm{d}t}\right|_{t=0} = \int_{D_0} \nabla \cdot f dx$.

Proof The volume $V(t)$ can be expressed in the following form using the definition of the Jacobian of a transformation as

$$V(t) = \int_{D_0} \left| \frac{\partial \phi(t, x)}{\partial x} \right| dx$$

Expanding Taylor series of $\phi(t,x)$ in the neighborhood of $t=0$, we get

$$\phi(t,x) = x + f(x)t + O\left(t^2\right)$$
$$\Rightarrow \frac{\partial\phi}{\partial x} = I + \frac{\partial f}{\partial x}t + O\left(t^2\right)$$

Here I is the $n \times n$ identity matrix and

$$\left|\frac{\partial\phi}{\partial x}\right| = \left|I + \frac{\partial f}{\partial x}t\right| + O\left(t^2\right)$$
$$= 1 + \text{trace}\left(\frac{\partial f}{\partial x}\right)t + O\left(t^2\right) \text{ [Using expansion of the determinant]}$$

Now, trace $\left(\frac{\partial f}{\partial x}\right) = \nabla \cdot f$, so we have

$$V(t) = V(0) + \int_{D_0} t\nabla \cdot f dx + O\left(t^2\right)$$

This gives $\frac{\mathrm{d}v}{\mathrm{d}t}\Big|_{t=0} = \int_{D_0} \nabla \cdot f dx$.

Theorem (Liouville's Theorem) Suppose $\nabla \cdot f = 0$ for a vector field f. Then for any region $D_0 \subseteq \mathbb{R}^n$, the volume $V(t)$ generated by the flow $\phi(t,x)$ is $V(t) = V(0)$, $V(0)$ being the volume of D_0.

Proof Suppose the divergence of the vector field f is everywhere constant, that is, $\nabla \cdot f = c$. For arbitrary time t_0 the evolution equation for the volume is given as $\dot{V} = cV$. This gives $V(t) = V(0)e^{ct}$. When the vector field is divergence free, that is, $c = 0$, we get the result $\dot{V} = 0 \Rightarrow V(t) = V(0) = \text{constant}$. Thus we can say that the flow generated by a time independent system is volume preserving.

Examples of conservative and dissipative systems are presented below.

(a) Consider a linear and undamped pendulum represented as $\ddot{x} + x = 0$. This is an example of a conservative system. Setting $\dot{x} = y$, we can write it as a system of equations

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x \end{aligned}\right\}$$

The system may also be written in the compact form $\dot{x} = fx)$, where $f(x) = \binom{y}{-x}$. The divergence of the vector field f is given by $\nabla \cdot f = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(-x) = 0$. According to the definition, the system is conservative and the area occupied in the xy-phase plane is constant.

(b) The damped pendulum governed by $\ddot{x} + \alpha\dot{x} + \beta x = 0, \alpha, \beta > 0$ is an example of a dissipative system. Setting $\dot{x} = y$, we can write the system as

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -xy - \beta x \end{aligned}\right\}$$

The vector field is then expressed as $f(x) = \binom{y}{-\alpha y - \beta x}$. Now, $\vec{\nabla} \cdot f = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(-\alpha y - \beta x) = -\alpha < 0$, since $\alpha > 0$.

This shows that the divergence of the vector field is negative.

So the system is dissipative in nature and the area in the phase plane is decreasing as time goes on. This is the simplest linear oscillator with linear damping. It describes a spring-mass system with a damper in parallel. The spring force is proportional to the extension x of the spring and the damping or frictional force is proportional to the velocity $\dot{x}$. The two constants α and β are related to the stiffness of the spring and the degrees of friction in the damper, respectively. According to the above lemma, the change in phase area is given by

$$A(t) = cA(0)e^{-\alpha}, \alpha > 0 \text{ as } t \to \infty, c \text{ being a constant.}$$

Example Find the phase volume element for the systems **(i)** $\dot{x} = -x$, **(ii)** $\dot{x} = ax - bxy, \dot{y} = bxy - cy$ where $x, y \geq 0$ and a, b, c are positive constants.

Solution (i) The flow of the system $\dot{x} = -x$ is attracted toward the point $x = 0$. The time rate of change of volume element $V(t)$ under the flow is given as

$$\left.\frac{dV}{dt}\right|_{r=0} = -\int_{D(0)} dx = -V(0)$$
$$\text{or, } V(t) = V(0)e^{-t} \to 0 \text{ as } t \to \infty.$$

Hence the phase volume element $V(t)$ shrinks exponentially.

(ii) The given system is a Lotka-Volterra predator-prey population model. The rate of change in phase area $A(t)$ is given as

$$\begin{aligned} \frac{dV}{dt} &= -\int \vec{\nabla} : f \, dx \, dy \\ &= -\int (a - c - by + bx) dx \, dy \end{aligned}$$

This shows that a phase area periodically shrinks and expands.

1.8 Some Definitions

In this section we give some important preliminary definitions relating to flow of a system. The definitions given here are elaborately discussed in the later chapters for higher dimensional systems.

Invariant set A set $D \subset \mathbb{R}^n$ is said to be an invariant set under the flow ϕ_t if for any point $p \in D, \phi_t(p) \in D$ for all $t \in \mathbb{R}$. The set D is said to be positively invariant if $\phi_r(p) \in D$ for $t \geq 0$. Trajectories starting in an invariant set remain in the set for all times. An interval is called trapping if it is mapped into itself and is said to be invariant if it is mapped exactly onto itself. Moreover, if a bounded interval is trapping, then all of its trajectories are trapped inside and must converge to a closed, invariant, and bounded limit set. Basically these limit sets are the attractors of a system. So the periodic orbits are examples of invariant sets. We now define two limiting topological concepts which are relevant to the orbits of dynamical systems.

Limit points (ω - and α-limit points)

The asymptotic behavior of a trajectory may be related with limit points/sets or cycles and are termed as ω - and α-limit points/sets or cycles. We now give the definitions.

A point $p \in \mathbb{R}^n$ is called an ω-(resp. a α-) limit point if there exists a sequence $\{t_i\}$ with $t_i \to \infty$ (resp. $t_i \to -\infty$) such that $\phi(t_i, x) \to p$ as $i \to \infty$. The ω-limit set(cycle) is denoted by $\Lambda(x)$ and is defined as

$$\Lambda(x) = \left\{x \in \mathbb{R}^n \mid \exists \{t_i\} \text{ with } t_i \to \infty \text{ and } \phi(t_i, x) \to p \text{ as } i \to \infty\right\}.$$

Similarly, the α-limit set (cycle), $\mu(x)$, is defined as

$$\mu(x) = \left\{x \in \mathbb{R}^n \mid \exists \{t_i\} \text{ with } t_i \to -\infty \text{ and } \phi(t_i, x) \to p \text{ as } i \to \infty\right\}.$$

For example, consider a flow $\phi(t, x)$ on $\mathbb{R}^2$ generated by the system $\dot{r} = cr(1-r), \dot{\theta} = 1, c$ being a positive constant. For $x \neq 0$, let p be any point of the closed orbit C and take $\{t_i\}_{i=1}^{\infty}$ to be the sequence of $t > 0$. The trajectory through x crosses the radial line through p. So, $t_i \to \infty$ as $i \to \infty$ and
$\phi(t_i, x) \to p$ as $i \to \infty$. If x lies in the closed orbit C, then $\phi(t_i, x) = p$ for each i. Hence every point of C is a ω-limit point of x and so $\Lambda(x) = C$ for every $x \neq 0$. When $|x| \leq 1$, the sequence $\{t_i\}_{i=1}^{\infty}$ with $t < 0$ gives the α-limit set $\mu(x) = \begin{cases} \{0\} & \text{for } |x| < 1 \\ \text{closed orbit} & \text{for } |x| = 1 \end{cases}$.

When $|x| > 1$, there is no sequence $\{t_i\}_{i=1}^{\infty}$, with $t_i \to \infty$ as $i \to \infty$, such that $\phi(t_i, x)$ exists as $i \to \infty$. So, $\mu(x)$ is empty when $|x| > 1$. The closed orbit C is called a limit cycle of the system.

The trajectory of a system through a point x is the set $\gamma(x) = \bigcup_{t \in \mathbb{R}} \phi(t, x)$ and the corresponding positive semi-trajectory $\gamma^+(x)$ and negative semi-trajectory $\gamma^-(x)$ are defined as follows:

$$\gamma^+(x) = \bigcup_{t \geq 0} \phi(t, x) \text{ and } \gamma^-(x) = \bigcup_{t \leq 0} \phi(t, x).$$

Attracting set A closed invariant set $D \subset \mathbb{R}^n$ for a flow ϕ_r is said to be an attracting set if there exists some neighborhood U in D such that $\forall t \geq 0, \phi(t, U) \subset U$ and $\bigcap_{t>0} \phi(t, U) = D$.

Absorbing set A positive invariant compact subset $B \subseteq \mathbb{R}^n$ is said to be an absorbing set if there exists a bounded subset C of $\mathbb{R}^n$ with $C \supset B$ such that $t_C > 0 \Rightarrow \phi(t, C) \subset B \forall t \geq t_C$ (see the book by Wiggins [7] for details).

Trapping zone An open set U in an invariant set $D \subset \mathbb{R}^n$ in an attracting set for a flow generated by a system is called a trapping zone. Let a set A be closed and invariant. The set A is said to be stable if and only if every neighborhood of A contains a neighborhood U of A which is trapping.

Basin of attraction The domain (called as basin of attraction) of an attracting set D is defined as $\bigcup_{t \leq 0} \phi(t, U)$ where U is any open set in $D \subset \mathbb{R}^n$.

Consider the one-dimensional system $\dot{x} = -x^4 \sin(\pi/x)$. It has countably infinite set of fixed points at $x^* = 0, \pm\frac{1}{n}, n = 1, 2, 3, \ldots$. Now,

$$f(x) = -x^4 \sin(\pi/x) \Rightarrow f'(x) = -4x^3 \sin(\pi/x) + \pi x^2 \cos(\pi/x)$$
$$\Rightarrow f'(x^*)\Big|_{x-\pm\frac{1}{n}} = \frac{\pi}{n^2} \cos(n\pi) = \frac{\pi}{n^2}(-1)^n.$$

The fixed point $x^* = 0$ is neither attracting nor repelling. The interval $[-1, 1]$ is an attracting set of the given system. The fixed points $x^* = \pm\frac{1}{2n}, n = 1, 2, \ldots$ are repelling while the fixed points $x^* = \pm\frac{1}{(2\pi-1)}, n = 1, 2, \ldots$ are attracting.

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