Chapter 2

Local Bifurcations of Co-dimension 1

Introduction

The theory of bifurcations in dynamical systems aims to describe the changes in the phase portraits of vector fields depending on a parameter $\mu \in \mathbb{R}^k$:

$$\dot{x} = f(x, \mu). \tag{2.1}$$

A value μ^* of the parameter μ is said to be a bifurcation value if the vector field $f(x, \mu^*)$ is not topologically equivalent to $f(x, \mu)$ for all μ in a neighborhood of μ^*.

Definition 2.0.1 (Topological equivalence)**.** *Two dynamical systems $\dot{x} = f(x)$ and $\dot{y} = g(y)$, where f is defined on an open set $U \subset \mathbb{R}^n$, are said to be *topologically equivalent* if there exists a homeomorphism $h : U \to V$ that maps the trajectories of the first system to those of the second, preserving the direction of time.*

If φ and ψ are the flows of the two systems, then for $y = h(x)$, we have:

$$\psi(t, h(x)) = h \circ \varphi(t, x).$$

Definition 2.0.2 (Structural stability)**.** *A vector field $f \in C^1(U)$ is said to be *structurally stable* if there exists $\varepsilon > 0$ such that any $g \in C^1(U)$ satisfying:*

$$\|f - g\| < \varepsilon,$$

is topologically equivalent to f.

2.1 Some Bifurcations in Dimension 1

2.1.1 Saddle-node bifurcation

Consider the differential equation:

$$\dot{x} = \mu - x^2 = f(x, \mu), \tag{2.2}$$

where μ is a real parameter. Three cases arise depending on the value of μ:

1. If $\mu < 0$: there are no equilibrium points, and $\dot{x}$ is always negative.

2. If $\mu = 0$: the origin is the only equilibrium point, with:

$$\dot{x} = -x^2 < 0, \quad \forall x.$$

3. If $\mu > 0$: two equilibrium points exist: $x_1^* = \sqrt{\mu}$ and $x_2^* = -\sqrt{\mu}$. The stability is determined by:

$$\frac{\partial f}{\partial x}(x_1^*) = -2\sqrt{\mu} < 0 \quad \text{(stable)},$$

$$\frac{\partial f}{\partial x}(x_2^*) = 2\sqrt{\mu} > 0 \quad \text{(unstable)}.$$

2.1.2 Supercritical pitchfork bifurcation

Consider the differential equation:

$$\dot{x} = \mu x - x^3 = f(x, \mu),$$

where μ is a real parameter. Three cases are possible:

1. $\mu < 0$: The origin is the only equilibrium point and is stable.
2. $\mu = 0$: The origin remains the only equilibrium point and is stable.
3. $\mu > 0$: Three equilibrium points appear: the origin (unstable), and $x_1^* = \sqrt{\mu}$, $x_2^* = -\sqrt{\mu}$ (both stable).

2.1.3 Transcritical bifurcation

Consider the differential equation:

$$\dot{x} = \mu x + x^2,$$

where μ is a real parameter. Two equilibrium points always exist: 0 and $x^* = -\mu$. We distinguish three cases:

1. $\mu < 0$: The origin is stable, while $x^* = -\mu$ is unstable.
2. $\mu = 0$: The two points merge into a single semi-stable equilibrium point.
3. $\mu > 0$: The origin becomes unstable, while x^* is stable.

2.2 Some Bifurcations in Dimension 2

2.2.1 Saddle-node bifurcation in 2D

Consider the system:

$$\begin{cases} \dot{x} = \mu + x^2, \\ \dot{y} = -y. \end{cases}$$

The system has two equilibrium points for $\mu < 0$:

$$E_1 = (-\sqrt{-\mu}, 0), \quad E_2 = (\sqrt{-\mu}, 0).$$

The Jacobian matrix at a point (x, y) is given by:

$$\mathcal{A}(x, y) = \begin{pmatrix} 2x & 0 \\ 0 & -1 \end{pmatrix}.$$

At the equilibrium points:

$$\mathcal{A}(\pm\sqrt{-\mu},0) = \begin{pmatrix} \pm 2\sqrt{-\mu} & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus, E_1 is a stable node, while E_2 is a saddle (unstable).

For $\mu = 0$, the origin is the only equilibrium point and is semi-stable. For $\mu > 0$, there are no equilibrium points.

The phase portraits for different values of μ are illustrated below:

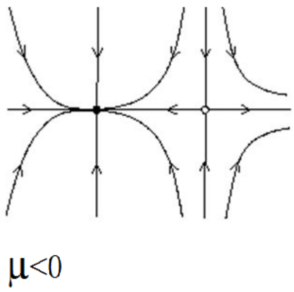


Figure 2.1: Phase portrait for $\mu < 0$.

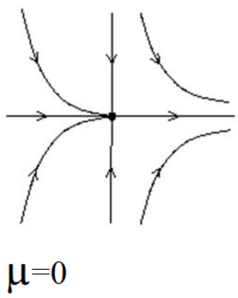


Figure 2.2: Phase portrait for $\mu = 0$.

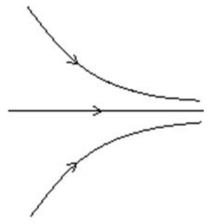


Figure 2.3: Phase portrait for $\mu > 0$.

2.3 Hopf Bifurcation

Consider the differential system:

$$\begin{cases} \dot{x} &= y + x(\mu - x^2 - y^2), \\ \dot{y} &= -x + y(\mu - x^2 - y^2) \end{cases}$$

This system has a unique equilibrium point at the origin, $(0, 0)$, for all values of μ.
In polar coordinates, the system becomes:

$$\begin{cases} \dot{r} &= r(\mu - r^2), \\ \dot{\theta} &= -1 \end{cases}$$

We have three possibilities depending on the sign of μ:

1. $\mu < 0$: In this case, $\dot{r} < 0$ for all $r \neq 0$, and the trajectories spiral toward the origin, which is a stable focus.

2. $\mu = 0$: In this case, $\dot{r} = -r^3 < 0$ for all $r \neq 0$, and the trajectories spiral toward the origin, which remains a stable focus.

3. $\mu > 0$: In this case, the first equation has two equilibrium points: $r = 0$ and $r = \sqrt{\mu}$. The origin is unstable, and the second equilibrium point at $r = \sqrt{\mu}$ is stable. This leads to a closed trajectory, a stable limit cycle.

Theorem 2.3.1. ***Supercritical Hopf Bifurcation***
Consider the differential system:

$$\begin{cases} x' = f(x, y, \mu), \\ y' = g(x, y, \mu), \end{cases}$$

where μ is a real parameter. Suppose that the origin is an equilibrium point for all values of μ. Let the linear part of the system be represented by:

$$\mathcal{A}(0, 0) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} (0, 0).$$

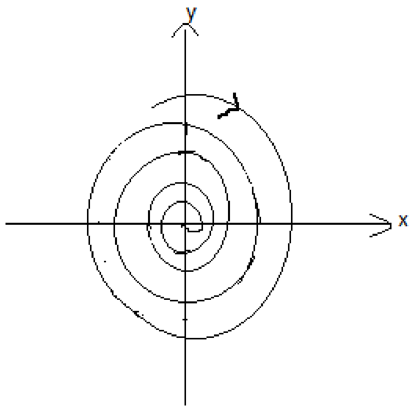


Figure 2.4: $\mu < 0$.

In general, a Hopf bifurcation occurs when the determinant is non-negative and the trace can change sign as μ varies. Let $\lambda_1(\mu), \lambda_2(\mu)$ be the eigenvalues of $\mathcal{A}(0,0)$. We define:

$$\lambda_{1,2}(\mu) = \alpha(\mu) = \pm\beta(\mu).$$

Theorem 2.3.2. ***Hopf Bifurcation***
Suppose the following three conditions hold:

1. *There exists μ_c such that $\alpha(\mu_c) = 0$.*
2. $\beta(\mu_c) \neq 0$.
3. $\frac{\partial\alpha}{\partial\mu}(\mu_c) > 0$.

Then, we can conclude:

a) $\mu = \mu_c$ is a bifurcation value.

b) There exists $\mu_1 \leq \mu_c$ such that for $\mu \in [\mu_1, \mu_c]$, the origin is a stable focus.

c) For any neighborhood of μ around the origin, there exists $\mu_2 > \mu_c$ such that for $\mu \in (\mu_c, \mu_2]$, the origin is an unstable focus surrounded by a stable limit cycle whose amplitude increases as $\sqrt{\mu - \mu_c}$.

2.4 Subcritical Hopf Bifurcation

When the limit cycle that appears is unstable, the origin is unstable. This type of bifurcation is called subcritical. There is a method by Marsden and McCracken to analyze this. Consider the Jacobian matrix:

$$A = \begin{pmatrix} \frac{df}{dx} & \frac{df}{dy} \\ \frac{dg}{dx} & \frac{dg}{dy} \end{pmatrix}.$$

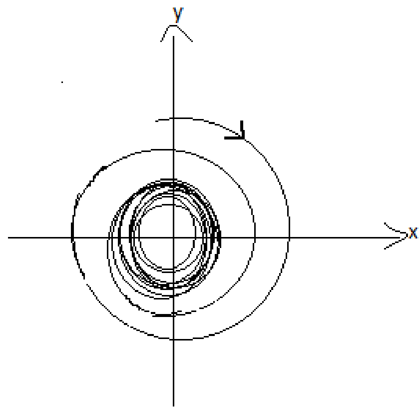


Figure 2.5: $\mu = 0$.

We seek a change of basis matrix P that brings the linear part of the system to Jordan form when $\mu = \mu_c$:

$$D = P^{-1}AP = \begin{pmatrix} 0 & B^* \\ -B^* & 0 \end{pmatrix}.$$

The system in the new coordinates (u, v) can be written as:

$$\begin{cases} u' = h(u, v), \\ v' = k(u, v), \end{cases}$$

where h and k are functions of u and v. The relationship between (u, v) and (x, y) is:

$$\begin{pmatrix} u \\ v \end{pmatrix} = P^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = P \begin{pmatrix} u \\ v \end{pmatrix}.$$

One can then compute an index I called the Marsden-McCracken index:

Definition 2.4.1. *The Marsden-McCracken index is defined by:*

$$I = \beta^*(h_{uuu}+h_{uvv}+k_{uuv}+k_{vvv})+(h_{uu}\times k_{uu}-h_{uu}\times k_{uv}+k_{uu}+k_{uv}+k_{vv}\times k_{uv}-h_{vv}\times h_{uv}-h_{vv}\times k_{vv}).$$

If $I < 0$, the cycle is stable.
If $I > 0$, the cycle is unstable.
If $I = 0$, the bifurcation may be degenerate, and in this case, there is no limit cycle, but there may be a center at the origin at the bifurcation.

In the case of a subcritical bifurcation where $I > 0$, the theorem can be written as: Under the same conditions as in the previous theorem, we conclude that if $\frac{\partial \alpha}{\partial \mu}(\mu_c) < 0$, then:

a) $\mu = \mu_c$ is a bifurcation value.

b) There exists $\mu_1 < \mu_c$ such that for $\mu \in [\mu_1, \mu_c]$, the origin is an unstable focus.

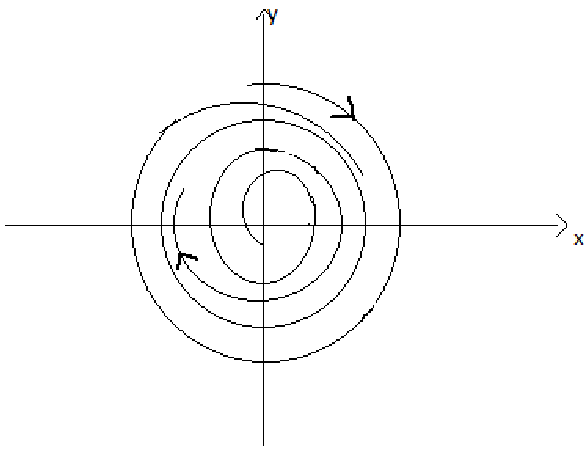
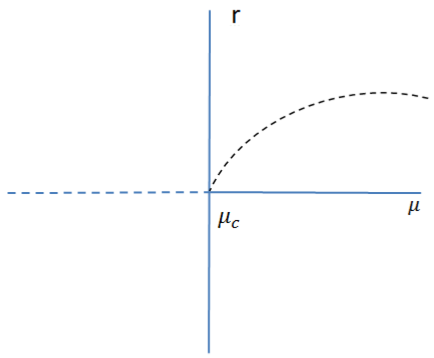


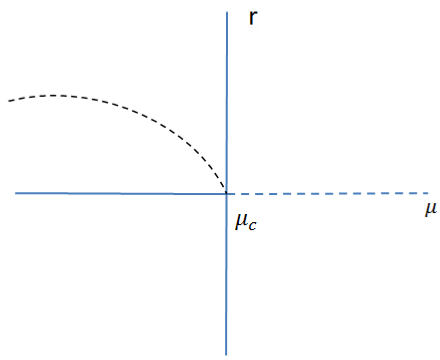
Figure 2.6: $\mu > 0$.

c) There exists $\mu_2 > \mu_c$ such that for $\mu \in (\mu_2, \mu_c]$, the origin is a stable focus surrounded by an unstable cycle whose amplitude increases with $\sqrt{\mu - \mu_c}$.



Subcritical Hopf bifurcation diagram $I > 0$

$$\frac{\partial \alpha}{\partial \mu}(\mu_c) < 0$$



Subcritical Hopf bifurcation diagram $I > 0$:

$$\frac{\partial \alpha}{\partial \mu}(\mu_c) > 0$$