People's Democratic Republic of Algeria

Ministry of Higher Education and scientific Research

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Institute of mathematics and computer sciences

Department of mathematics


# Serie of exercise
# Introduction To Dynamics Systems .

## Master 1 (first year) fundamental and applied mathematics

## The first semester

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# CHAPTER 1

# LINEAR SYSTEMS

## exercise 7

Find the eigenvalues and eigenvectors of the matrix $A$ and show that $B = P^{-1}AP$ is a diagonal matrix. Solve the linear system $\dot{\mathbf{y}} = B\mathbf{y}$ and then solve $\dot{\mathbf{x}} = A\mathbf{x}$ using the above corollary. And then sketch the phase portraits in both the $\mathbf{x}$ plane and $\mathbf{y}$ plane.

(a) $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$.

## solution

(a) $\lambda_1 = 2, \lambda_2 = 4, v_1 = (1, -1)^T, \mathbf{v}_2 = (1, 1)^T, \mathrm{P} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \mathrm{P}^{-1} = 1/2 \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

and $\mathrm{B} = \mathrm{P}^{-1}\mathrm{AP} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$

$$y(t) = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{4t} \end{bmatrix} y_0.$$

$$x(t) = P \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{4t} \end{bmatrix} P^{-1} x_0 = 1/2 \begin{bmatrix} e^{2t} + e^{4t} & e^{4t} - e^{2t} \\ e^{4t} - e^{2t} & e^{4t} + e^{2t} \end{bmatrix} x_0.$$

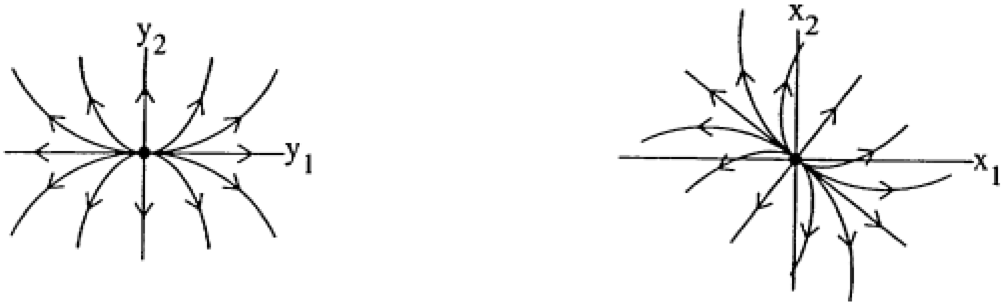



Figure 1.1: Phase portrait

(b) $\lambda_1 = 4, \lambda_2 = -2, v_1 = (1,1)^{\mathrm{T}}, v_2 = (1,-1)^{\mathrm{T}}$,

$$y(t) = \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{bmatrix} y_0,$$

$$\mathbf{x}(t) = 1/2 \begin{bmatrix} e^{4t} + e^{-2t} & e^{4t} - e^{-2t} \\ e^{4t} - e^{-2t} & e^{4t} + e^{-2t} \end{bmatrix} x_0.$$

(c) $\lambda_1 = -2, \lambda_2 = 0, v_1 = (1,-1)^{\mathrm{T}}, v_2 = (1,1)^{\mathrm{T}}$,

$$y(t) = \begin{bmatrix} e^{-2t} & 0 \\ 0 & 1 \end{bmatrix} y_0,$$

$$\mathbf{x}(t) = 1/2 \begin{bmatrix} e^{-2t} + 1 & 1 - e^{-2t} \\ 1 - e^{-2t} & 1 + e^{-2t} \end{bmatrix} \mathbf{x}_0$$

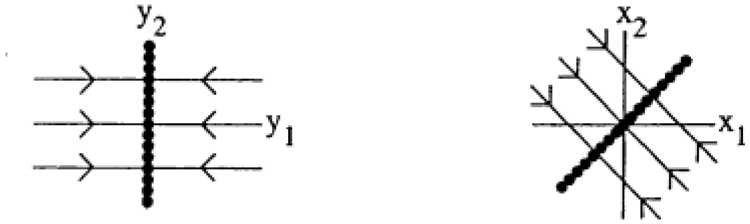



Figure 1.2: Phase portrait

## exercise 8

2. Find the eigenvalues and eigenvectors for the matrix $A$, solve the linear system $\dot{\mathbf{x}} = A\mathbf{x}$, determine the stable and unstable subspaces for the linear system, and sketch the phase portrait for

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix} \mathbf{x}$$

## solution

$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = -1, v_1 = (2, -2, 1)^{\mathrm{T}}, \mathbf{v}_2 = (0, 1, 0)^{\mathrm{T}}, \mathbf{v}_3 = (0, 0, 1)^{\mathrm{T}}$

$$\mathbf{y}(t) = \begin{bmatrix} e^t & & \\ & e^{2t} & \\ & & e^{-t} \end{bmatrix} \mathbf{y}_0, \quad \mathbf{x}(t) = 1/2 \begin{bmatrix} 2e^t & 0 & 0 \\ 2\left(e^{2t} - e^t\right) & 2e^{2t} & 0 \\ e^t - e^{-t} & 0 & 2e^{-t} \end{bmatrix} \mathbf{x}_0,$$

$$E^s = \operatorname{Span}\{\mathbf{v}_3\}, E^u = \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2\}.$$

## exercise 9

Write the following linear differential equations with constant coefficients in the form of the linear system and solve:

(a) $\ddot{x} + \dot{x} - 2x = 0$

(b) $\ddot{x} + x = 0$

(c) $\dddot{x} - 2\ddot{x} - \dot{x} + 2x = 0$

Hint: Let $x_1 = x, x_2 = \dot{x}_1$, etc.

## solution

$\dot{\mathbf{x}} = \mathrm{A}\mathrm{x}$

(a) $\mathrm{A} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(c) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix}$.

## exercise 10

solve the initial value problem

$$\dot{\mathbf{x}} = A\mathbf{x}$$

$$\mathbf{x}(0) = \mathbf{x}_0$$

(a) with $A$ given by 1 (a) above and $x_0 = (1, 2)^T$ (b) with $A$ given in problem 2 above and $\mathbf{x}_0 = (1, 2, 3)^T$.

## solution

(a) $x(t) = 1/2\left(3e^{41} - e^{21}, 3e^{41} + e^{21}\right)$
(b) $x(t) = 1/2\left(2e^{t}, 6e^{2t} - 2e^{t}, e^{t} + 5e^{-1}\right)$.

## exercise 11

Let the $n \times n$ matrix $A$ have real, distinct eigenvalues. Find conditions on the eigenvalues that are necessary and sufficient for $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$ where $\mathbf{x}(t)$ is any solution of $\dot{\mathbf{x}} = A\mathbf{x}$.

## solution

$\lim_{t\to\infty} x(t) = 0$ iff $\lambda_j < 0$ for $j = 1, 2, 3, \cdots, n$.

## exercise 12

Let the $n \times n$ matrix $A$ have real, distinct eigenvalues. Let $\phi(t, \mathbf{x}_0)$ be the solution of the initial value problem

$$\dot{\mathbf{x}} = A\mathbf{x}$$
$$\mathbf{x}(0) = \mathbf{x}_0.$$

Show that for each fixed $t \in \mathbf{R}$,

$$\lim_{\mathbf{y}_0 \to \mathbf{x}_0} \phi(t, \mathbf{y}_0) = \phi(t, \mathbf{x}_0).$$

This shows that the solution $\phi(t, \mathbf{x}_0)$ is a continuous function of the initial condition.

## solution

$\phi(\mathrm{t}, \mathrm{x}_0) = \mathrm{P}\begin{bmatrix} \mathrm{e}^{\lambda_1 1} & & \\ & \ddots & \\ & & \mathrm{e}^{\lambda_0 1} \end{bmatrix}\mathrm{P}^{-1}x_0$ and $\lim_{\lambda_0 \to x_0} \phi(\mathrm{t}, \mathrm{y}_0) = \phi(\mathrm{t}, \mathrm{x}_0)$ since $\lim_{y_0 \to x_0} y_0 = x_0$ according to the definition of the limit.

## exercise 13

Let the $2 \times 2$ matrix $A$ have real, distinct eigenvalues $\lambda$ and $\mu$. Suppose that an eigenvector of $\lambda$ is $(1, 0)^T$ and an eigenvector of $\mu$ is $(-1, 1)^T$. Sketch the phase portraits of $\dot{\mathbf{x}} = A\mathbf{x}$ for the following cases:

(a) $0 < \lambda < \mu$

(b) $0 < \mu < \lambda$

(c) $\lambda < \mu < 0$

(d) $\lambda < 0 < \mu$

(e) $\mu < 0 < \lambda$

(f) $\lambda = 0, \mu > 0$.

## solution

## exercise 14

Compute the operator norm of the linear transformation defined by the following matrices:

(a) $\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$

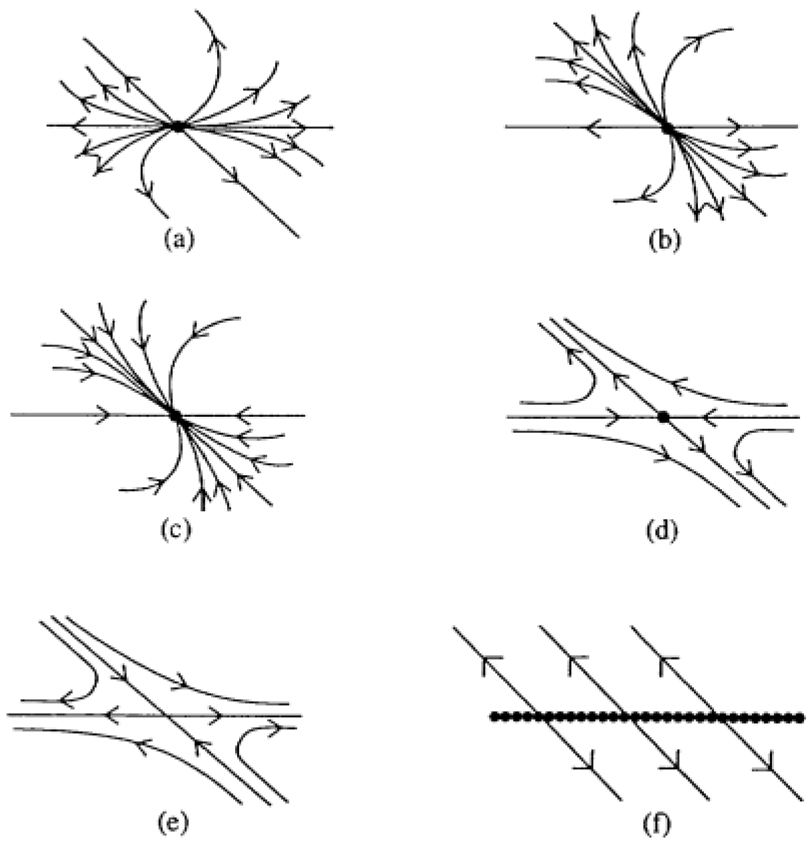



Figure 1.3: Phase portrait

(b) $\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$.

Hint: In (c) maximize $|A\mathbf{x}|^2 = 26x_1^2 + 10x_1x_2 + x_2^2$ subject to the constraint $x_1^2 + x_2^2 = 1$ and use the result of exercise 2; or use the fact that $\|A\| = \left[ \text{Max eigenvalue of } A^T A \right]^{1/2}$. Follow this same hint for (b).

## solution

**(a)** $||A|| = \max_{||\leq 1} |Ax| = \max_{w\leq 1} \sqrt{4x^2 + 9y^2} \leq 3|x|$; but for $x = (0,1)^\top, |Ax| = |-3| = 3$; thus, $||A|| = 3$.

**(b)** the hint for (c), we can maximize $|Ax|^2 = x^2 + 4xy + 5y^2$ subject to the constraint $x^2 + y^2 = 1$ to find $x^2 = (2 \pm \sqrt{2})/4$ and $y^2 = 1 - x^2$ which leads to $||A|| = 2.4142136$; or since $A^\top = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ with eigenvalues $3 \pm 2\sqrt{2}$, we have $||A|| = \sqrt{3 + 2\sqrt{2}} = 1 + \sqrt{2}$.

**(c)** We can either maximize $|Ax|^2 = 26x^2 + 10xy + y^2$ subject to the constraint $x^2 + y^2 = 1$; or find the eigenvalues of $AA^T = \begin{bmatrix} 26 & 5 \\ 5 & 1 \end{bmatrix}$ which are $(27 \pm \sqrt{725})/2$; in either case, $||A|| = 5.1925824\cdots$.

## exercise 15

Show that the operator norm of a linear transformation $T$ on $\mathbf{R}^n$ satisfies

$$||T|| = \max_{|\mathbf{x}|=1} |T(\mathbf{x})| = \sup_{\mathbf{x}\neq 0} \frac{|T(\mathbf{x})|}{|\mathbf{x}|}$$

## solution

**2**. By definition, $||T|| = \max_{x|\leq 1} |T(\mathbf{x})|$. Thus, $||T|| \geq \max_{\{x|=1} |T(\mathbf{x})|$. But $\max_{\dot{x}=1} |T(x)| = \sup_{x=0} \frac{|T(x)|}{|x|}$ since if $|x| = a$ and we set $y = x/a$ for $x \neq 0$, then $|y| = |x|/a = 1$ and since $T$ is linear,

$\sup_{x\neq 0} \frac{|T(x)|}{|x|} = \sup_{x\neq 0} \frac{|T(x)|}{a} = \sup_{x\neq 0} \left|T\left(\frac{x}{a}\right)\right| = \max_{|y|=1} |T(y)|\cdot$.

Thus, $||T\big| \leq \sup_{\alpha<x\leq 1} \frac{|T(x)|}{|x|} \leq \sup_{x\neq 0} \frac{|T(x)|}{|x|} = \max_{|x|=1} |T(\mathbf{x})|$. It follows that $||\mathrm{T}|| = \max_{|x|=1} |\mathrm{T}(\mathbf{x})| = \sup_{\mathbf{x}\neq 0} |\mathrm{T}(\mathbf{x})|/|\mathbf{x}|$.

## exercise 16

Use the lemma in section 3.1 to show that if $T$ is an invertible linear transformation then $||T|| > 0$ and

$$\left\|T^{-1}\right\| \geq \frac{1}{||T||}$$

## solution

If T is invertible, then there exists an inverse, $T^{-1}$, such that $TT^{-1} = 1$ and therefore $\left\|TT^{-1}\right\| = 1$. By the lemma in Section 3, $1 = \left\|TT^{-1}\right\| \leq \|T\| \left\|T^{-1}\right\|$. This implies that $\|T\| > 0$, $\left\|T^{-1}\right\| > 0$, and $\left\|T^{-1}\right\| \geq \frac{1}{\|T\|}$.

## exercise 17

If $T$ is a linear transformation on $\mathbf{R}^n$ with $\|T - I\| < 1$, prove that $T$ is invertible and that the series $\sum_{k=0}^{\infty}(I - T)^k$ converges absolutely to $T^{-1}$. Hint: Use the geometric series.

## solution

Given $T \in L(\mathbf{R}^n)$ with $\|I - T\| < 1$.

Let $a = \|I - T\| < 1$ and the geometric series $\Sigma a^k$ converges.

Thus, by the Weierstrass M-Test, $\sum_{k=0}^{\infty}(I - T)^k$ converges absolutely to $S \in L(\mathbf{R}^n)$. By induction it follows that $T[I + (I - T) + \cdots + (I - T)^{\cdot}] = I - (I - T)^{n+1}$.

Thus, $TS = T\sum_{k=0}^{\infty}(I - T)^k = \sum_{k=0}^{\infty} T(I - T)^k = \lim_{n\to\infty}\sum_{k=0}^{n} T(I - T)^k = \lim_{n\to\infty}\left[I - (I - T)^{n+1}\right] = 1$ since $\lim_{n\to\infty}\|I - T\|^{n+1} = 0$ which implies that $\lim_{n\to\infty}(I - T)^{n+1} = 0$

since $0 \leq \left\|(I - T)^{n+1}\right\| \leq \|(I - T)\|^{a+1}$. Therefore $S = T^{-1}$.

## exercise 18

Compute the exponentials of the following matrices:

(a) $\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$ (e) $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ (f) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

## solution

. (a) $e^A = \begin{bmatrix} e^2 & 0 \\ 0 & e^{-3} \end{bmatrix}$.

(b) The eigenvalues and eigenvectors of $A$ are $\lambda_1 = 1, \lambda_2 = -1, v_1 = (1,0)^T, v_2 = (-1,1)^T$; thus, $e^A = P\begin{bmatrix} e & 0 \\ 0 & e^{-1} \end{bmatrix}P^{-1} = \begin{bmatrix} e & e - e^{-1} \\ 0 & e^{-1} \end{bmatrix}$ where $P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

(c) $e^A = e\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$ by Corollary 4 .

(d) The eigenvalues and eigenvectors of $A$ are $\lambda_1 = 2, \lambda_2 = -1, v_1 = (2,1)^T, \mathbf{v}_2 = (1,1)^T$; thus,

$e^A = P\begin{bmatrix} e^2 & 0 \\ 0 & e^{-1} \end{bmatrix}P^{-1} = \begin{bmatrix} 2e^2 - e^{-1} & 2e^{-1} - 2e^2 \\ e^2 - e^{-1} & 2e^{-1} - e^2 \end{bmatrix}$ where $P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

(e) $e^{\wedge} = e^2\begin{bmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{bmatrix}$ by Corollary 3 .

(f) The eigenvalues and eigenvectors of $A$ are $\lambda_1 = 1, \lambda_2 = -1, v_1 = (1,1)^T, v_2 = (-1,1)^T$; thus $e^A = P\begin{bmatrix} e & 0 \\ 0 & e^{-1} \end{bmatrix}P^{-1} = \begin{bmatrix} \cosh(1) & \sinh(1) \\ \sinh(1) & \cosh(1) \end{bmatrix}$ with $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Note that $A^2 = 1$ and from Definition it therefore follows that $e^A = 1(1 + 1/2! + 1/4! + \cdots) + A(1 + 1/3! + 1/5! + \cdots) = I\cosh(1) + A\sinh(1)$. This remark also applies to part (b).

## exercise 19

**(a)** For each matrix in exercise 18 find the eigenvalues of $e^A$.

**(b)** Show that if $\mathbf{x}$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda$, then $\mathbf{x}$ is also an eigenvector of $e^A$ corresponding to the eigenvalue $e^\lambda$.

**(c)** If $A = P\operatorname{diag}\left[\lambda_j\right]P^{-1}$, use Corollary 1 to show that

$$\det e^A = e^{\operatorname{trace} A}$$

Also, using the results in the last paragraph of this section, show that this formula holds for any $2 \times 2$ matrix $A$.

## solution

**(a)** The eigenvalues are $e^2, e^{-3}; e, e^{-1}; e, e; e^2, e^{-1}; e^{2+i} = e^2[\cos(1) \pm i\sin(1)]; e, e^{-1}$.

**(b)** If $Ax = \lambda x$, then $e^A x = \lim_{k\to\infty}\left[I + A + A^2/2! + \cdots + A^k/k!\right]x = \lim_{k\to 0}\left[x + \lambda x + \lambda^2 x/2! + \cdots + \lambda^k \mathbf{x}/k!\right] = e^\lambda \mathbf{x}$.

**(c)** If $A = P\operatorname{diag}\left[\lambda_j\right]P^{-1}$, then by Corollary 1 , $\det e^A = \det\left\{P\operatorname{diag}\left[e^{\lambda_1}\right]P^{-1}\right\} = \det\{\operatorname{diag}\left[e^{\lambda_i}\right]\} = e^{\lambda_1}\cdots e^{\lambda_k} = e^{\text{trees}}$.

For a $2\times 2$ matrix $A$ with repeated eigenvalues $\lambda$, we have $\det e^\lambda = \det\begin{bmatrix} e^\lambda & e^\lambda \\ 0 & e^\lambda \end{bmatrix} = e^{\lambda\lambda} = e^{\text{troceA}}$; and for $2\times 2$ matrix $A$ with complex eigenvalues, $\lambda = a \pm ib$, we have $dete^A = \det\begin{bmatrix} e^a\cos b & -e^a\sin b \\ e^a\sin b & e^a\cos b \end{bmatrix} = e^{2_a} = e^{\text{ancecA}}$ (since the trace $A = \lambda_1 + \lambda_2 = (a + ib) + (a - ib) = 2a$ in this case ).

## exercise 20

Compute the exponentials of the following matrices:

$$(a)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad (b)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \quad (c)\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

Hint: Write the matrices in (b) and (c) as a diagonal matrix $S$ plus a matrix $N$. Show that $S$ and $N$ commute and compute $e^S$ as in part (a) and $e^N$ by using the definition.

## solution

**(a)**

$e^A = \mathrm{diag}\left[e, e^2, e^3\right]$.

**(b)**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$ = N + S and NS = SN so that by Proposition 2,

$$e^A = \mathrm{diag}\left[c, e^2, e^2\right]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e & 0 & 0 \\ 0 & c^2 & e^2 \\ 0 & 0 & e^2 \end{bmatrix}$$ since $N^2 = 0$ implies that $e^N = 1 + N$.

**(c)** $$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$ = N + S and NS = SN so that by Proposition 2

$$e^A = e^s e^N = e^2\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1/2 & 1 & 1 \end{bmatrix}$$ since $N^3 = 0$ implies that $e^N = I + N + N^2/2$.

## exercise 21

Find $2 \times 2$ matrices $A$ and $B$ such that $e^{A+B} \neq e^A e^B$.

## solution

For $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ we have $AB = 0 \neq BA = B$, $e^{A+B} = \begin{bmatrix} c & 0 \\ e-1 & 1 \end{bmatrix} \neq e^A e^B = \begin{bmatrix} e & 0 \\ 1 & 1 \end{bmatrix}$.

## exercise 22

Let $T$ be a linear operator on $\mathbf{R}^n$ that leaves a subspace $E \subset \mathbf{R}^n$ invariant; i.e., for all $\mathbf{x} \in E, T(\mathbf{x}) \in E$. Show that $e^T$ also leaves $E$ invariant.

## solution

If $T(x) \in E$ for all $x \in E$, then by induction $T^2(x) \in E, \cdots, T^x(x) \in E$
and therefore $e^T(x) = \lim_{k\to\infty}[1 + T + \cdots + T^k/k!]x = \lim_{k\to\infty}\left[x + T(x) + \cdots + \frac{Tk(x)}{k!}\right] \in E$ since any subspace $E$ of $R^n$ is complete and since $\mathrm{\mathbf{x}_k = \mathbf{x} + T(\mathbf{x}) + \cdots + T^k(\mathbf{x})/k!}$ is a Cauchy sequence in E.

## exercise 22

Find the eigenvalues and eigenvectors of the matrix $A$ and show that $B = P^{-1}AP$ is a diagonal matrix. Solve the linear system $\dot{\mathbf{y}} = B\mathbf{y}$ and then solve $\dot{\mathbf{x}} = A\mathbf{x}$ using the above corollary. And then sketch the phase portraits in both the $\mathbf{x}$ plane and $\mathbf{y}$ plane.

$$\begin{aligned} \dot{x}_1 &= -x_1 - 3x_2 \\ \dot{x}_2 &= 2x_2 \end{aligned} \tag{1.1}$$

## solution

Consider the linear system

$$\begin{aligned} \dot{x}_1 &= -x_1 - 3x_2 \\ \dot{x}_2 &= 2x_2 \end{aligned}$$

which can be written in the form 1.1 with the matrix $A = \begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix}$ The eigenvalues of $A$ are $\lambda_1 = -1$ and $\lambda_2 = 2$. A pair of corresponding eigenvectors is given by $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$
The matrix $P$ and its inverse are then given by

$$P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

The student should verify that

$$P^{-1}AP = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

Then under the coordinate transformation $\mathbf{y} = P^{-1}\mathbf{x}$, we obtain the uncoupled linear system

$$\begin{aligned} \dot{y}_1 &= -y_1 \\ \dot{y}_2 &= 2y_2 \end{aligned}$$

which has the general solution $y_1(t) = c_1 e^{-t}, y_2(t) = c_2 e^{2t}$. The phase portrait for this system is given in Figure 1 . And according to the above corollary, the general solution to the original linear system of this example is given by

$$\mathbf{x}(t) = P \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} P^{-1}\mathbf{c}$$

where $\mathbf{c} = \mathbf{x}(0)$, or equivalently by

$$\begin{aligned} x_1(t) &= c_1 e^{-t} + c_2 \left(e^{-t} - e^{2t}\right) \\ x_2(t) &= c_2 e^{2t} \end{aligned} \tag{1.2}$$

The phase portrait for the linear system of this example can be found by sketching the solution curves defined by 1.2. It is shown in Figure 2.

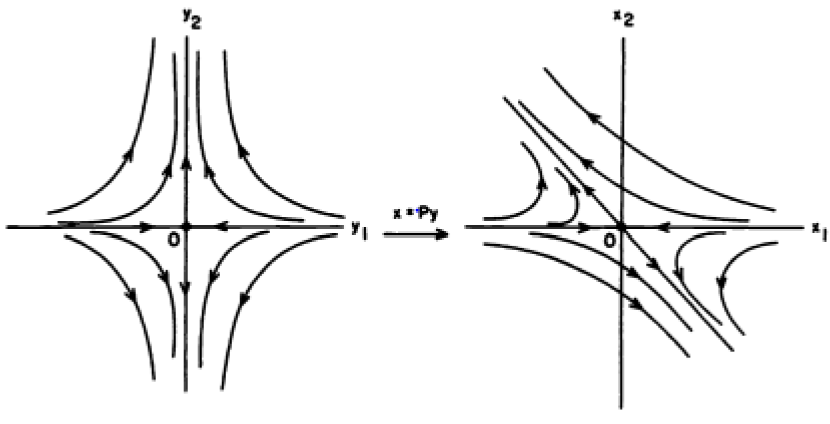



**Figure 1** **Figure 2**

The phase portrait in Figure 2 can also be obtained from the phase portrait in Figure 1 by applying the linear transformation of coordinates $\mathbf{x} = P\mathbf{y}$. Note that the subspaces spanned by the eigenvectors $\mathbf{v}_1$

and $\mathbf{v}_2$ of the matrix A determine the stable and unstable subspaces of the linear system **??**

## exercise 23

Solve the initial value problem

$$\dot{\mathbf{x}} = A\mathbf{x}$$
$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

for

$$A = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}$$

## solution

The solution is given by

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 = e^{-2t}\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = e^{-2t}\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}.$$

It follows that $|\mathbf{x}(t)| = e^{-2t}$ and that the angle $\theta(t) = \tan^{-1} x_2(t)/x_1(t) = t$. The solution curve therefore spirals into the origin as shown in Figure 1 below.Then the fact that $\mathbf{x}(t)$ is a solution of (1)

$$\begin{aligned}\mathbf{y}'(t) &= -Ae^{-At}\mathbf{x}(t) + e^{-At}\mathbf{x}'(t) \\ &= -Ae^{-At}\mathbf{x}(t) + e^{-At}A\mathbf{x}(t) \\ &= 0\end{aligned}$$

for all $t \in \mathbf{R}$ since $e^{-At}$ and A commute. Thus, $\mathbf{y}(t)$ is a constant. Setting $t = 0$ shows that $y(t) = x_0$ and therefore any solution of the initial value problem (1) is given by $\mathbf{x}(t) = e^{At}\mathbf{y}(t) = e^{At}\mathbf{x}_0$.

## exercise 24

**Solve the initial value problem**

$$\dot{\mathbf{x}} = A\mathbf{x}$$
$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

for

$$A = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}$$

## solution

the solution is given by

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 = e^{-2t}\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = e^{-2t}\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}.$$

It follows that $|\mathbf{x}(t)| = e^{-2t}$ and that the angle $\theta(t) = \tan^{-1} x_2(t)/x_1(t) = t$. The solution curve therefore spirals into the origin as shown in Figure 1 below.

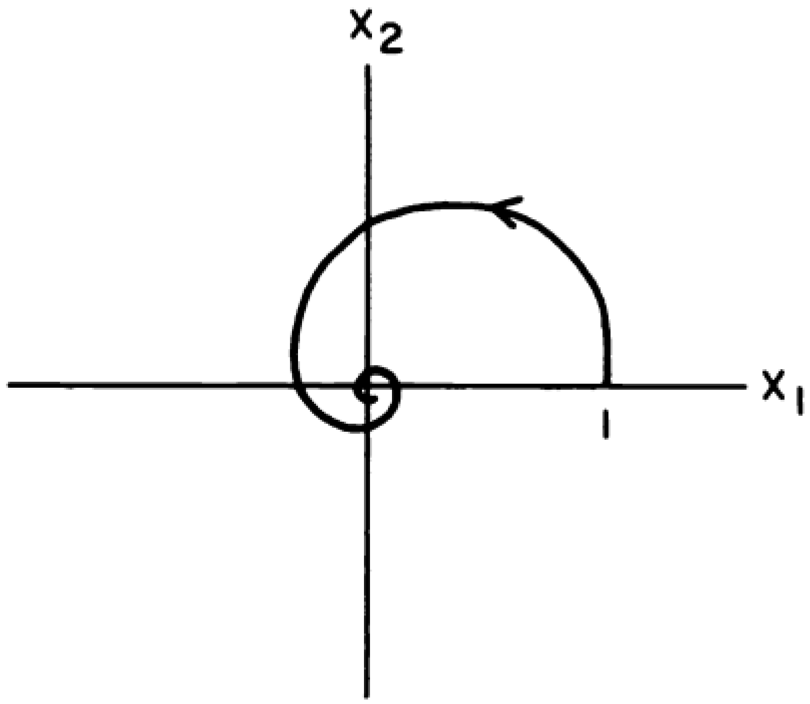



**Figure 1**

## exercise 25

Solve the linear system

$$\dot{\mathrm{x}} = A\mathrm{x}$$

with

$$A = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix}$$

## solution

has a center at the origin since the matrix $A$ has eigenvalues $\lambda = \pm 2i$. According to the theorem in Section 1.6, the invertible matrix

$$P = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{with} \quad P^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$$

reduces $A$ to the matrix

$$B = P^{-1}AP = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

The solution to the linear system $\dot{\mathbf{x}} = A\mathbf{x}$ is then given by

$$\mathbf{x}(t) = P \begin{bmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{bmatrix} P^{-1}\mathbf{c} = \begin{bmatrix} \cos 2t & -2\sin 2t \\ 1/2 \sin 2t & \cos 2t \end{bmatrix} \mathbf{c}$$

where $\mathbf{c} = \mathbf{x}(0)$, or equivalently by

$$x_1(t) = c_1 \cos 2t - 2c_2 \sin 2t$$

$$x_2(t) = 1/2 c_1 \sin 2t + c_2 \cos 2t.$$

It is then easily shown that the solutions satisfy

$$x_1^2(t) + 4x_2^2(t) = c_1^2 + 4c_2^2$$

for all $t \in \mathbf{R}$; i.e., the trajectories of this system lie on ellipses as shown in Figure 5.

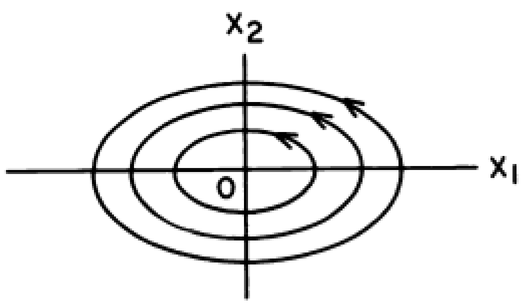



**Figure 5.** A center at the origin.

## exercise 26

**Example. Solve the initial value problem (1) for**

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

## solution

The matrix $A$ has the complex eigenvalues $\lambda_1 = 1 + i$ and $\lambda_2 = 2 + i$ (as well as $\bar{\lambda}_1 = 1 - i$ and $\bar{\lambda}_2 = 2 - i$ ). A corresponding pair of complex eigenvectors is

$$w_1 = u_1 + iv_1 = \begin{bmatrix} i \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } w_2 = u_2 + iv_2 = \begin{bmatrix} 0 \\ 0 \\ 1+i \\ 1 \end{bmatrix}.$$

The matrix

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{u}_1 & \mathbf{v}_2 & \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is invertible,

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$P^{-1}AP = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The solution to the initial value problem (1) is given by

$$\mathbf{x}(t) = P\begin{bmatrix} e^t\cos t & -e^t\sin t & 0 & 0 \\ e^t\sin t & e^t\cos t & 0 & 0 \\ 0 & 0 & e^{2t}\cos t & -e^{2t}\sin t \\ 0 & 0 & e^{2t}\sin t & e^{2t}\cos t \end{bmatrix} P^{-1}\mathbf{x}_0$$

$$= \begin{bmatrix} e^t\cos t & -e^t\sin t & 0 & 0 \\ e^t\sin t & e^t\cos t & 0 & 0 \\ 0 & 0 & e^{2t}(\cos t + \sin t) & -2e^{2t}\sin t \\ 0 & 0 & e^{2t}\sin t & e^{2t}(\cos t - \sin t) \end{bmatrix} \mathbf{x}_0$$

In case $A$ has both real and complex eigenvalues and they are distinct, we have the following result: If $A$ has distinct real eigenvalues $\lambda_j$ and corresponding eigenvectors $\mathbf{v}_j$, $j = 1, \ldots, k$ and distinct complex eigenvalues $\lambda_j = a_j + ib_j$ and $\bar{\lambda}_j = a_j - ib_j$ and corresponding eigenvectors $\mathbf{w}_j = u_j + i\mathbf{v}_j$ and $\bar{w}_j = u_j - i\mathbf{v}_j$, $j = k+1, \ldots, n$, then the matrix

$$P = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_k & \mathbf{v}_{k+1} & \mathbf{u}_{k+1} & \cdots & \mathbf{v}_n & \mathbf{u}_n \end{bmatrix}$$

is invertible and

$$P^{-1}AP = \mathrm{diag}\,[\lambda_1, \ldots, \lambda_k, B_{k+1}, \ldots, B_n]$$

where the $2 \times 2$ blocks

$$B_j = \begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix}$$

## exercise 27

Solve the initial value problem

$$\dot{\mathbf{x}} = A\mathbf{x}$$

$$\mathbf{x}(0) = \mathbf{x}_0$$

with

$$A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

## solution

has eigenvalues $\lambda_1 = -3, \lambda_2 = 2 + i$ (and $\bar{\lambda}_2 = 2 - i$ ). The corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{w}_2 = u_2 + i\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1+i \\ 1 \end{bmatrix}$$

Thus

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

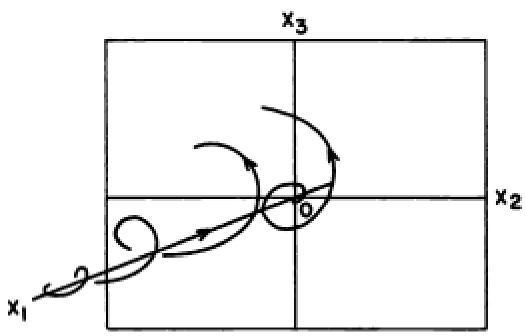
and

$$P^{-1}AP = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

The solution of the initial value problem (1) is given by

$$\begin{aligned} \mathbf{x}(t) &= P \begin{bmatrix} e^{-3t} & 0 & 0 \\ 0 & e^{2t}\cos t & -e^{2t}\sin t \\ 0 & e^{2t}\sin t & e^{2t}\cos t \end{bmatrix} P^{-1}x_0 \\ &= \begin{bmatrix} e^{-3t} & 0 & 0 \\ 0 & e^{2t}(\cos t + \sin t) & -2e^{2t}\sin t \\ 0 & e^{2t}\sin t & e^{2t}(\cos t - \sin t) \end{bmatrix} x_0 \end{aligned}$$

The stable subspace $E^s$ is the $x_1$-axis and the unstable subspace $E^u$ is the $x_2, x_3$ plane. The phase portrait is given in this Figure .



## exercise 28

Solve the initial value problem

$$\dot{\mathbf{x}} = A\mathbf{x}$$

$$\mathbf{x}(0) = \mathbf{x}_0$$

with

$$A = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$$

It is easy to determine that $A$ has an eigenvalue $\lambda = 2$ of multiplicity 2 ; i.e., $\lambda_1 = \lambda_2 = 2$. Thus,

$$S = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

and

$$N = A - S = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

It is easy to compute $N^2 = 0$ and the solution of the initial value problem for (1) is therefore given by

$$\begin{aligned} \mathbf{x}(t) &= e^{At}\mathbf{x}_0 = e^{2t}[I + Nt]\mathbf{x}_0 \\ &= e^{2t}\begin{bmatrix} 1+t & t \\ -t & 1-t \end{bmatrix}\mathbf{x}_0 \end{aligned}$$

## exercise 29

Solve the initial value problem

$$\dot{\mathbf{x}} = A\mathbf{x}$$

$$\mathbf{x}(0) = \mathbf{x}_0$$

with

$$A = \begin{bmatrix} 0 & -2 & -1 & -1 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

## solution

In this case, the matrix $A$ has an eigenvalue $\lambda = 1$ of multiplicity 4 . Thus, $S = I_4$,

$$N = A - S = \begin{bmatrix} -1 & -2 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and it is easy to compute

$$N^2 = \begin{bmatrix} -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and $N^3 = 0$; i.e., $N$ is nilpotent of order 3. The solution of the initial value problem for (1) is therefore given by

$$\begin{aligned} \mathbf{x}(t) &= e^t \left[ I + Nt + N^2 t^2/2 \right] \mathbf{x}_0 \\ &= e^t \begin{bmatrix} 1 - t - t^2/2 & -2t - t^2/2 & -t - t^2/2 & -t - t^2/2 \\ t & 1 + t & t & t \\ t^2/2 & t + t^2/2 & 1 + t^2/2 & t^2/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}_0 \end{aligned}$$

## exercise 30

Solve the initial value problem

$$\dot{\mathbf{x}} = A\mathbf{x}$$

$$\mathbf{x}(0) = \mathbf{x}_0$$

with

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

## solution

It is easy to see that $A$ has the eigenvalues $\lambda_1 = 1, \lambda_2 = \lambda_3 = 2$. And it is not difficult to find the corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Nonzero multiples of these eigenvectors are the only eigenvectors of $A$ corresponding to $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = 2$ respectively. We therefore must find one generalized eigenvector corresponding to $\lambda = 2$ and independent of $\mathbf{v_2}$ by solving

$$(A-2I)^2 v = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \mathbf{v} = 0$$

We see that we can choose $\mathbf{v}_3 = (0,1,0)^T$. Thus,

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

We then compute

$$S = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix},$$

$$N = A - S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix},$$

and $N^2 = 0$. The solution is then given by

$$\begin{aligned} x(t) &= P \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} P^{-1}[I + Nt]x_0 \\ &= \begin{bmatrix} e^t & 0 & 0 \\ e^t - e^{2t} & e^{2t} & 0 \\ -2e^t + (2-t)e^{2t} & te^{2t} & e^{2t} \end{bmatrix} x_0 \end{aligned}$$

In the case of multiple complex eigenvalues, we have the following theorem also proved in Appendix III of Hirsch and Smale [H/S]:

Let $A$ be a real $2n \times 2n$ matrix with complex eigenvalues $\lambda_j = a_j + ib_j$ and $\bar{\lambda}_j = a_j - ib_j, j = 1, \dots, n$. Then there exists generalized complex eigenvectors $\mathbf{w}_j = \mathbf{u}_j + i\mathbf{v}_j$ and $\overline{\mathbf{w}}_j = \mathbf{u}_j - i\mathbf{v}_j, i = 1, \dots, n$ such that $\{\mathbf{u}_1, \mathbf{v}_1, \dots, \mathbf{u}_n, \mathbf{v}_n\}$ is a basis for $\mathbf{R}^{2n}$. For any such basis, the matrix $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{u}_1 & \cdots & \mathbf{v}_n & \mathbf{u}_n \end{bmatrix}$ is invertible,

$$A = S + N$$

where

$$P^{-1}SP = \text{diag}\begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix}$$

the matrix $N = A - S$ is nilpotent of order $k \leq 2n$

## exercise 32

Solve the initial value problem

$$\dot{\mathbf{x}} = A\mathbf{x}$$

$$\mathbf{x}(0) = \mathbf{x}_0$$

with

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 2 & 0 & 1 & 0 \end{bmatrix}$$

## solution

The matrix $A$ has eigenvalues $\lambda = i$ and $\bar{\lambda} = -i$ of multiplicity 2 . The equation

$$(A - \lambda I)\mathbf{w} = \begin{bmatrix} -i & -1 & 0 & 0 \\ 1 & -i & 0 & 0 \\ 0 & 0 & -i & -1 \\ 2 & 0 & 1 & -i \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = 0$$

is equivalent to $z_1 = z_2 = 0$ and $z_3 = iz_4$. Thus, we have one eigenvector $w_1 = (0, 0, i, 1)^T$. Also, the equation

$$(A - \lambda I)^2 w = \begin{bmatrix} -2 & 2i & 0 & 0 \\ -2i & -2 & 0 & 0 \\ -2 & 0 & -2 & 2i \\ -4i & -2 & -2i & -2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = 0$$

is equivalent to $z_1 = iz_2$ and $z_3 = iz_4 - z_1$. We therefore choose the generalized eigenvector $\mathbf{w}_2 = (i, 1, 0, 1)$. Then $\mathbf{u}_1 = (0, 0, 0, 1)^T, \mathbf{v}_1 = (0, 0, 1, 0)^T$, $u_2 = (0, 1, 0, 1)^T, v_2 = (1, 0, 0, 0)^T$, and according to the above theorem,

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$S = P \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} P^{-1} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix},$$

$$N = A - S = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

and $N^2 = 0$. Thus, the solution to the initial value problem

$$\begin{aligned} \mathbf{x}(t) &= \begin{bmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{bmatrix} P^{-1}[I + Nt]\mathbf{x}_0 \\ &= \begin{bmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ -t\sin t & \sin t - t\cos t & \cos t & -\sin t \\ \sin t + t\cos t & -t\sin t & \sin t & \cos t \end{bmatrix} \mathbf{x}_0 \end{aligned}$$

# exercise 33

Find the stable, unstable and center subspaces $E^s$,$E^u$ and $E^C$of the linear system

$$\dot{\mathbf{x}} = A\mathbf{x}$$

with The matrix

$$A = \begin{bmatrix} -2 & -1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

## solution

has eigenvectors

$$w_1 = u_1 + i\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{corresponding to } \lambda_1 = -2 + i$$

and

$$\mathbf{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{corresponding to } \lambda_2 = 3.$$

The stable subspace $E^s$ of (1) is the $x_1, x_2$ plane and the unstable subspace $E^u$ of (1) is the $x_3$-axis. The phase portrait for the system (1) is shown in this Figure for this exercise.

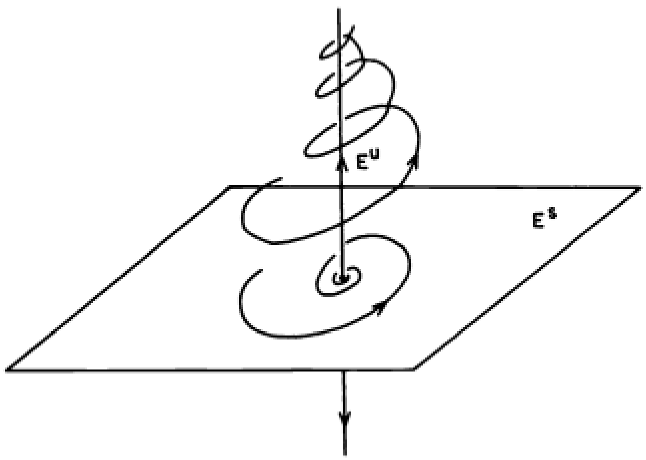



**Figure: The stable and unstable subspaces $E^s$ and $E^u$ of the linear system (1).**

## exercise 34

Find the stable, unstable and center subspaces $E^s$,$E^u$ and $E^C$of the linear system (1) with the matrix

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

## solution

has $\lambda_1 = i, u_1 = (0,1,0)^T, v_1 = (1,0,0)^T, \lambda_2 = 2$ and $u_2 = (0,0,1)^T$. The center subspace of (1) is the $x_1, x_2$ plane and the unstable subspace of (1) is the $x_3$-axis. The phase portrait for the system (1) is shown in Figure 2 for this example. Note that all solutions lie on the cylinders $x_1^2 + x_2^2 = c^2$.

In these examples we see that all solutions in $E^s$ approach the equilibrium point $\mathbf{x} = 0$ as $t \to \infty$ and that all solutions in $E^u$ approach the equilibrium point $\mathbf{x} = \mathbf{0}$ as $t \to -\infty$. Also, in the above example the solutions in $E^c$ are bounded and if $\mathbf{x}(0) \neq 0$, then they are bounded away from $\mathbf{x} = \mathbf{0}$ for all $t \in \mathbf{R}$. We shall see that these statements about $E^s$ and $E^u$ are true in general; however, solutions in $E^c$ need not be bounded as the next example shows.

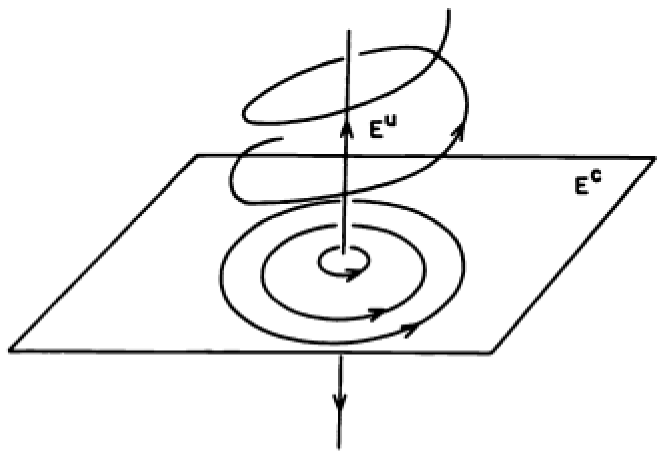



**Figure 2. The center and unstable subspaces $E^c$ and $E^u$ of the linear system (1).**

## exercise 35

Find the stable, unstable and center subspaces $E^s$,$E^u$ and $E^C$of the linear system (1) with the matrix

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; \quad \text{i.e.,} \quad \begin{aligned} \dot{x}_1 &= 0 \\ \dot{x}_2 &= x_1 \end{aligned}$$

## solution

We have $\lambda_1 = \lambda_2 = 0, u_1 = (0,1)^T$ is an eigenvector and $u_2 = (1,0)^T$ is a generalized eigenvector corresponding to $\lambda = 0$. Thus $E^c = \mathbf{R}^2$. The solution of (1) with $\mathbf{x}(0) = c = (c_1, c_2)^T$ is easily found to be

$$x_1(t) = c_1$$

$$x_2(t) = c_1 t + c_2.$$

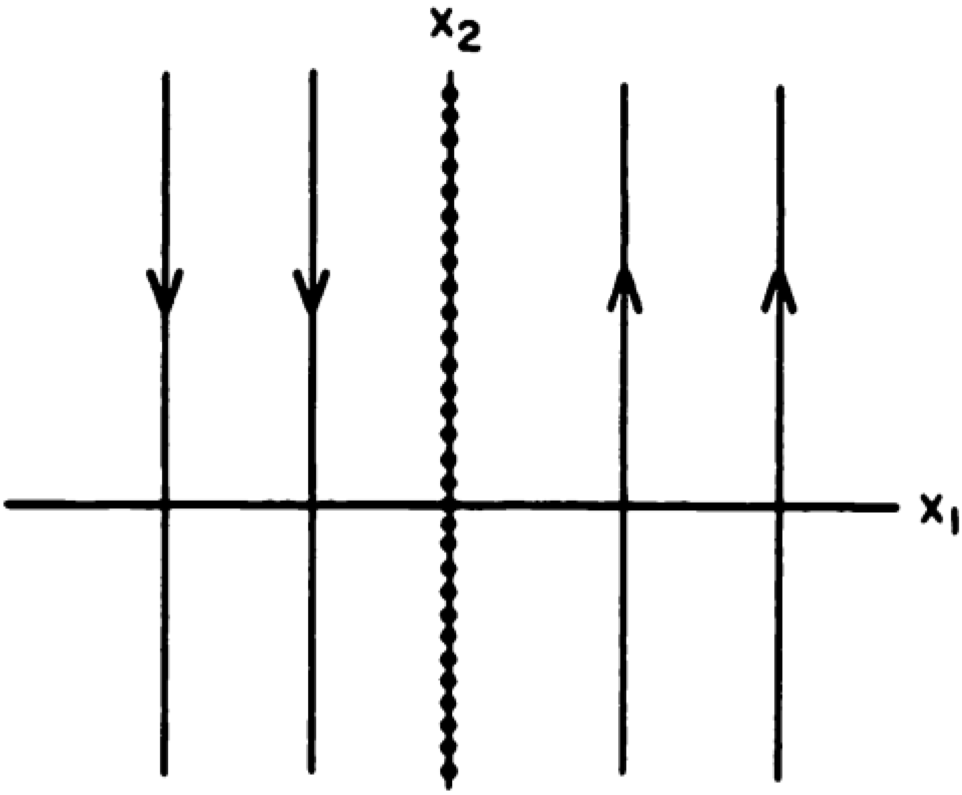



**Figure 3. The center subspace $E^c$ for (1).**

## exercise 36

Find the stable, unstable and center subspaces $E^s$,$E^u$ and $E^C$of the linear system (1) with the matrix

$$A = \begin{bmatrix} -2 & -1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

## solution

We have eigenvalues $\lambda_1 = -2 + i$ and $\lambda_2 = -3$ and the same eigenvectors as in Example 1. $E^s = \mathbf{R}^3$ and the origin is a sink for this example. The phase portrait is shown in Figure 4.

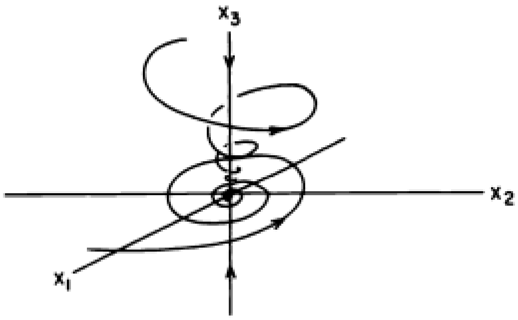



**Figure 4.** A linear system with a sink at the origin.

## exercise 37

Solve the forced harmonic oscillator problem

$$\ddot{x} + x = f(t).$$

## solution

This can be written as the nonhomogeneous system

$$\dot{x}_1 = -x_2$$
$$\dot{x}_2 = x_1 + f(t)$$

or equivalently in the form (1) with

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b}(t) = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}.$$

In this case

$$e^{At} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} = R(t),$$

a rotation matrix; and

$$e^{-At} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} = R(-t).$$

The solution of the above system with initial condition $\mathbf{x}(0) = \mathbf{x}_0$ is thus given by

$$\begin{aligned} \mathbf{x}(t) &= e^{At}\mathbf{x}_0 + e^{At}\int_0^t e^{-A\tau}\mathbf{b}(\tau)d\tau \\ &= R(t)\mathbf{x}_0 + R(t)\int_0^t \begin{bmatrix} f(\tau)\sin\tau \\ f(\tau)\cos\tau \end{bmatrix} d\tau. \end{aligned}$$

It follows that the solution $x(t) = x_1(t)$ of the original forced harmonic oscillator problem is given by

$$x(t) = x(0)\cos t - \dot{x}(0)\sin t + \int_0^t f(\tau)\sin(\tau - t)d\tau.$$