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Department of mathematics

Course Of Introduction To Dynamics Systems.

Master 1 (first year) fundamental and applied mathematics

The first semester

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CONTENTS

CHAPTER 1

STABILITY THEORY OF LINEAR SYSTEMS

This chapter presents a study of linear systems of ordinary differential equations:

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{1.1}$$

where $\mathbf{x} \in \mathbf{R}^n, A$ is an $n \times n$ matrix and

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix}$$

It is shown that the solution of the linear system 1.1 together with the initial condition $\mathbf{x}(0) = \mathbf{x}_0$ is given by

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0$$

where e^{At} is an $n \times n$ matrix function defined by its Taylor series. A good portion of this chapter is concerned with the computation of the matrix e^{At} in terms of the eigenvalues and eigenvectors of the square matrix A. Throughout this cour all vectors will be written as column vectors and A^T will denote the transpose of the matrix A.

1.1 Uncoupled Linear Systems

The method of separation of variables can be used to solve the first-order linear differential equation

$$\dot{x} = ax$$

The general solution is given by

$$x(t) = ce^{at}$$

where the constant $c = x(0)$, the value of the function $x(t)$ at time $t = 0$.

Now consider the uncoupled linear system

$$\dot{x}_1 = -x_1$$
$$\dot{x}_2 = 2x_2$$

This system can be written in matrix form as (1.1). where

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

Note that in this case A is a diagonal matrix, $A = \mathrm{diag}[-1, 2]$, and in general whenever A is a diagonal matrix, the system 1.1 reduces to an uncoupled linear system. The general solution of the above uncoupled linear system can once again be found by the method of separation of variables. It is given by

$$x_1(t) = c_1 e^{-t} \tag{1.2}$$
$$x_2(t) = c_2 e^{2t} \tag{1.3}$$

or equivalently by

$$\mathbf{x}(t) = \begin{bmatrix} \mathrm{e}^{-t} & 0 \\ 0 & \mathrm{e}^{2t} \end{bmatrix} \mathbf{c} \tag{1.4}$$

where $\mathbf{c} = \mathbf{x}(0)$. Note that the solution curves 1.2, 4 lie on the algebraic curves $y = k/x^2$ where the constant $k = c_1^2 c_2$. The solution 1.2, 4 or 1.4 defines a motion along these curves; i.e., each point $\mathbf{c} \in \mathbf{R}^2$ moves to the point $\mathbf{x}(t) \in \mathbf{R}^2$ given by 1.4 after time t. This motion can be described geometrically by drawing the solution curves 1.2, 4 in the x_1, x_2 plane, referred to as the phase plane, and by using arrows to indicate the direction of the motion along these curves with increasing time t; cf. Figure 1. For $c_1 = c_2 = 0, x_1(t) = 0$ and $x_2(t) = 0$ for all $t \in \mathbf{R}$ and the origin is referred to as an equilibrium point in this example. Note that solutions starting on the x_1-axis approach the origin as $t \to \infty$ and that solutions starting on the x_2-axis approach the origin as $t \to -\infty$.

The phase portrait of a system of differential equations such as 1.1 with $\mathbf{x} \in \mathbf{R}^n$ is the set of all solution curves of 1.1 in the phase space $\mathbf{R}^n$. Figure 1 gives a geometrical representation of the phase portrait of the uncoupled linear system considered above. The dynamical system defined by the linear system 1.1 in this example is simply the mapping $\phi : \mathbf{R} \times \mathbf{R}^2 \to \mathbf{R}^2$ defined by the solution $\mathbf{x}(t, \mathbf{c})$ given by 1.4; i.e.,

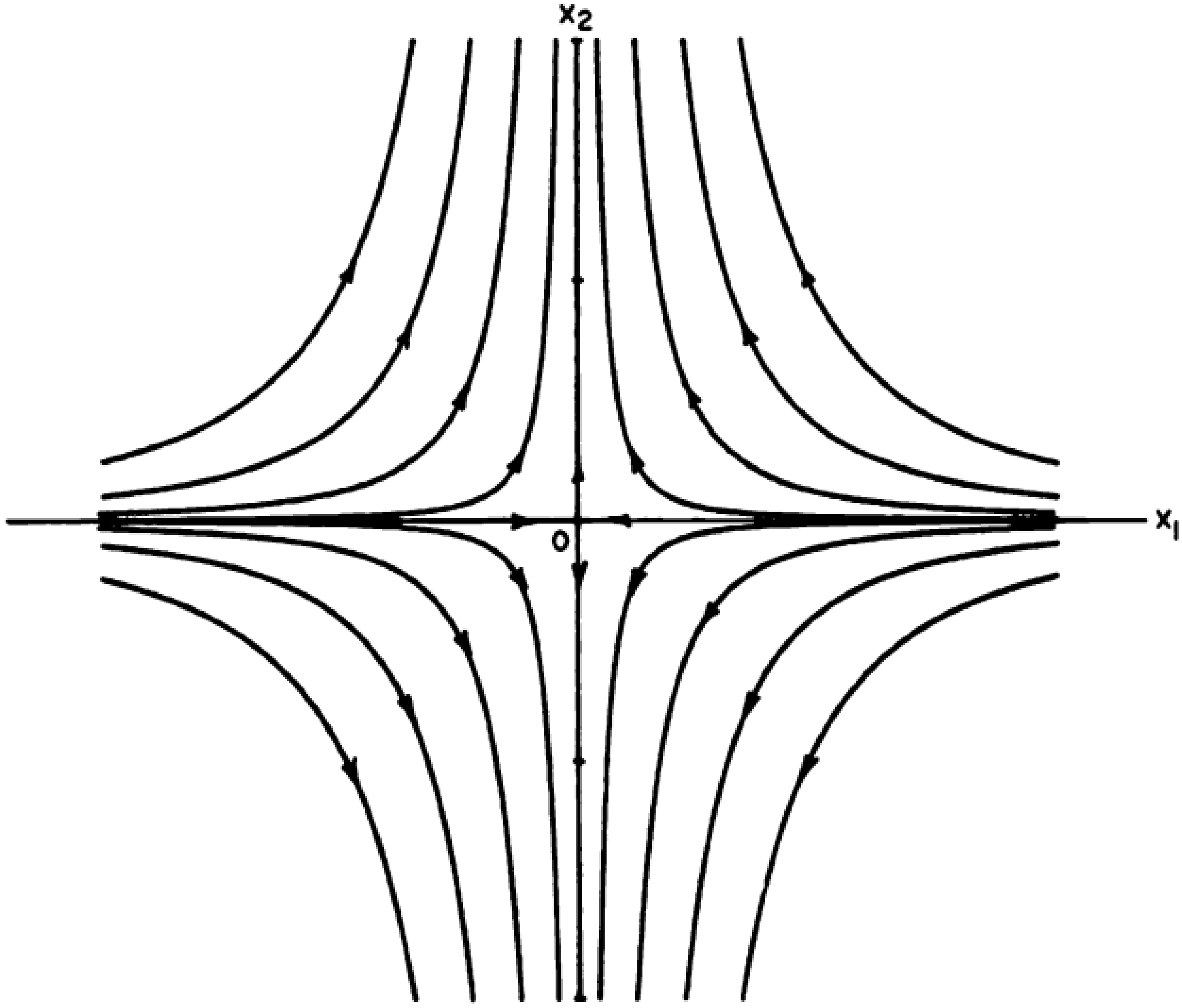

Figure 1

Geometrically, the dynamical system describes the motion of the points in phase space along the solution curves defined by the system of differential equations.

The function

$$\mathbf{f}(\mathbf{x}) = A\mathbf{x}$$

on the right-hand side of 1.1 defines a mapping $\mathbf{f} : \mathbf{R}^2 \to \mathbf{R}^2$ (linear in this case).
This mapping (which need not be linear) defines a vector field on R^2; i.e., to each point $x \in R^2$, the mapping f assigns a vector $f(x)$. If we draw each vector $f(\mathbf{x})$ with its initial point at the point $\mathbf{x} \in R^2$, we obtain a geometrical representation of the vector field as shown in Figure 2.

Note that at each point $\mathbf{x}$ in the phase space $\mathbf{R}^2$, the solution curves 1.2 are tangent to the vectors in the vector field $A\mathbf{x}$. This follows since at time $t = t_0$, the velocity vector $\mathbf{v}_0 = \dot{\mathbf{x}}(t_0)$ is tangent to the curve $\mathbf{x} = \mathbf{x}(t)$ at the point $\mathrm{x}_0 = \mathbf{x}(t_0)$ and since $\dot{\mathrm{x}} = A\mathbf{x}$ along the solution curves. Consider the following uncoupled linear system in $\mathbf{R}^3$:

$$\begin{aligned} \dot{x}_1 &= x_1 \\ \dot{x}_2 &= x_2 \\ \dot{x}_3 &= -x_3 \end{aligned} \tag{1.5}$$

The general solution is given by

$$\begin{aligned} x_1(t) &= c_1 e^t \\ x_2(t) &= c_2 \mathrm{e}^t \\ x_3(t) &= c_3 \mathrm{e}^{-t} \end{aligned}$$

And the phase portrait for this system is shown in Figure 3 above. The x_1, x_2 plane is referred to as the unstable subspace of the system (1.5) and
the x_3 axis is called the stable subspace of the system (1.5). Precise definitions of the stable and unstable subspaces of a linear system will be given in the next section.

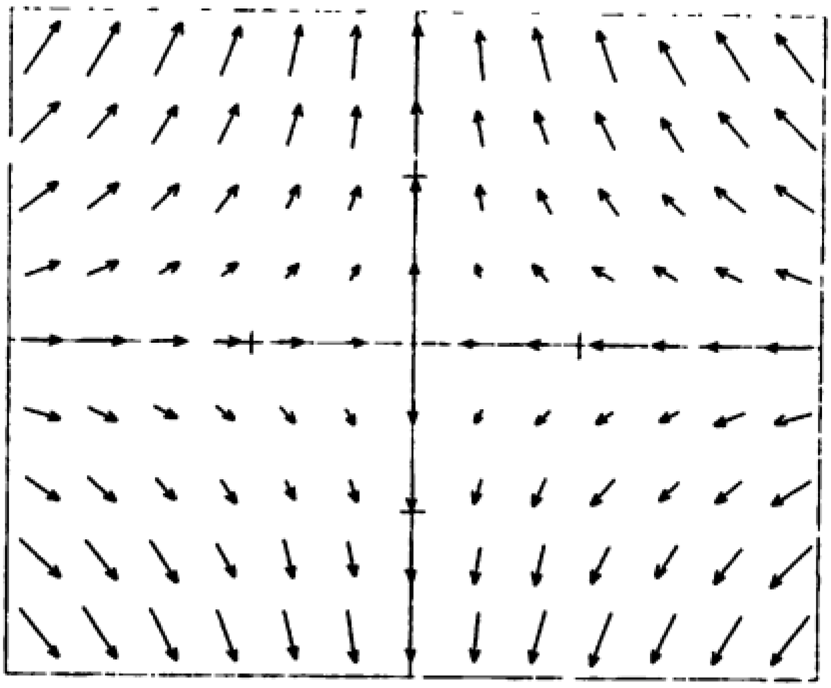

Figure 2

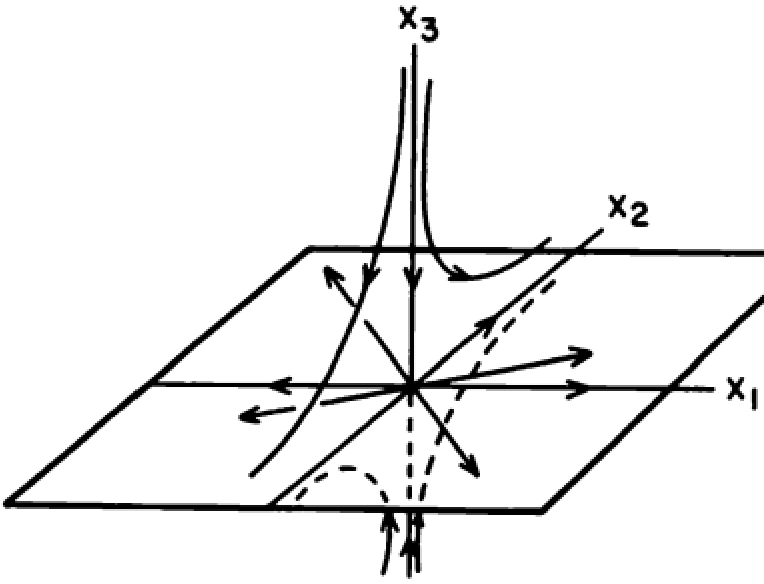

Figure 3

1.2 Diagonalization

The algebraic technique of diagonalizing a square matrix **A** can be used to reduce the linear system (1.1) to an uncoupled linear system. We first consider the case when A has real, distinct eigenvalues. The following theorem from linear algebra then allows us to solve the linear system (1.1).

Theorem: If the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of an $n \times n$ matrix A are real and distinct, then any set of corresponding eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ forms a basis for $\mathbf{R}^n$, the matrix $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \cdots \mathbf{v}_n \end{bmatrix}$ is invertible and

$$P^{-1}AP = \operatorname{diag}[\lambda_1, \ldots, \lambda_n]$$

This theorem says that if a linear transformation $T : \mathbf{R}^n \to \mathbf{R}^n$ is represented by the $n \times n$ matrix A with respect to the standard basis $\{e_1, e_2, \ldots, e_n\}$ for $\mathbf{R}^n$, then with respect to any basis of eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}, T$ is represented by the diagonal matrix of eigenvalues, $\operatorname{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]$. A proof of this theorem can be found, for example, in Lowenthal [Lo]. In order to reduce the system (1.1) to an uncoupled linear system using the above theorem, define the linear transformation of coordinates

$$\mathbf{y} = P^{-1}\mathbf{x}$$

where P is the invertible matrix defined in the theorem. Then

$$\mathbf{x} = P\mathbf{y},$$
$$\dot{\mathbf{y}} = P^{-1}\dot{\mathbf{x}} = P^{-1}A\mathbf{x} = P^{-1}AP\mathbf{y}$$

and, according to the above theorem, we obtain the uncoupled linear system

$$\dot{\mathbf{y}} = \operatorname{diag}\left[\lambda_1, \ldots, \lambda_n\right]\mathbf{y}$$

This uncoupled linear system has the solution

$$\mathbf{y}(t) = \operatorname{diag}\left[e^{\lambda_1 t}, \ldots, e^{\lambda_n t}\right]\mathbf{y}(0)$$

(Cf. problem 4 in Problem Set 1.) And then since $\mathbf{y}(0) = P^{-1}\mathbf{x}(0)$ and $\mathbf{x}(t) = P\mathbf{y}(t)$, it follows that (1.1) has the solution

$$\mathbf{x}(t) = PE(t)P^{-1}\mathbf{x}(0) \tag{1.6}$$

where $E(t)$ is the diagonal matrix

$$E(t) = \operatorname{diag}\left[e^{\lambda_1 t}, \ldots, e^{\lambda_n t}\right]$$

Corollary. Under the hypotheses of the above theorem, the solution of the linear system (1.1) is given by the function $\mathbf{x}(t)$ defined by (1.6).

Example 1.2.1 *Consider the linear system*

$$\dot{x}_1 = -x_1 - 3x_2$$
$$\dot{x}_2 = 2x_2$$

which can be written in the form (1.1) with the matrix

$$A = \begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix}$$

The eigenvalues of A are $\lambda_1 = -1$ *and* $\lambda_2 = 2$. *A pair of corresponding eigenvectors is given by*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The matrix P and its inverse are then given by

$$P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

The student should verify that

$$P^{-1}AP = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

Then under the coordinate transformation $\mathbf{y} = P^{-1}\mathbf{x}$*, we obtain the uncoupled linear system*

$$\begin{aligned} \dot{y}_1 &= -y_1 \\ \dot{y}_2 &= 2y_2 \end{aligned}$$

which has the general solution $y_1(t) = c_1e^{-t}, y_2(t) = c_2e^{2t}$*. The phase portrait for this system is given in Figure 1 in Section 0.1 which is reproduced below. And according to the above corollary, the general solution to the original linear system of this example is given by*

$$\mathbf{x}(t) = P \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} P^{-1}\mathbf{c}$$

where $\mathbf{c} = \mathbf{x}(0)$*, or equivalently by*

$$\begin{aligned} x_1(t) &= c_1e^{-t} + c_2\left(e^{-t} - e^{2t}\right) \\ x_2(t) &= c_2e^{2t} \end{aligned} \tag{3}$$

The phase portrait for the linear system of this example can be found by sketching the solution curves defined by (3). It is shown in Figure 2. The phase portrait in Figure 2 can also be obtained from the phase portrait in Figure 1 by applying the linear transformation of coordinates $\mathbf{x} = P\mathbf{y}$*. Note that the subspaces spanned by the eigenvectors* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *of the matrix A determine the stable and unstable subspaces of the linear system (1.1) according to the following definition: Suppose that the* $n \times n$ *matrix A has k negative eigenvalues* $\lambda_1, \ldots, \lambda_k$ *and* $n - k$ *positive eigenvalues* $\lambda_{k+1}, \ldots, \lambda_n$ *and that these eigenvalues are distinct. Let* $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ *be a corresponding set of eigenvectors. Then the stable and unstable subspaces of the linear system (1.1),* E^s *and* E^u*, are the linear* *subspaces spanned by* $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ *and* $\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ *respectively; i.e.,*

$$E^s = \mathrm{Span}\,\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$$

$$E^u = \mathrm{Span}\,\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$$

If the matrix A has pure imaginary eigenvalues, then there is also a center subspace E^c*; cf. Problem 2(c) in Section 0.1. The stable, unstable and center subspaces are defined for the general case in Section 0.9.*

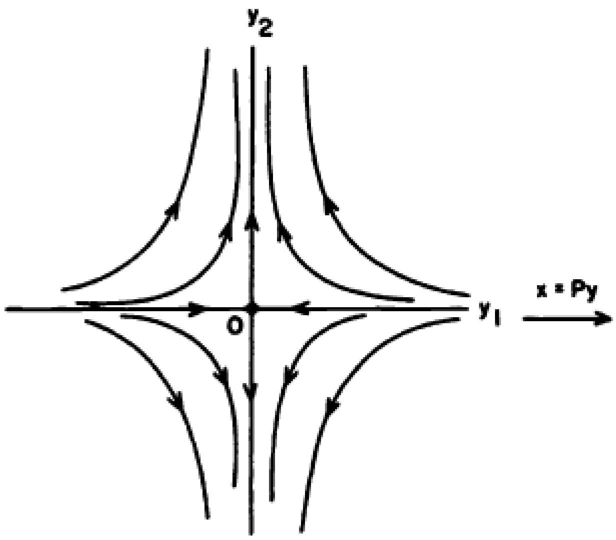

Figure 1

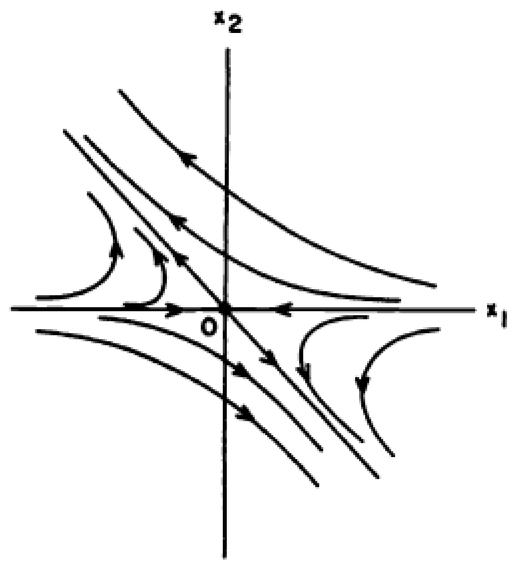

Figure 2

1.3 Exponentials of Operators

In order to define the exponential of a linear operator $T : \mathbf{R}^n \to \mathbf{R}^n$, it is necessary to define the concept of convergence in the linear space $L(\mathbf{R}^n)$ of linear operators on $\mathbf{R}^n$. This is done using the operator norm of T defined by

$$\|T\| = \max_{|\mathbf{x}| \leq 1} |T(\mathbf{x})|$$

where $|\mathbf{x}|$ denotes the Euclidean norm of $\mathbf{x} \in \mathbf{R}^n$; i.e.,

$$|\mathbf{x}| = \sqrt{x_1^2 + \cdots + x_n^2}$$

The operator norm has all of the usual properties of a norm, namely, for $S, T \in L(\mathbf{R}^n)$

(a) $\|T\| \geq 0$ and $\|T\| = 0$ iff $T = 0$

(b) $\|kT\| = |k|\|T\|$ for $k \in \mathbf{R}$

(c) $\|S + T\| \leq \|S\| + \|T\|$.

It follows from the Cauchy-Schwarz inequality that if $T \in L(\mathbf{R}^n)$ is represented by the matrix A with respect to the standard basis for $\mathbf{R}^n$, then $\|A\| \leq \sqrt{n}\ell$ where ℓ is the maximum length of the rows of A.

The convergence of a sequence of operators $T_k \in L(\mathbf{R}^n)$ is then defined in terms of the operator norm as follows:

Definition 1. A sequence of linear operators $T_k \in L(\mathbf{R}^n)$ is said to converge to a linear operator $T \in L(\mathbf{R}^n)$ as $k \to \infty$, i.e.,

$$\lim_{k\to\infty} T_k = T$$

if for all $\varepsilon > 0$ there exists an N such that for $k \geq N$, $\|T - T_k\| < \varepsilon$.

Lemma. For $S, T \in L(\mathbf{R}^n)$ and $\mathbf{x} \in \mathbf{R}^n$,

(1) $|T(\mathbf{x})| \leq \|T\||\mathbf{x}|$

(2) $\|TS\| \leq \|T\|\|S\|$

(3) $\left\|T^k\right\| \leq \|T\|^k$ for $k = 0, 1, 2, \ldots$.

Proof. (1) is obviously true for $\mathbf{x} = \mathbf{0}$. For $\mathbf{x} \neq 0$ define the unit vector $\mathbf{y} = \mathbf{x}/|\mathbf{x}|$. Then from the definition of the operator norm,

$$\|T\| \geq |T(\mathbf{y})| = \frac{1}{|\mathbf{x}|}|T(\mathbf{x})|$$

(2) For $|\mathbf{x}| \leq 1$, it follows from (1) that

$$\begin{aligned} |T(S(\mathbf{x}))| &\leq \|T\||S(\mathbf{x})| \\ &\leq \|T\|\|S\||\mathbf{x}| \\ &\leq \|T\|\|S\|. \end{aligned}$$

Therefore,

$$\|TS\| = \max_{|\mathbf{x}|\leq 1} |TS(\mathbf{x})| \leq \|T\|\|S\|$$

and (3) is an immediate consequence of (2).

Theorem. Given $T \in L(\mathbf{R}^n)$ and $t_0 > 0$, the series

$$\sum_{k=0}^{\infty} \frac{T^k t^k}{k!}$$

is absolutely and uniformly convergent for all $|t| \leq t_0$.

Proof. Let $\|T\| = a$. It then follows from the above lemma that for $|t| \leq t_0$,

$$\left\|\frac{T^k t^k}{k!}\right\| \le \frac{\|T\|^k |t|^k}{k!} \le \frac{a^k t_0^k}{k!}$$

But

$$\sum_{k=0}^{\infty} \frac{a^k t_0^k}{k!} = e^{a t_0}$$

It therefore follows from the Weierstrass M-Test that the series

$$\sum_{k=0}^{\infty} \frac{T^k t^k}{k!}$$

is absolutely and uniformly convergent for all $|t| \le t_0$; cf. [R], p.148.

The exponential of the linear operator T is then defined by the absolutely convergent series

$$e^T = \sum_{k=0}^{\infty} \frac{T^k}{k!}$$

It follows from properties of limits that e^T is a linear operator on $\mathbf{R}^n$ and it follows as in the proof of the above theorem that $\left\|e^T\right\| \le e^{\|T\|}$.

Since our main interest in this chapter is the solution of linear systems of the form

$$\dot{\mathbf{x}} = A\mathbf{x}$$

we shall assume that the linear transformation T on $\mathbf{R}^n$ is represented by the $n \times n$ matrix A with respect to the standard basis for $\mathbf{R}^n$ and define the exponential e^{At}.

Definition 2. Let A be an $n \times n$ matrix. Then for $t \in \mathbf{R}$,

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

For an $n \times n$ matrix A, e^{At} is an $n \times n$ matrix which can be computed in terms of the eigenvalues and eigenvectors of A. This will be carried out
in the remainder of this chapter. As in the proof of the above theorem $\left\|e^{At}\right\| \le e^{\|A\||t|}$ where $\|A\| = \|T\|$ and T is the linear transformation $T(\mathbf{x}) = A\mathbf{x}$.

We next establish some basic properties of the linear transformation e^T in order to facilitate the computation of e^T or of the $n \times n$ matrix e^A.

Proposition 1. If P and T are linear transformations on $\mathbf{R}^n$ and $S = PTP^{-1}$, then $e^S = Pe^T P^{-1}$.

Proof. It follows from the definition of e^S that

$$e^S = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{\left(PTP^{-1}\right)^k}{k!} = P \lim_{n\to\infty} \sum_{k=0}^{n} \frac{T^k}{k!} P^{-1} = Pe^T P^{-1}$$

The next result follows directly from Proposition 1 and Definition 2.

Corollary 1. If $P^{-1}AP = \operatorname{diag}\left[\lambda_j\right]$ then $e^{At} = P \operatorname{diag}\left[e^{\lambda_j t}\right] P^{-1}$.

Proposition 2. If S and T are linear transformations on $\mathbf{R}^n$ which commute, i.e., which satisfy $ST = TS$, then $\mathrm{e}^{S+T} = e^S e^T$.

Proof. If $ST = TS$, then by the binomial theorem

$$(S+T)^n = n! \sum_{j+k=n} \frac{S^j T^k}{j!k!}$$

Therefore,

$$e^{S+T} = \sum_{n=0}^{\infty} \sum_{j+k=n} \frac{S^j T^k}{j!k!} = \sum_{j=0}^{\infty} \frac{S^j}{j!} \sum_{k=0}^{\infty} \frac{T^k}{k!} = \mathrm{e}^S \mathrm{e}^T$$

We have used the fact that the product of two absolutely convergent series is an absolutely convergent series which is given by its Cauchy product; cf. [*R*], p. 74.

Upon setting $S = -T$ in Proposition 2, we obtain

Corollary 2. If T is a linear transformation on $\mathbf{R}^n$, the inverse of the linear transformation e^T is given by $\left(e^T\right)^{-1} = e^{-T}$.

Corollary 3. If

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

then

$$e^A = e^a \begin{bmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{bmatrix}$$

Proof. If $\lambda = a + ib$, it follows by induction that

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}^k = \begin{bmatrix} \operatorname{Re}\left(\lambda^k\right) & -\operatorname{Im}\left(\lambda^k\right) \\ \operatorname{Im}\left(\lambda^k\right) & \operatorname{Re}\left(\lambda^k\right) \end{bmatrix}$$

where Re and Im denote the real and imaginary parts of the complex number λ respectively. Thus,

$$
\begin{aligned}
\mathrm{e}^{A} &= \sum_{k=0}^{\infty} \begin{bmatrix} \mathrm{Re}\left(\frac{\lambda_k}{k!}\right) & -\mathrm{Im}\left(\frac{\lambda^k}{k!}\right) \\ \mathrm{Im}\left(\frac{\lambda^k}{k!}\right) & \mathrm{Re}\left(\frac{\lambda^k}{k!}\right) \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{Re}\left(\mathrm{e}^{\lambda}\right) & -\mathrm{Im}\left(e^{\lambda}\right) \\ \mathrm{Im}\left(e^{\lambda}\right) & \mathrm{Re}\left(e^{\lambda}\right) \end{bmatrix} \\
&= e^{a} \begin{bmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{bmatrix}
\end{aligned}
$$

Note that if $a = 0$ in Corollary 3, then e^A is simply a rotation through b radians.

Corollary 4. If

$$
A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}
$$

then

$$
e^{A} = e^{a} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}
$$

Proof. Write $A = aI + B$ where

$$
B = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}
$$

Then aI commutes with B and by Proposition 2,

$$
e^{A} = e^{aI} e^{B} = e^{a} e^{B}
$$

And from the definition

$$
e^{B} = I + B + B^{2}/2! + \cdots = I + B
$$

since by direct computation $B^2 = B^3 = \cdots = 0$.

We can now compute the matrix e^{At} for any 2×2 matrix A. In Section 1.8 of this chapter it is shown that there is an invertible 2×2 matrix P (whose columns consist of generalized eigenvectors of A) such that the matrix

$$
B = P^{-1} A P
$$

has one of the following forms

$$B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \quad B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad \text{or} \quad B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

It then follows from the above corollaries and Definition 2 that

$$e^{Bt} = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix}, \quad e^{Bt} = e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad e^{Bt} = e^{at}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$$

respectively. And by Proposition 1, the matrix e^{At} is then given by

$$e^{At} = Pe^{Bt}P^{-1}$$

As we shall see in Section 1.4, finding the matrix e^{At} is equivalent to solving the linear system (1) in Section 1.1.

1.4 The Fundamental Theorem for Linear Systems

Let A be an $n \times n$ matrix. In this section we establish the fundamental fact that for $x_0 \in \mathbb{R}^n$ the initial value problem

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} \\ \mathbf{x}(0) &= \mathbf{x}_0. \end{aligned} \tag{1}$$

has a unique solution for all $t \in R^n$ which is given by

$$x(t) = \exp(At)X_0. \tag{2}$$

Notice the similarity in the form of the solution (2) and the solution $x(t) = \exp(At)X_0$ of the elementary first-order differential equation $x' = ax$ and initial condition $\mathbf{x}(0) = \mathbf{x}_0$

In order to prove this theorem, we first compute the derivative of the exponential function e^{At} using the basic fact from analysis that two convergent limit processes can be interchanged if one of them converges uniformly.

Lemma. Let A be a square matrix, then

$$\frac{d}{dt}e^{At} = Ae^{At}.$$

Proof. Since A commutes with itself, it follows from Proposition 2 and Definition 2 in Section 3 that

$$\begin{aligned}\frac{d}{dt}e^{At} &= \lim_{h\to 0}\frac{e^{A(t+h)}-e^{At}}{h}\\ &= \lim_{h\to 0} e^{At}\frac{\left(e^{Ah}-I\right)}{h}\\ &= e^{At}\lim_{h\to 0}\lim_{k\to\infty}\left(A+\frac{A^2h}{2!}+\cdots+\frac{A^kh^{k-1}}{k!}\right)\\ &= Ae^{At}.\end{aligned}$$

The last equality follows since by the theorem in Section 1.3 the series defining e^{Ah} converges uniformly for $|h| \le 1$ and we can therefore interchange the two limits.

Theorem(The Fundamental Theorem for Linear Systems).
Let A be an $n \times n$ matrix. Then for a given $\mathbf{x}_0 \in \mathbf{R}^n$, the initial value problem

$$\begin{aligned}\dot{\mathbf{x}} &= A\mathbf{x}\\ \mathbf{x}(0) &= \mathbf{x}_0\end{aligned} \tag{1}$$

has a unique solution given by

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0. \tag{2}$$

Proof. By the preceding lemma, if $\mathbf{x}(t) = e^{At}\mathbf{x}_0$, then

$$\mathbf{x}'(t) = \frac{d}{dt}e^{At}\mathbf{x}_0 = Ae^{At}\mathbf{x}_0 = A\mathbf{x}(t)$$

for all $t \in \mathbf{R}$. Also, $\mathbf{x}(0) = I\mathbf{x}_0 = \mathbf{x}_0$. Thus $\mathbf{x}(t) = e^{At}\mathbf{x}_0$ is a solution. To see that this is the only solution, let $\mathbf{x}(t)$ be any solution of the initial value problem (1) and set

$$\mathbf{y}(t) = e^{-At}\mathbf{x}(t).$$

Then from the above lemma and the fact that $\mathbf{x}(t)$ is a solution of (1)

$$\begin{aligned}\mathbf{y}'(t) &= -Ae^{-At}\mathbf{x}(t) + e^{-At}\mathbf{x}'(t)\\ &= -Ae^{-At}\mathbf{x}(t) + e^{-At}A\mathbf{x}(t)\\ &= 0\end{aligned}$$

for all $t \in \mathbf{R}$ since e^{-At} and A commute. Thus, $\mathbf{y}(t)$ is a constant. Setting $t = 0$ shows that $y(t) = x_0$ and therefore any solution of the initial value problem (1) is given by $\mathbf{x}(t) = e^{At}\mathbf{y}(t) = e^{At}\mathbf{x}_0$. This completes the proof of the theorem.

Example 1.4.1 *Solve the initial value problem*

$$\dot{\mathbf{x}} = A\mathbf{x}$$

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

for

$$A = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}$$

and sketch the solution curve in the phase plane $\mathbf{R}^2$. By the above theorem and Corollary 3 of the last section, the solution is given by

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 = e^{-2t}\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = e^{-2t}\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}.$$

It follows that $|\mathbf{x}(t)| = e^{-2t}$ and that the angle $\theta(t) = \tan^{-1} x_2(t)/x_1(t) = t$. The solution curve therefore spirals into the origin as shown in Figure 1 below.

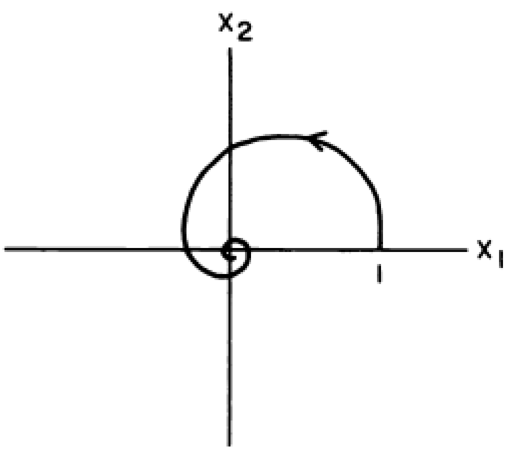

Figure 1

1.5 Linear Systems in R^2

In this section we discuss the various phase portraits that are possible for the linear system

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{1}$$

when $\mathbf{x} \in \mathbf{R}^2$ and A is a 2×2 matrix. We begin by describing the phase portraits for the linear system

$$\dot{\mathbf{x}} = B\mathbf{x} \tag{2}$$

where the matrix $B = P^{-1}AP$ has one of the forms given at the end of Section 1.3. The phase portrait for the linear system (1) above is then obtained from the phase portrait for (2) under the linear transformation of coordinates $\mathbf{x} = P\mathbf{y}$ as in Figures 1 and 2 in Section 1.2. First of all, if

$$B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \quad B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \quad \text{or} \quad B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

it follows from the fundamental theorem in Section 1.4 and the form of the matrix e^{Bt} computed in Section 1.3 that the solution of the initial value problem (2) with $\mathbf{x}(0) = x_0$ is given by

$$\mathbf{x}(t) = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix} \mathbf{x}_0, \quad \mathbf{x}(t) = e^{\lambda t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mathbf{x}_0,$$

or

$$\mathbf{x}(t) = e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix} \mathbf{x}_0$$

respectively. We now list the various phase portraits that result from these solutions, grouped according to their topological type with a finer classification of sources and sinks into various types of unstable and stable nodes and foci:

Case I. $B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ with $\lambda < 0 < \mu$.

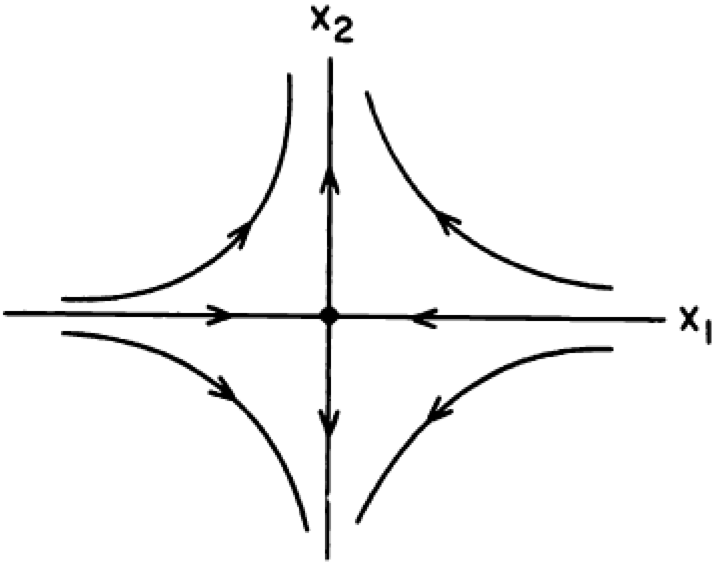

Figure 1. A saddle at the origin.

The phase portrait for the linear system (2) in this case is given in Figure 1. See the first example in Section 1.1. The system (2) is said to have a saddle at the origin in this case. If $\mu < 0 < \lambda$, the arrows in Figure 1 are reversed. Whenever A has two real eigenvalues of opposite sign, $\lambda < 0 < \mu$, the phase portrait for the linear system (1) is linearly equivalent to the phase portrait shown in Figure 1; i.e., it is obtained from Figure 1 by a linear transformation of coordinates; and the stable and unstable subspaces of (1) are determined by the eigenvectors of A as in the Example in Section 1.2. The four non-zero trajectories or solution curves that approach the equilibrium point at the origin as $t \to \pm\infty$ are called separatrices of the system.

Case II. $B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ with $\lambda \leq \mu < 0$ or $B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ with $\lambda < 0$. The phase portraits for the linear system (2) in these cases are given in Figure 2. Cf. the phase portraits in Problems 1(a), (b) and (c) of Problem Set 1 respectively. The origin is referred to as a stable node in each of these cases. It is called a proper node in the first case with $\lambda = \mu$ and an improper node in the other two cases. If $\lambda \geq \mu > 0$ or if $\lambda > 0$ in Case II, the arrows in Figure 2 are reversed and the origin is referred to as an unstable node. Whenever A has two negative eigenvalues $\lambda \leq \mu < 0$, the phase portrait of the linear system (1) is linearly equivalent to one of the phase portraits shown in Figure 2. The stability of the node is determined by the sign of the eigenvalues: stable if $\lambda \leq \mu < 0$ and unstable if $\lambda \geq \mu > 0$. Note that each trajectory in Figure 2 approaches the equilibrium point at the origin along a well-defined tangent line $\theta = \theta_0$, determined by an eigenvector of A, as $t \to \infty$.

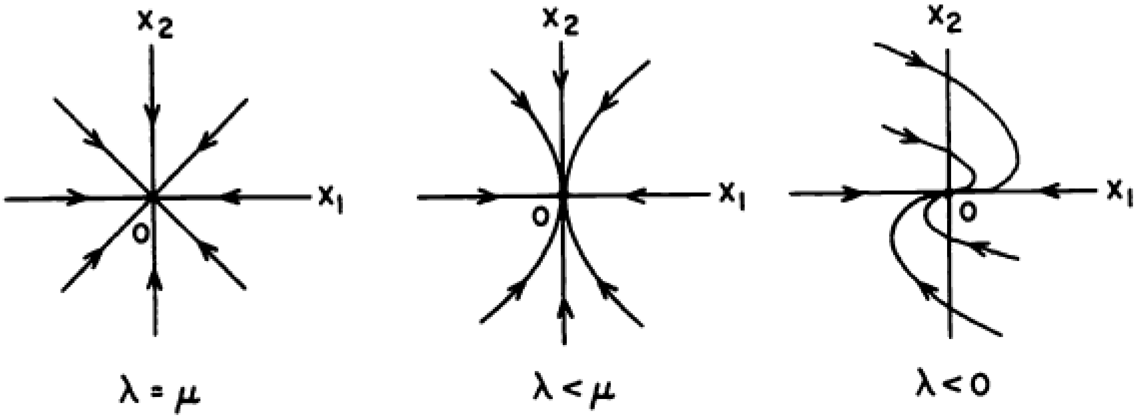

Figure 2. A stable node at the origin.

Case III. $B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ with $a < 0$. $b > 0$ $b < 0$

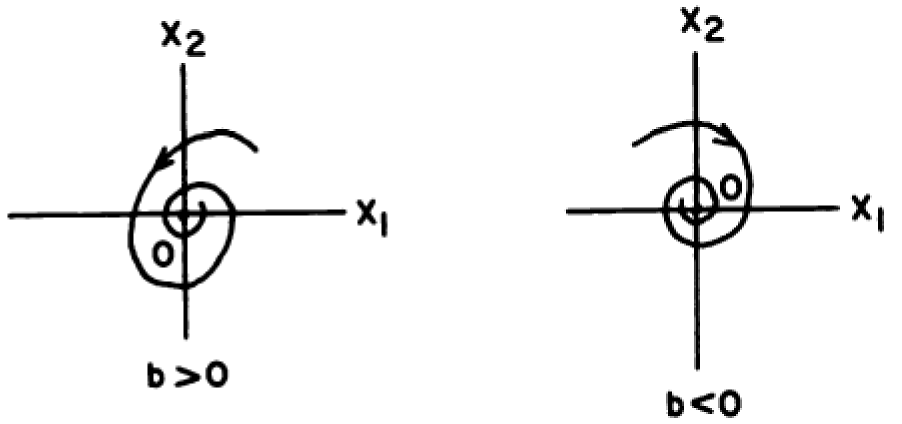

Figure 3. A stable focus at the origin.

The phase portrait for the linear system (2) in this case is given in Figure 3. Cf. Problem 9. The origin is referred to as a stable focus in these cases. If $a > 0$, the trajectories spiral away from the origin with increasing t and the origin is called an unstable focus. Whenever A has a pair of complex conjugate eigenvalues with nonzero real part, $a \pm ib$, with $a < 0$, the phase portraits for the system (1) is linearly equivalent to one of the phase portraits shown in Figure 3. Note that the trajectories in Figure 3 do not approach the origin along well-defined tangent lines; i.e., the angle $\theta(t)$ that the vector $\mathbf{x}(t)$ makes with the x_1-axis does not approach a constant θ_0 as $t \to \infty$, but rather $|\theta(t)| \to \infty$ as $t \to \infty$ and $|\mathbf{x}(t)| \to 0$ as $t \to \infty$ in this case.

Case IV. $B = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}$ The phase portrait for the linear system (2) in this case is given in Figure 4. Cf. Problem 1(d) in Problem Set 1. The system (2) is said to have a center at the origin in this case. Whenever A has a pair of pure imaginary complex conjugate eigenvalues, $\pm ib$, the phase portrait of the linear system (1) is linearly equivalent to one of the phase portraits shown in Figure 4. Note that the trajectories or solution curves in Figure 4 lie on circles $|\mathbf{x}(t)| =$ constant. In general, the trajectories of the system (1) will lie on ellipses and the solution $\mathbf{x}(t)$ of (1) will satisfy $m \leq |\mathbf{x}(t)| \leq M$ for all $t \in \mathbf{R}$; cf. the following Example. The angle $\theta(t)$ also satisfies $|\theta(t)| \to \infty$ as $t \to \infty$ in this case.

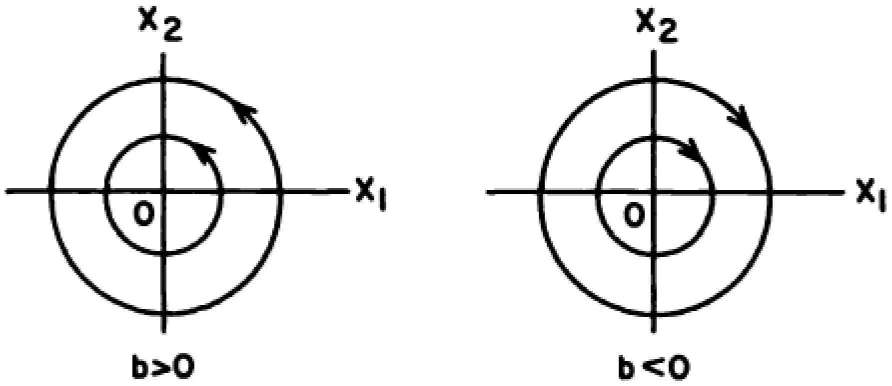

Figure 4. A center at the origin.

If one (or both) of the eigenvalues of A is zero, i.e., if $\det A = 0$, the origin is called a degenerate equilibrium point of (1). The various portraits for the linear system (1) are determined in Problem 4 in this case.

Example 1.5.1 ***(A linear system with a center at the origin)***.
The linear system

$$\dot{\mathrm{x}} = A\mathrm{x}$$

with

$$A = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix}$$

has a center at the origin since the matrix A has eigenvalues $\lambda = \pm 2i$. *According to the theorem in Section 1.6, the invertible matrix*

$$P = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad \textit{with} \quad P^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$$

reduces A to the matrix

$$B = P^{-1}AP = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

The student should verify the calculation. The solution to the linear system $\dot{\mathbf{x}} = A\mathbf{x}$, *as determined by Sections 1.3 and 1.4 , is then given by*

$$\mathbf{x}(t) = P \begin{bmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{bmatrix} P^{-1}\mathbf{c} = \begin{bmatrix} \cos 2t & -2\sin 2t \\ 1/2 \sin 2t & \cos 2t \end{bmatrix} \mathbf{c}$$

where $\mathbf{c} = \mathbf{x}(0)$, *or equivalently by*

$$x_1(t) = c_1 \cos 2t - 2c_2 \sin 2t$$

$$x_2(t) = 1/2 c_1 \sin 2t + c_2 \cos 2t.$$

It is then easily shown that the solutions satisfy

$$x_1^2(t) + 4x_2^2(t) = c_1^2 + 4c_2^2$$

for all $t \in \mathbf{R}$; i.e., the trajectories of this system lie on ellipses as shown in Figure 5.

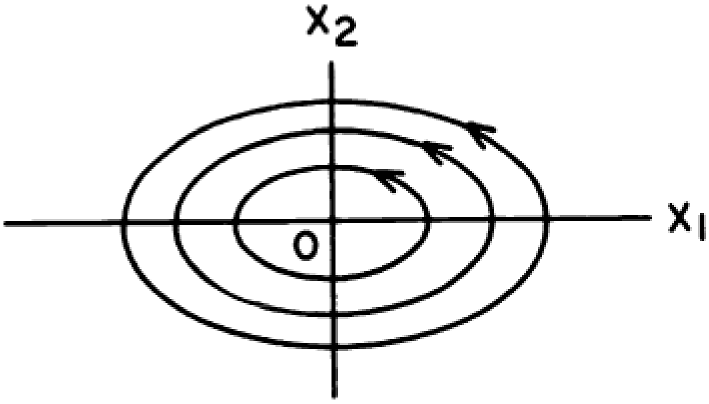

Figure 5. A center at the origin.

Definition 1. The linear system (1) is said to have a saddle, a node, a focus or a center at the origin if the matrix A is similar to one of the matrices B in Cases I, II, III or IV respectively, i.e., if its phase portrait is linearly equivalent to one of the phase portraits in Figures 1, 2, 3 or 4 respectively.

Remark If the matrix A is similar to the matrix B, i.e., if there is a nonsingular matrix P such that $P^{-1}AP = B$, then the system (1) is transformed into the system (2) by the linear transformation of coordinates $\mathbf{x} = P\mathbf{y}$. If B has the form III, then the phase portrait for the system (2) consists of either a counterclockwise motion (if $b > 0$) or a clockwise motion (if $b < 0$) on either circles (if $a = 0$) or spirals (if $a \neq 0$). Furthermore, the direction of rotation of trajectories in the phase portraits for the systems (1) and (2) will be the same if $\det P > 0$ (i.e., if P is orientation preserving) and it will be opposite if $\det P < 0$ (i.e., if P is orientation reversing).

For $\det A \neq 0$ there is an easy method for determining if the linear system has a saddle, node, focus or center at the origin. This is given in the next theorem. Note that if $\det A \neq 0$ then $A\mathbf{x} = 0$ if $\mathbf{x} = 0$; i.e., the origin is the only equilibrium point of the linear system (1) when $\det A \neq 0$. If the origin is a focus or a center, the sign σ of $\dot{x}_2$ for $x_2 = 0$ (and for small $x_1 > 0$) can be used to determine whether the motion is counterclockwise (if $\sigma > 0$) or clockwise (if $\sigma < 0$).

Theorem. Let $\delta = \det A$ and $\tau = \text{trace } A$ and consider the linear system

$$\dot{\mathbf{x}} = A\mathbf{x}. \tag{1}$$

(a) If $\delta < 0$ then (1) has a saddle at the origin.

(b) If $\delta > 0$ and $\tau^2 - 4\delta \geq 0$ then (1) has a node at the origin; it is stable if $\tau < 0$ and unstable if $\tau > 0$.

(c) If $\delta > 0, \tau^2 - 4\delta < 0$, and $\tau \neq 0$ then (1) has a focus at the origin; it is stable if $\tau < 0$ and unstable if $\tau > 0$.
(d) If $\delta > 0$ and $\tau = 0$ then (1) has a center at the origin.

Note that in case (b), $\tau^2 \geq 4|\delta| > 0$; i.e., $\tau \neq 0$.
Proof The eigenvalues of the matrix A are given by

$$\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2}$$

Thus (a) if $\delta < 0$ there are two real eigenvalues of opposite sign.
(b) If $\delta > 0$ and $\tau^2 - 4\delta \geq 0$ then there are two real eigenvalues of the same sign as τ;
(c) if $\delta > 0, \tau^2 - 4\delta < 0$ and $\tau \neq 0$ then there are two complex conjugate eigenvalues $\lambda = a \pm ib$ and, as will be shown in Section 1.6, A is similar to the matrix B in Case III above with $a = \tau/2$; and
(d) if $\delta > 0$ and $\tau = 0$ then there are two pure imaginary complex conjugate eigenvalues. Thus, cases a, b, c and d correspond to the Cases I, II, III and IV discussed above and we have a saddle, node, focus or center respectively.
Definition 2. A stable node or focus of (1) is called a sink of the linear system and an unstable node or focus of (1) is called a source of the linear system.

The above results can be summarized in a "bifurcation diagram," shown in Figure 6, which separates the (τ, δ)-plane into three components in which the solutions of the linear system (1) have the same "qualitative structure". In describing the topological behavior or qualitative structure of the solution set of a linear system, we do not distinguish between nodes and foci, but only if they are stable or unstable. There are eight different topological types of behavior that are possible for a linear system according to whether $\delta \neq 0$ and it has a source, a sink, a center or a saddle or whether $\delta = 0$ and it has one of the four types of behavior determined in Problem 4.

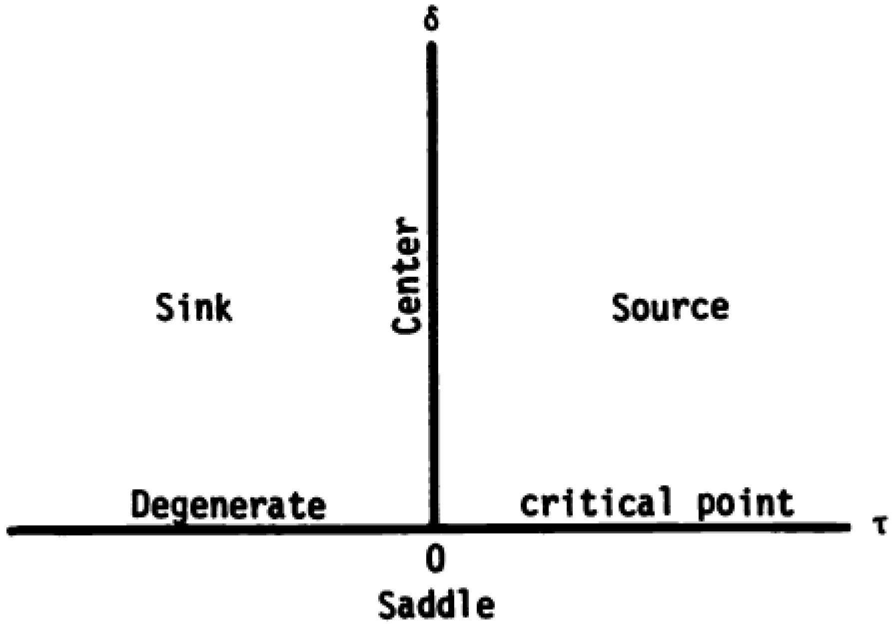

Figure 6. A bifurcation diagram for the linear system (1).

1.6 Complex Eigenvalues

If the $2n \times 2n$ real matrix A has complex eigenvalues, then they occur in complex conjugate pairs and if A has $2n$ distinct complex eigenvalues, the following theorem from linear algebra proved in Hirsch and Smale [H/S] allows us to solve the linear system (1.1).

Theorem. If the $2n \times 2n$ real matrix A has $2n$ distinct complex eigenualues $\lambda_j = a_j + ib_j$ and $\bar{\lambda}_j = a_j - ib_j$ and corresponding complex eigenvectors
$w_j = u_j + iv_j$ and $\bar{w}_j = u_j - iv_j$, $j = 1, \ldots, n$, then $\{u_1, v_1, \ldots, u_n, v_n\}$ is a basis for $\mathbf{R}^{2n}$, the matrix

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{u}_1 & \mathbf{v}_2 & \mathbf{u}_2 & \cdots & \mathbf{v}_\mathrm{n} & \mathbf{u}_n \end{bmatrix}$$

is invertible and

$$P^{-1}AP = \text{ding}\begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix},$$

a real $2n \times 2n$ matrix with 2×2 blocks along the diagonal.

Remark. Note that if instead of the matrix P we use the invertible matrix

$$Q = \begin{bmatrix} \mathbf{u}_1 & \mathbf{v}_1 & \mathbf{u}_2 & \mathbf{v}_2 & \cdots & \mathbf{u}_n & \mathbf{v}_n \end{bmatrix}$$

then

$$Q^{-1}AQ = \operatorname{diag}\begin{bmatrix} a_j & b_j \\ -b_j & a_j \end{bmatrix}.$$

The next corollary then follows from the above theorem and the fundamental theorem in Section 1.4.

Corollary. Under the hypotheses of the above theorem, the solution of the instial value problem

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} \\ \mathbf{x}(0) &= x_0 \end{aligned} \tag{1}$$

is given by

$$\mathbf{x}(t) = P \operatorname{diag} e^{a_j t} \begin{bmatrix} \cos b_j t & -\sin b_j t \\ \sin b_j t & \cos b_j t \end{bmatrix} P^{-1} x_0.$$

Note that the matrix

$$R = \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$$

represents a rotation through bt radians.

Example 1.6.1 *Solve the initial value problem (1) for*

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

The matrix A has the complex eigenvalues $\lambda_1 = 1 + i$ and $\lambda_2 = 2 + i$ (as well as $\bar{\lambda}_1 = 1 - i$ and $\bar{\lambda}_2 = 2 - i$). A corresponding pair of complex eigenvectors is

$$w_1 = u_1 + iv_1 = \begin{bmatrix} i \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } w_2 = u_2 + iv_2 = \begin{bmatrix} 0 \\ 0 \\ 1+i \\ 1 \end{bmatrix}.$$

The matrix

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{u}_1 & \mathbf{v}_2 & \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is invertible,

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$P^{-1}AP = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The solution to the initial value problem (1) is given by

$$\mathbf{x}(t) = P \begin{bmatrix} e^t \cos t & -e^t \sin t & 0 & 0 \\ e^t \sin t & e^t \cos t & 0 & 0 \\ 0 & 0 & e^{2t} \cos t & -e^{2t} \sin t \\ 0 & 0 & e^{2t} \sin t & e^{2t} \cos t \end{bmatrix} P^{-1}\mathbf{x}_0$$

$$= \begin{bmatrix} e^t \cos t & -e^t \sin t & 0 & 0 \\ e^t \sin t & e^t \cos t & 0 & 0 \\ 0 & 0 & e^{2t}(\cos t + \sin t) & -2e^{2t} \sin t \\ 0 & 0 & e^{2t} \sin t & e^{2t}(\cos t - \sin t) \end{bmatrix} \mathbf{x}_0$$

In case A has both real and complex eigenvalues and they are distinct, we have the following result: If A has distinct real eigenvalues λ_j and corresponding eigenvectors $\mathbf{v}_j$, $j = 1, \ldots, k$ and distinct complex eigenvalues $\lambda_j = a_j + ib_j$ and $\bar{\lambda}_j = a_j - ib_j$ and corresponding eigenvectors $\mathbf{w}_j = u_j + i\mathbf{v}_j$ and $\bar{w}_j = u_j - i\mathbf{v}_j$, $j = k+1, \ldots, n$, then the matrix

$$P = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_k & \mathbf{v}_{k+1} & \mathbf{u}_{k+1} & \cdots & \mathbf{v}_n & \mathbf{u}_n \end{bmatrix}$$

is invertible and

$$P^{-1}AP = \operatorname{diag}[\lambda_1, \ldots, \lambda_k, B_{k+1}, \ldots, B_n]$$

where the 2×2 *blocks*

$$B_j = \begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix}$$

for $j = k+1, \ldots, n$. We illustrate this result with an example.

Example 1.6.2 *The matrix*

$$A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

has eigenvalues $\lambda_1 = -3, \lambda_2 = 2 + i$ (and $\bar{\lambda}_2 = 2 - i$). The corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{w}_2 = u_2 + i\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1+i \\ 1 \end{bmatrix}$$

Thus

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$P^{-1}AP = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

The solution of the initial value problem (1) is given by

$$\begin{aligned} \mathbf{x}(t) &= P \begin{bmatrix} e^{-3t} & 0 & 0 \\ 0 & e^{2t}\cos t & -e^{2t}\sin t \\ 0 & e^{2t}\sin t & e^{2t}\cos t \end{bmatrix} P^{-1} x_0 \\ &= \begin{bmatrix} e^{-3t} & 0 & 0 \\ 0 & e^{2t}(\cos t + \sin t) & -2e^{2t}\sin t \\ 0 & e^{2t}\sin t & e^{2t}(\cos t - \sin t) \end{bmatrix} x_0 \end{aligned}$$

The stable subspace E^s is the x_1-axis and the unstable subspace E^u is the x_2, x_3 plane. The phase portrait is given in Figure 1.

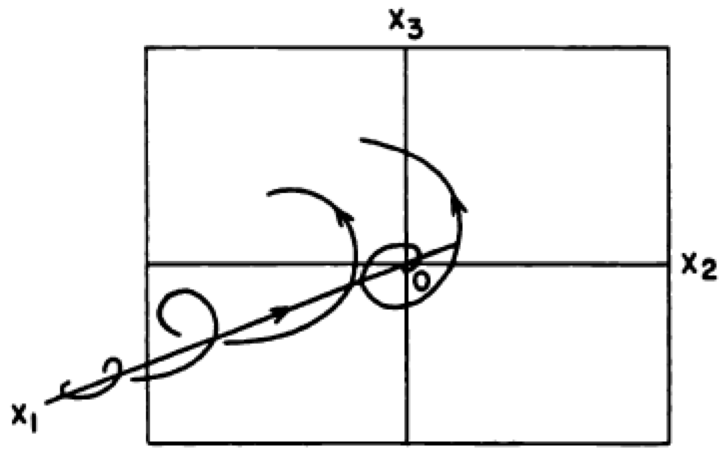

Figure 1

1.7 Multiple Eigenvalues

The fundamental theorem for linear systems in Section 1.4 tells us that the solution of the linear system (1.1) together with the initial condition $\mathbf{x}(0) = x_0$ is given by

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0$$

We have seen how to find the $n \times n$ matrix e^{At} when A has distinct eigenvalues. We now complete the picture by showing how to find e^{At}, i.e., how to solve the linear system (1), when A has multiple eigenvalues.

Definition 1 Let λ be an eigenvalue of the $n \times n$ matrix A of multiplicity $m \leq n$. Then for $k = 1, \ldots, m$, any nonzero solution $\mathbf{v}$ of

$$(A - \lambda I)^k \mathbf{v} = 0$$

is called a generalized eigenvector of A.

Definition 2 An $n \times n$ matrix N is said to be nilpotent of order k if $N^{k-1} \neq 0$ and $N^k = 0$.

The following theorem is proved, for example, in Appendix III of Hirsch and Smale [H/S].

Theorem 1. Let A be a real $n \times n$ matrix with real eigenvalues $\lambda_1, \ldots, \lambda_n$ repeated according to their multiplicity. Then there exists a basis of generalized eigenvectors for $\mathbf{R}^n$. And if $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is any basis of generalized eigenvectors for $\mathbf{R}^n$, the matrix $P = [\mathbf{v}_1 \cdots \mathbf{v}_n]$ is invertible,

$$A = S + N$$

where

$$P^{-1}SP = \text{diag}\left[\lambda_j\right]$$

the matrix $N = A - S$ is nilpotent of order $k \le n$, and S and N commute, i.e., $SN = NS$.

This theorem together with the propositions in Section 1.3 and the fundamental theorem in Section 1.4 then lead to the following result.

Corollary 1. Under the hypotheses of the above theorem, the linear system (1.1), together with the initial condition $\mathbf{x}(0) = \mathbf{x}_0$, has the solution

$$\mathbf{x}(t) = P\,\text{diag}\left[e^{\lambda_j t}\right]P^{-1}\left[I + Nt + \cdots + \frac{N^{k-1}t^{k-1}}{(k-1)!}\right]\mathbf{x}_0$$

If λ is an eigenvalue of multiplicity n of an $n \times n$ matrix A, then the above results are particularly easy to apply since in this case

$$S = \text{diag}[\lambda]$$

with respect to the usual basis for $\mathbf{R}^n$ and

$$N = A - S$$

The solution to the initial value problem (1) together with $\mathbf{x}(0) = \mathbf{x}_0$ is therefore given by

$$\mathbf{x}(t) = e^{\lambda t}\left[I + Nt + \cdots + \frac{N^k t^k}{k!}\right]\mathbf{x}_0$$

Let us consider two examples where the $n \times n$ matrix A has an eigenvalue of multiplicity n. In these examples, we do not need to compute a basis of generalized eigenvectors to solve the initial value problem!

Theorem 2. Let A be a real $2n \times 2n$ matrix with complex eigenvalues $\lambda_j = a_j + ib_j$ and $\bar{\lambda}_j = a_j - ib_j$, $j = 1, \ldots, n$. Then there exists generalized complex eigenvectors $\mathbf{w}_j = \mathbf{u}_j + i\mathbf{v}_j$ and $\overline{\mathbf{w}}_j = \mathbf{u}_j - i\mathbf{v}_j$, $i = 1, \ldots, n$ such that $\{\mathbf{u}_1, \mathbf{v}_1, \ldots, \mathbf{u}_n, \mathbf{v}_n\}$ is a basis for $\mathbf{R}^{2n}$. For any such basis, the matrix $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{u}_1 & \cdots & \mathbf{v}_n & \mathbf{u}_n \end{bmatrix}$ is invertible,

$$A = S + N$$

where

$$P^{-1}SP = \text{diag}\begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix}$$

the matrix $N = A - S$ is nilpotent of order $k \leq 2n$, and S and N commute.

The next corollary follows from the fundamental theorem in Section 1.4 and the results in Section 1.3:

Corollary 2. Under the hypotheses of the above theorem, the solution of the initial value problem (1.1), together with $\mathbf{x}(0) = \mathbf{x}_0$, is given by

$$\mathbf{x}(t) = P \operatorname{diag} e^{a_j t} \begin{bmatrix} \cos b_j t & -\sin b_j t \\ \sin b_j t & \cos b_j t \end{bmatrix} P^{-1} \left[I + \cdots + \frac{N^k t^k}{k!} \right] \mathbf{x}_0$$

1.8 Stability Theory

In this section we define the stable, unstable and center subspace, E^s, E^u and E^c respectively, of a linear system (1.1). Recall that E^s and E^u were defined in Section 1.2 in the case when A had distinct eigenvalues. We also establish some important properties of these subspaces in this section. Let $\mathbf{w}_j = \mathbf{u}_j + i\mathbf{v}_j$; be a generalized eigenvector of the (real) matrix A corresponding to an eigenvalue $\lambda_j = a_j + ib_j$. Note that if $b_j = 0$ then $\mathbf{v}_j = \mathbf{0}$. And let

$$B = \{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \mathbf{v}_{k+1}, \ldots, \mathbf{u}_m, \mathbf{v}_m\}$$

be a basis of $\mathbf{R}^n$ (with $n = 2m - k$) as established by Theorems 1 and 2 and the Remark in Section 1.7.

Definition 1. Let $\lambda_j = a_j + ib_j, w_j = u_j + i\mathbf{v}_j$ and B be as described above. Then

$$E^a = \operatorname{Span}\left\{\mathbf{u}_j, \mathbf{v}_j \mid a_j < 0\right\}$$
$$E^c = \operatorname{Span}\left\{\mathbf{u}_j, \mathbf{v}_j \mid a_j = 0\right\}$$

and

$$E^u = \operatorname{Span}\left\{\mathbf{u}_j, \mathbf{v}_j \mid a_j > 0\right\};$$

i.e., E^s, E^c and E^u are the subspaces of $\mathbf{R}^n$ spanned by the real and imaginary parts of the generalized eigenvectors $\mathbf{w}_j$ corresponding to eigenvalues λ_j with negative, zero and positive real parts respectively.

Definition 2. If all eigenvalues of the $n \times n$ matrix A have nonzero real part, then the flow $e^{At}; \mathbf{R^n} \to \mathbf{R^n}$ is called a hyperbolic flow and (1.1) is called a hyperbolic linear system.

Definition 3. A subspace $E \subset \mathbf{R}^n$ is said to be invariant with respect to the flow $e^{At} : \mathbf{R}^n \to \mathbf{R}^n$ if $e^{At}E \subset E$ for all $t \in \mathbf{R}$.

We next show that the stable, unstable and center subspaces, E^s, E^u and E^c of (1.1) are invariant under the flow e^{At} of the linear system (1.1); i.e., any solution starting in E^u, E^u or E^c at time $t = 0$ remains in E^u, E^u or E^c respectively for all $t \in \mathbf{R}$.

Lemma. Let E be the generalized eigenspace of A corresponding to an eigenvalue λ. Then $AE \subset E$.

Proof. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a basis of generalized eigenvectors for E. Then given $\mathbf{v} \in E$,

$$\mathbf{v} = \sum_{j=1}^{k} c_j \mathbf{v}_j$$

and by linearity

$$A\mathbf{v} = \sum_{j=1}^{k} c_j A v_j$$

Now since each $\mathbf{v}_j$ satisfies

$$(A - \lambda I)^{k_j} \mathbf{v}_j = \mathbf{0}$$

for some minimal k_j, we have

$$(A - \lambda I)\mathbf{v}_j = \mathbf{V}_j$$

where $\mathbf{V}_j \in \operatorname{Ker}(A - \lambda I)^{k_j - 1} \subset E$. Thus, it follows by induction that $A\mathbf{v}_j = \lambda \mathbf{v}_j + \mathbf{V}_j \in E$ and since E is a subspace of $\mathbf{R^n}$, it follows that

$$\sum_{j=1}^{k} c_j A v_j \in E$$

i.e., $A\mathbf{v} \in E$ and therefore $AE \subset$ E.

Theorem 1. Let A be a real $n \times n$ matrix. Then

$$\mathbf{R}^n = E^s \oplus E^u \oplus E^c$$

where E^s, E^u and E^c are the stable, unstable and center subspaces of (1.1) respectively; furthermore, E^a, E^u and E^c are invariant with respect to the flow e^{At} of (1.1) respectively.

Proof. Since $B = \{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \mathbf{v}_{k+1}, \ldots, \mathbf{u}_m, \mathbf{v}_m\}$ described at the beginning of this section is a basis for $\boldsymbol{R^n}$, it follows from the definition of E^s, E^u and E^c that

$$\mathbf{R}^n = E^s \oplus E^u \oplus E^c$$

If $x_0 \in E^s$ then

$$\mathbf{x}_0 = \sum_{j=1}^{n_4} c_j \mathbf{V}_j$$

where $\mathbf{V}_j = \mathbf{v}_j$ or $\mathbf{u}_j$ and $\left\{\mathbf{V}_j\right\}_{j=1}^{\mathbf{n}_*} \subset B$ is a basis for the stable subspace E^s as described in Definition 1 .

Then by the linearity of e^{At}, it follows that

$$e^{At}\mathrm{x}_0 = \sum_{j=1}^{n_4} c_j e^{At}\mathbf{V}_j$$

But

$$e^{At}\mathbf{V}_j = \lim_{k\to\infty}\left[I + At + \cdots + \frac{A^k t^k}{k!}\right]\mathbf{V}_j \in E^*$$

since for $j = 1, \ldots, n_s$ by the above lemma $A^k\mathbf{V}_j \in E^s$ and since E^s is complete. Thus, for all $t \in \mathbf{R}$, $e^{At}x_0 \in E^s$ and therefore $e^{At}E^s \subset E^s$; i.e., E^s is invariant under the flow e^{At}. It can similarly be shown that E^w and E^c are invariant under the flow e^{At}.

We next generalize the definition of sinks and sources of two-dimensional systems given in Section 1.5.

Definition 4. If all of the eigenvalues of A have negative (positive) real parts, the origin is called a sink (source) for the linear system (1.1).

Theorem 2. The following statements are equivalent:

(a) For all $\mathbf{x}_0 \in \mathbf{R}^n$, $\lim_{t\to\infty} e^{At}\mathbf{x}_0 = 0$ and for $\mathbf{x}_0 \neq 0$, $\lim_{t\to-\infty}\left|e^{At}\mathbf{x}_0\right| = \infty$.

(b) All eigenvalues of A have negative real part.

(c) There are positive constants a, c, m and M such that for all $\mathbf{x}_0 \in \mathbf{R}^n$

$$\left|e^{At}\mathbf{x}_0\right| \leq Me^{-ct}\left|\mathbf{x}_0\right|$$

for $t \geq 0$ and

$$\left|e^{At}\mathbf{x}_0\right| \geq me^{-at}\left|\mathbf{x}_0\right|$$

for $t \leq 0$.

Proof (a $\Rightarrow$ b **):** If one of the eigenvalues $\lambda = a + ib$ has positive real part, $a > 0$, then by the theorem and corollary in Section 1.8, there exists an $\mathbf{x}_0 \in \mathbf{R}^n$, $\mathbf{x}_0 \neq \mathbf{0}$, such that $\left|e^{At}\mathbf{x}_0\right| \geq e^{at}\left|\mathbf{x}_0\right|$. Therefore $\left|e^{At}\mathbf{x}_0\right| \to \infty$ as $t \to \infty$ i.e.,

$$\lim_{t\to\infty} e^{At}\mathbf{x}_0 \neq 0.$$

And if one of the eigenvalues of A has zero real part, say $\lambda = ib$, then by the corollary in Section 1.8, there exists $\mathrm{x}_0 \in \mathbf{R}^n$, $\mathbf{x}_0 \neq 0$ such that at least one component of the solution is of the form $ct^k \cos bt$ or $ct^k \sin bt$ with $k \geq 0$. And once again

$$\lim_{t\to\infty} e^{At}\mathbf{x}_0 \neq 0.$$

Thus, if not all of the eigenvalues of A have negative real part, there exists $\mathbf{x}_0 \in \mathbf{R}^n$ such that $e^{A1}\mathbf{x}_0 \nrightarrow \mathbf{0}$ as $t \to \infty$; i.e., a $\Rightarrow$ b. (b $\Rightarrow$ c) : If all of the eigenvalues of A have negative real part, then it follows from the Jordan canonical form theorem and its corollary in Section 1.8 that there exist positive constants a, c, m and M such that for all $\mathbf{x}_0 \in \mathbf{R}^n$ $\left|e^{At}\mathbf{x}_0\right| \leq Me^{-ct}\left|\mathbf{x}_0\right|$ for $t \geq 0$ and $\left|e^{At}\mathbf{x}_0\right| \geq me^{-at}\left|\mathbf{x}_0\right|$ for $t \leq 0$. (c $\Rightarrow$ a):

If this last pair of inequalities is satisfied for all $x_0 \in \mathbf{R}^n$, it follows by taking the limit as $t \to \pm\infty$ on each side of the above inequalities that

$$\lim_{t\to\infty} \left|e^{At}\mathbf{x}_0\right| = 0 \text{ and that } \quad \lim_{t\to-\infty} \left|e^{At}\mathbf{x}_0\right| = \infty$$

for $x_0 \neq 0$. This completes the proof of Theorem 2.
The next theorem is proved in exactly the same manner as Theorem 2 above using the theorem and its corollary in Section 1.8.

Theorem 3. The following statements are equivalent:
(a) For all $\mathbf{x}_0 \in \mathbf{R}^n \cdot \lim_{t\to-\infty} e^{At}\mathbf{x}_0 = 0$ and for $\mathbf{x}_0 \neq 0, \lim_{t\to\infty} \left|e^{At}\mathbf{x}_0\right| = \infty$.
(b) All eigenvalues of A have positive real part.
(c) There are positive constants a, c, m and M such that for all $\mathbf{x}_0 \in \mathbf{R}^n$

$$\left|e^{At}\mathbf{x}_0\right| \leq Me^{ct} \left|\mathbf{x}_0\right|$$

for $t \leq 0$ and

$$\left|e^{At}x_0\right| \geq me^{at} \left|\mathbf{x}_0\right|$$

for $t \geq 0$.
Corollary. If $\mathbf{x}_0 \in E^s$, then $e^{At}\mathbf{x}_0 \in E^s$ for all $t \in \mathbf{R}$ and

$$\lim_{t\to\infty} e^{At}x_0 = 0.$$

And if $\mathrm{x}_0 \in E^u$, then $e^{At}\mathrm{x}_0 \in E^u$ for all $t \in \mathbf{R}$ and

$$\lim_{t\to-\infty} e^{At}x_0 = 0.$$

Thus, we see that all solutions of (1) which start in the stable manifold E^s of (1) remain in E^* for all t and approach the origin exponentially fast as $t \to \infty$; and all solutions of (1) which start in the unstable manifold E^u of (1) remain in E^u for all t and approach the origin exponentially fast as $t \to -\infty$.

1.9 Nonhomogeneous Linear Systems

In this section we solve the nonhomogeneous linear system

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}(t) \tag{1.7}$$

where A is an $n \times n$ matrix and $\mathbf{b}(t)$ is a continuous vector valued function.

Definition. A fundamental matrix solution of (1.1) is any nonsingular $n \times n$ matrix function $\Phi(t)$ that seatisfies

$$\Phi'(t) = A\Phi(t) \text{ for all } t \in \mathbf{R}.$$

Note that according to the lemma in Section 1.4, $\Phi(t) = e^{At}$ is a fundamental matrix solution which satisfies $\Phi(0) = I$, the $n \times n$ identity matrix. Furthermore, any fundamental matrix solution $\Phi(t)$ of (1.7) is given by $\Phi(t) = e^{At}C$ for some nonsingular matrix C. Once we have found a fundamental matrix solution of (1.7), it is easy to solve the nonhomogeneous system (1.1). The result is given in the following theorem.

Theorem 1. If $\Phi(t)$ is any fundamental matrix solution of (1.7), then the solution of the nonhomogeneous linear system (1.1) and the initial condition $\mathbf{x}(0) = \mathbf{x}_0$ is unique and is given by

$$\mathbf{x}(t) = \Phi(t)\Phi^{-1}(0)\mathbf{x}_0 + \int_0^t \Phi(t)\Phi^{-1}(\tau)\mathbf{b}(\tau)d\tau. \tag{1.8}$$

Proof. For the function $\mathbf{x}(t)$ defined above,

$$\begin{aligned} \mathbf{x}'(t) =& \Phi'(t)\Phi^{-1}(0)\mathbf{x}_0 + \Phi(t)\Phi^{-1}(t)\mathbf{b}(t) \\ &+ \int_0^t \Phi'(t)\Phi^{-1}(\tau)\mathbf{b}(\tau)d\tau \end{aligned}$$

And since $\Phi(t)$ is a fundamental matrix solution of (1.1), it follows that

$$\begin{aligned} \mathbf{x}'(t) &= A\left[\Phi(t)\Phi^{-1}(0)\mathbf{x}_0 + \int_0^t \Phi(t)\Phi^{-1}(\tau)\mathbf{b}(\tau)d\tau\right] + \mathbf{b}(t) \\ &= A\mathbf{x}(t) + \mathbf{b}(t) \end{aligned}$$

for all $t \in \mathbf{R}$. And this completes the proof of the theorem.

Remark 1. If the matrix A in (1.7) is time dependent, $A = A(t)$, then exactly the same proof shows that the solution of the nonhomogenous linear system (1.7) and the initial condition $x(0) = x_0$ is given by (1.8) provided that $\Phi(t)$ is a fundamental matrix solution of (1.1) with a variable coefficient matrix $A = A(t)$. For the most part, we do not consider solutions of (1.1) with $A = A(t)$ in this book. The reader should consult [C/L], [H] or [W] for a discussion of this topic which requires series methods and the theory of special functions.

Remark 2. With $\Phi(t) = e^{At}$, the solution of the nonhomogeneous linear system (1.7), as given in the above theorem, has the form

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 + e^{At}\int_0^t e^{-A\tau}\mathbf{b}(\tau)d\tau.$$

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