Waves and vibrations

October 2024

Teaching objectives Teaching objectives Introduce the student to the phenomena of mechanical vibrations restricted to low amplitude oscillations for 1 or 2 degrees of freedom as well as to the study of the propagation of mechanical waves. Recommended prior knowledge : 1st year Mathematics and Physics concepts

Contents

List of Figures

List of Tables

Chapter 1

Vibrations

1.1 GENERALITIES “Introduction to Lagrange’s equations

1.1.1 General information on vibrations

Definition of a periodic movement

A movement is said to be periodic if it repeats identically to itself during equal intervals of time.

Examples: The movement of revolution of the Moon: The moon makes a complete cycle of revolution around the earth in approximately 29 days.

Heartbeats: the heartbeat is a succession of contractions and relaxations of the cardiac muscles that activate valves and cause the circulation of blood in the body.

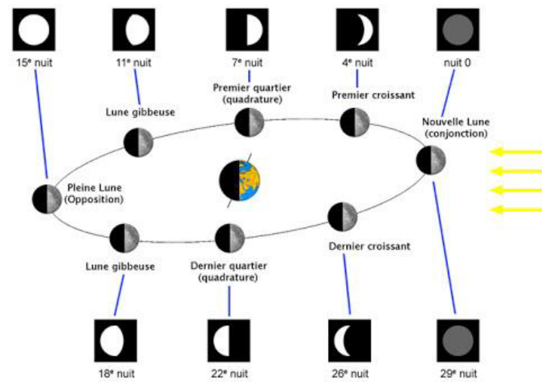

FIGURE 1.1: Movement of revolution of the moon

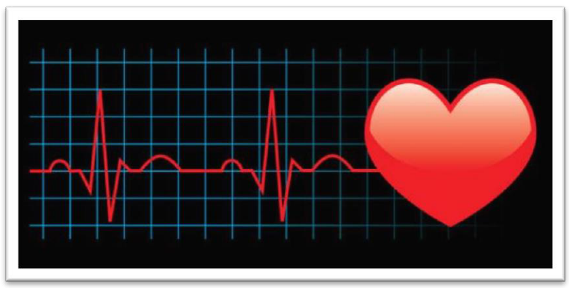

FIGURE 1.2: Electrocardiogram

Definition of an oscillation

Oscillation: any movement of a body that moves alternately from one side to the other from an equilibrium position. Oscillation refers to any repetitive motion around an equilibrium point. It can occur in mechanical systems (like a pendulum) or in other systems, such as electrical circuits (AC current) or biological rhythms. - Type of Motion: Oscillations typically describe smooth, periodic motions where the system swings back and forth in a regular pattern. - Examples: - A pendulum swinging back and forth. - The oscillation of an electrical signal in an alternating current (AC). - Seasonal cycles in nature or heartbeats in biology. - Broader Context : Oscillation can occur in "physical, chemical, electrical, or biological systems". It may or may not involve physical movement; for example, **light waves** oscillate but do not "vibrate" in the mechanical sense.

Subdivided of Oscillations:

Free oscillations an oscillator is free if it oscillates without external interventions (without friction) during its return to equilibrium -Damped oscillations the oscillator is subjected to friction forces that dissipate energy the oscillation damps and eventually stops. -Forced oscillations an oscillator is forced if an external action communicates energy to it. -Damped forced oscillations the external periodic force (excitation) compensates for the losses of snow removal by friction, the oscillations thus maintained do not dampen.

Definition of vibration:

Definition: Vibration specifically refers to the rapid, mechanical oscillations of a material or object. It involves the back-and-forth movement of particles or structures, often in response to an external force. - Type of Motion: Vibration is generally associated with fast, small-amplitude movements around an equilibrium position, often creating sound or heat as energy dissipates. - Examples: - The vibration of a guitar string when plucked. - The shaking of a mobile phone during a notification. - The vibrating motion of an engine or machinery. - Mechanical Nature: Vibration typically involves **mechanical systems** and is usually associated with physical objects. It is a subset of oscillation, specifically relating to mechanical systems.

Key differences:

1. Scope:
   - Oscillation is a more general term and can apply to any repetitive motion (mechanical, electrical, biological, etc.).
   - Vibration specifically refers to mechanical oscillations of a structure or object.

2. Speed and Amplitude:
   - Vibration is often faster and involves smaller movements, whereas oscillation can be slower and cover a larger range of motion.

3. Systems:
   - Oscillation applies to systems as varied as electrical circuits and mechanical objects.
   - Vibration is usually limited to mechanical or physical systems.
   - Oscillation is a general concept of periodic motion, applicable to various systems.
   - Vibration is a specific type of oscillation that refers to the mechanical movement of objects.

Definition of a periodic motion

Periodic motion is a type of motion that repeats itself at regular intervals of time. In other words, an object in periodic motion returns to the same position and state after a fixed time period, known as the period. - Examples: The swinging of a pendulum, the vibration of a guitar string, the orbit of planets around the sun, and the oscillations of a mass on a spring.

Mathematical Form:
- Periodic motion is often described mathematically by sine or cosine functions, which reflect the repetitive nature of the movement. For fast motions we use the frequency (f) expressed in hertz (HZ) it is related to the period by:

$$f = \frac{1}{T} \tag{1.1}$$

The number of revolutions per second is called pulsation ω (noted , measured in rad/s)

$$\omega = 2\pi f = \frac{2\pi}{T} \tag{1.2}$$

1Hz = 1 period per second
1KHz = $10^3 Hz$

$1MHz = 10^6 Hz$

$1GHz = 10^9 Hz$

Note:

1. An oscillator is said to be harmonic if the system evolves according to a periodic law of sinusoidal form (Figure 1-3).

$$x(t) = A\cos(t + \varphi) \tag{1.3}$$

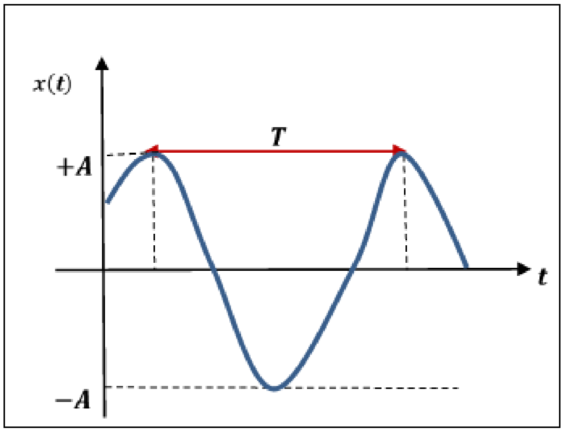

FIGURE 1.3: Representation of a harmonic oscillation

A: Amplitude of the oscillation (Maximum value of the displacement)

ω : Pulsation of the oscillation

φ : the initial phase (t = 0)

Speed: v The speed of the oscillating point M is the derivative of its displacement.

2- The oscillator is said to be non-harmonic if the system evolves according to a periodic law of any non-sinusoidal form figure 1.4.

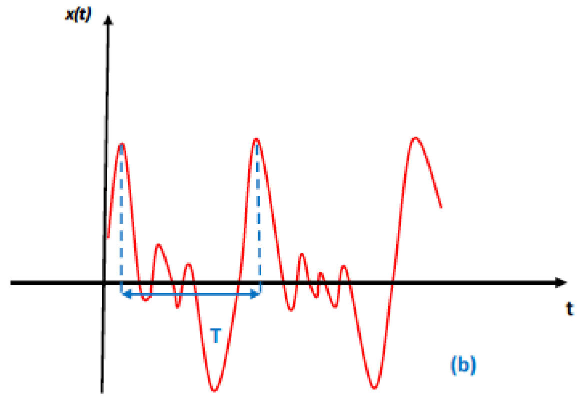

FIGURE 1.4: Representation of an anharmonic oscillation

1.1.2 Concept of energy

Key Concepts of Energy in Vibration and Oscillation:] The concept of energy in vibration and oscillation involves the continuous exchange between two primary forms of mechanical energy: kinetic energy (KE) and potential energy (PE). In systems undergoing vibration or oscillation, energy moves back and forth between these two forms as the object or system moves through its cycle.

Key Concepts of Energy in Vibration and Oscillation:

Total Mechanical Energy:

The total mechanical energy in a vibrating or oscillating system remains constant (assuming no energy loss due to friction or damping). This energy is the sum of kinetic and potential energy at any given point in the motion.

$$E_{Tot} = KE + PE = Const \tag{1.4}$$

Kinetic Energy (KE):

Kinetic energy is the energy of motion. It is at its maximum when the object moves the fastest, which typically occurs when the object passes through its equilibrium

position (the center of its motion).
Formula for kinetic energy:
$$KE = \frac{1}{2}mv^2 \tag{1.5}$$
$$v = \frac{\partial x}{\partial t} = \dot{x} \tag{1.6}$$
$$\Rightarrow KE = \frac{1}{2}m\dot{x}^2 \tag{1.7}$$
Where m is the mass of the object and v is its velocity.
- In an oscillating system, kinetic energy is highest when the velocity is greatest and zero at the turning points of motion.

Potential Energy (PE):

- Potential energy is the energy stored due to the position or configuration of the object. In oscillating systems, this can be elastic potential energy (in springs) or gravitational potential energy (in pendulums). - In an oscillating system, potential energy is maximum when the object is at the extreme points of its motion (the turning points) and zero at the equilibrium position. - Formula for potential energy in a mass-spring system:
$$PE = \frac{1}{2}kx^2 \tag{1.8}$$
Where k is the spring constant and x is the displacement from the equilibrium position.

Energy Exchange:

- During the motion, there is a continuous exchange between kinetic and potential energy.
- At the equilibrium position, the velocity is at its maximum, so kinetic energy is maximum and potential energy is zero.

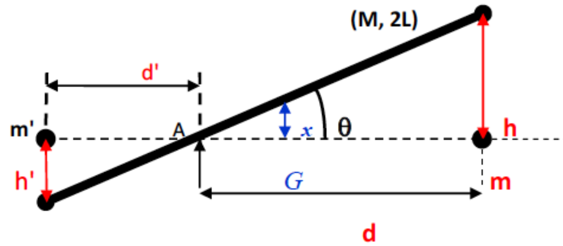

FIGURE 1.5: Double pendulum

- At the extreme points of the motion (maximum displacement), the velocity is zero, so kinetic energy is zero and potential energy is maximum.
- This energy transfer is what drives the oscillation or vibration.
- Damping and Energy Loss:
- In real systems, damping (due to friction or air resistance) can cause energy loss in the form of heat, leading to a gradual decrease in the amplitude of the oscillation.
- In a damped system, total mechanical energy decreases over time, eventually bringing the motion to a stop.

Examples:
- Simple Harmonic Oscillator: In a mass-spring system, the mass moves back and forth, converting potential energy stored in the spring into kinetic energy and vice versa.
- Pendulum: In a pendulum, gravitational potential energy and kinetic energy exchange as the pendulum swings from side to side.
- Vibration of a string: When a guitar string vibrates, the tension in the string stores potential energy, while the motion of the string translates that into kinetic energy.

Practical example Consider a mechanical system figure 1.5, below made up of two point masses (m and m') fixed to the free ends of a rod of mass M and length 2L. This system is a rotational movement relative to or fixed point A. Calculate the kinetic energy and the potential energy of the system: **Solution : The system is made up of 3 masses, so there are 3 kinetic**

energies and 3 potential energies:

1- The kinetic energy KE

$$KE_{Tot} = KE_M + KE_m + KE_{m'}$$

$$KE_M = \frac{1}{2} J_{M/A} \theta^2$$

$$J_{M/A} = \frac{1}{12} M(2L^2) + M(AG)^2$$

$$AG = \frac{L}{2}$$

$$J_{M/A} = \frac{1}{12} M(2L^2) + M(\frac{L}{2})^2 = \frac{7}{12} ML^2$$

$$\Rightarrow KE_M = \frac{1}{2} \frac{7}{12} ML^2 \theta^2$$

$$T_m = \frac{1}{2} J_{m/A} \theta^2$$

$$J_{m/A} = [0 + md^2]$$

$$d = L + \frac{L}{2} = \frac{3L}{2}$$

$$J_{m/A} = m \left(\frac{3L}{2} \right)^2$$

$$\Rightarrow T_m = \frac{1}{2} \left[\frac{4}{9} m \right] L^2 \theta^2$$

$$T_{m'} = \frac{1}{2} J_{m'/A} \theta^2$$

$$J_{m'/A} = [0 + m'd^2]$$

$$d = \frac{L}{2}$$

$$J_{m'/A} = m' \left(\frac{L}{2} \right)^2$$

$$\Rightarrow T_{m'} = \frac{1}{2} \left[\frac{1}{4} m' \right] L^2 \theta^2$$

$\mathrm{T}_{Tot} = \frac{1}{2} \frac{7}{12} M L^2 \theta^2 + \frac{1}{2} \left[\frac{4}{9} m \right] L^2 \theta^2 + \frac{1}{2} \left[\frac{4}{9} m' \right] L^2 \theta^2$

$$\Rightarrow T_{Tot} = \frac{1}{2} \left[\frac{7}{12} M + \frac{9}{4} m + \frac{1}{4} m' \right] L^2 \theta^2$$

2- The Potential energy PE

$$PE_{Tot} = PE_M + PE_m + PE_{m'}$$

$\mathrm{PE}_M = Mgx$

$$x = \frac{1}{2} L \sin\theta$$

$\mathrm{PE}_M = \frac{1}{2} MgL \sin\theta$

$\mathrm{PE}_m = mgh$

h= d$\sin\theta = \frac{3}{2}L\sin\theta$

$\mathrm{PE}_m = \frac{3}{2}mgL\sin\theta$

$$PE_{m'} = -m'gh'$$
$$h' = \frac{d}{3}\sin\theta = \frac{1}{2}L\sin\theta$$

$\mathrm{PE}_{m'} = -\frac{1}{2}m'gL\sin\theta$

$\mathrm{PE}_{Tot} = \frac{1}{2}MgL\sin\theta + \frac{3}{2}mgL\sin\theta - \frac{1}{2}m'gL\sin\theta$

$\mathrm{PE}_{Tot} = \frac{1}{2}gL\sin\theta(M + 3m - m')$

Equilibrium Conditions

The equilibrium condition set if the equilibrium

$$F = 0$$

if

$$x = x_0 \Rightarrow F\Big|_{x=x_0}$$

For a force derived from a potential,

$$F = -\frac{\partial U}{\partial x}$$

The equilibrium condition is written as:

$$\frac{\partial U}{\partial x}\Big|_{x=x_0} = 0 \tag{1.11}$$

There are two types of equilibrium:

1. Stable Equilibrium :
   Once the system is displaced from its equilibrium position, it returns to it. In this case, the restoring force is:

$$f = -C_x$$

with

C $\succ$ 0

$$C = -\frac{\partial f}{\partial x} = -\frac{\partial}{\partial x}\left(-\frac{\partial U}{\partial x}\right) = \frac{\partial^2 U}{\partial x^2} \tag{1.13}$$

$$\frac{\partial U}{\partial x}\Big|_{x=x_0} > 0 \tag{1.14}$$

This condition of stable equilibrium is the condition for oscillation.

2- Unstable Equilibrium :

The system does not return to its equilibrium when displaced. In this case, the unstable equilibrium condition is written as:

$$\frac{\partial U}{\partial x}\Big|_{x=x_0} < 0 \tag{1.15}$$

In the case of rotation, we replace:

$$x \to \theta$$

$$x_0 \to \theta_0$$

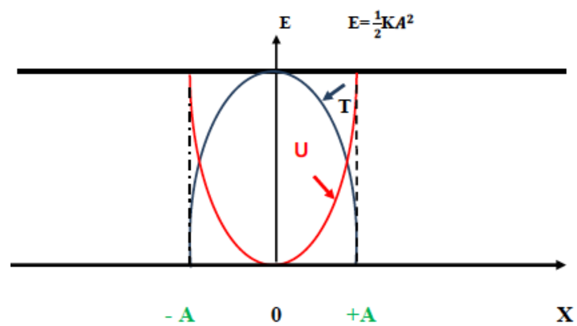

FIGURE 1.6: the variation of energies as a function of displacement

It is possible to graphically represent the evolution of three energies : potential energy kinetic energy and total (mechanical) energy, The figure1-6 . [1]

[1]When kinetic energy decreases, potential energy increases, and the opposite is also true. This phenomenon is known as the conservation of total energy in a system.

1.1.3 Lagrange Formalism

Generalized Coordinates

Generalized coordinates are the set of independent or related real viable coordinates allowing to describe and configure all the elements of a system at any time t.

Exemple

The position of a point M in space can be determined by 3 coordinates Along the axes (x, y, z)
The position of a solid body in space can be defined by six coordinates:

1. 03 coordinates relative to the center of gravity

2. 03 coordinates related to the Euler angles (θ, ϕ, ψ)

We designate by $q_1(t), q_2(t), q_3(t)....q_N(t)$**: The generalized coordinates.**
$\dot{q_1}, \dot{q_2}, \dot{q_3}.....\dot{q_N}$**: The generalized speeds.**

Degree of freedom

This is the number of independent coordinates needed to determine the position of each element of a system during its motion at any time: We write:

$$d = N - r \tag{1.16}$$

with:
d : Degree of freedom
N :Generalized number of coordinates

r :Number of relations, between the generalized coordinates (number of links)
Exemple: let a mechanical system consist of two points M_1 and M_1 connected by a rod of length L. Find the number of degrees of freedom.

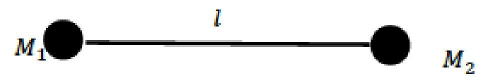

FIGURE 1.7: Exemple

$$\begin{cases} M_1(x_1, y_1, z_1) \\ M_2(x_2, y_2, z_2) \end{cases} \Rightarrow N = 6$$

$$l = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} = Cst \Rightarrow r = 1$$

so :

$$d = N - r = 6 - 1 = 5 \rightarrow d = 5$$

Lagrange formalism

This formalism is based on the Lagrange function

$$L = KE - PE$$

The set of equations of motion is written as:

$$\sum_{i=1}^{n} \left\{ \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \left(\frac{\partial L}{\partial q_i} \right) \right\} = 0 \qquad (1.17)$$

Where:
L: Lagrange Function or Lagrangian;
KE The Kinetic Energy of the System;

PE The Potential Energy of the System;

q_i The generalized coordinate and is the generalized velocity of the system.

For a system with one degree of freedom (N=1 or dof=1)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)-\left(\frac{\partial L}{\partial q}\right)=0$$

For a one-dimensional system, Lagrange's equation is written as :

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)-\left(\frac{\partial L}{\partial x}\right)=0$$

For a rotational movement θ :

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\left(\frac{\partial L}{\partial \theta}\right)=0$$

Chapter 2

Linear systems with a single degree of freedom

2.1 Introduction : Study of undamped free oscillations

A system that oscillates without the influence of any external force is known as a free oscillator. Such systems are considered conservative.

2.1.1 Complex representation and Definition of Fourier series

To facilitate calculations we transform the sinusoidal quantities into exponential form using the Euler form:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

The periodic quantity can be expressed by the sums of the sine and cosine functions in order to manipulate it physically and mathematically. This sum is called a

Fourier series. The Fourier series of a periodic function $f(t)$ with a period T is defined by:

$$f(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)]$$

- Where a_0, a_n, and b_n are the Fourier coefficients.

$$a_0 = \frac{1}{T} \int_0^T f(t)\, dt$$

$$a_n = \frac{2}{T} \int_0^T f(t) \cos(n\omega t)\, dt$$

$$b_n = \frac{2}{T} \int_0^T f(t) \sin(n\omega t)\, dt$$

The angular frequency (ω) is called the fundamental frequency :

$$T = \frac{2\pi}{\omega}$$

- The frequencies ($n\omega$), multiples of ω, are called harmonics.

2.1.2 Study of the mechanical system

To Obtained the differential equation can be determined by: 1. second Newton's law 2. Conservation of energy 3. Lagrange's method.

Example : A mass m attached to the free end of a spring and moving without friction in a vertical direction.

Newton's dynamic principle applied to mass

$$\sum \overrightarrow{F} = m\overrightarrow{a}$$

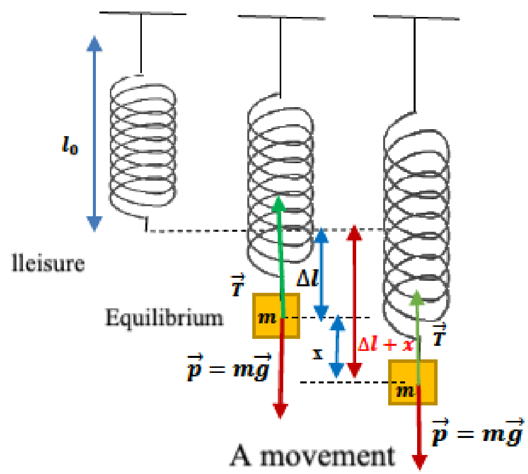

FIGURE 2.1: Harmonic oscillator

Projection on the ox axis:

$$m\vec{g} + \vec{T} = m\vec{a}$$

$$m\ddot{x} = mg - T$$

$$\Leftrightarrow m\ddot{x} = mg - k(x + \Delta l)$$

$$\Leftrightarrow m\ddot{x} = mg - kx - k\Delta l$$

In equilibrium

$$\sum \overrightarrow{F} = \overrightarrow{0}$$

$$m\vec{g} + \vec{T} = \vec{0}$$

$$mg - k(\Delta l) = 0$$

Equilibrium conditions $\Rightarrow$

$$m\ddot{x} = -kx + \underbrace{mg - k\Delta l}_{0}$$

$$m\ddot{x} = -kx$$

$$\Leftrightarrow m\ddot{x} + kx = 0$$

$$\Rightarrow \ddot{x} + \omega_0^2 x = 0$$

This is the differential equation of motion

Energy Conservation

$$E_{Tot} = T + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

In a free system, the mechanical (or total) energy is conserved, thus:

$$\frac{\mathrm{d}E_{Tot}}{\mathrm{d}t} = 0 \Rightarrow m\ddot{x}\dot{x} + k\dot{x}x = 0$$

$$\Rightarrow \ddot{x} + \frac{k}{m}x = 0 \Leftrightarrow \ddot{x} + \omega_0^2 x = 0$$

Lagrange Method

The Lagrangian of the system L is:

$$L = T - U$$

Where T is the kinetic energy and U is the potential energy. So L equal :

$$L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

The Lagrangian formalism:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = 0 \tag{2.1}$$

By substituting (I.18) in .

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = & \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\mathrm{d}}{\mathrm{d}t}(m\ddot{x}) & = m\ddot{x} \\ \frac{\partial L}{\partial x} & = -kx \end{cases}$$

2.1.3 Solution of the Differential Equation

The solution of the equation:

$$\ddot{x} + \frac{k}{m}x = 0 \Leftrightarrow \ddot{x} + \omega_0^2 x = 0$$

is of the form :

$$x(t) = Ae^{rt}$$

where r is a real number and A is a positive constant.

$$\dot{x} = Are^{rt}///\ddot{x} = Ar^2e^{rt} \Leftrightarrow (r^2 - \omega_0^2)Ae^{rt} = 0$$

$$\begin{Bmatrix} r_1 = & i\omega_0 \\ r_2 = & -i\omega_0 \end{Bmatrix}$$

We have two solutions:

$$x_1(t) = A_1e^{r_1} = A_1e^{i\omega_0 t}///x_2(t) = A_2e^{r_2} = A_2e^{-i\omega_0 t}$$

$$x(t) = A(e^{i\omega_0 t} + e^{-}i\omega_0 t)$$

The general solution of the equation of motion :

$$x(t) = x_1(t) + x_2(t) = A(e^{i\omega_0 t} + e^{-}i\omega_0 t)$$

According to Euler's relation :

$$e^{\pm}i\omega_0 t = \cos(\omega_0 t) \pm i\sin(\omega_0 t)$$

SO :

$$x(t) = A_1(\cos(\omega_0 t) + i\sin(\omega_0 t)) + A_2(\cos(\omega_0 t) - i\sin(\omega_0 t))$$

$$x(t) = (A_1 + A_2)\cos(\omega_0 t + (A_1 - A_2)i\sin(\omega_0 t)$$

$$x(t) = B\cos\omega_0 t + C\sin\omega_0 t$$

Where :

$$\begin{cases} B = & \cos\theta \\ C = & \sin\theta \end{cases}$$

So:

$$x(t) = D\cos\theta\cos\omega_0 t + D\sin\theta\sin\omega_0 t = D\cos(\omega_0 t + \varphi)$$

D and φ are constants deduced from the initial conditions. The oscillations are

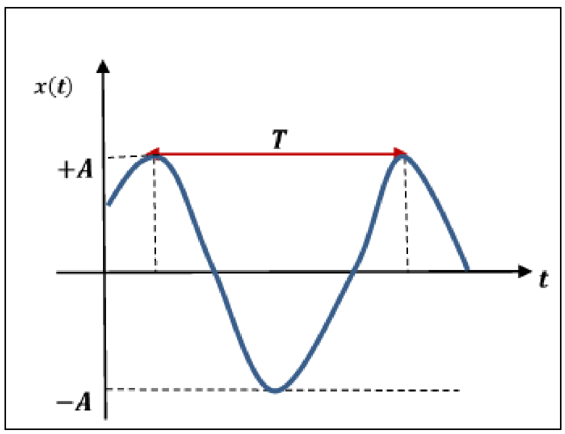

FIGURE 2.2: Sinusoidal motion

sinusoidal in amplitude and natural period:

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$$

Study of the electrical system

Let us consider an electrical circuit $C\ L$

$$\sum_i V_i = 0$$

$$V_C + V_L = 0$$

$$L\frac{di}{dt} + \frac{q}{c} = 0$$

where :

$$i = \frac{dq}{qi} \Rightarrow L\ddot{q} + \frac{q}{c} = 0 \Leftrightarrow \ddot{q} + \frac{1}{LC}q = 0$$

$$\Rightarrow \ddot{q} + \omega_0^2 = 0$$

$$q(t) = Q\cos(\omega_0 t + \varphi)$$

with :

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Energetic Aspects

$$E_m = T + U$$

$$\begin{cases} T = \frac{1}{2}m\dot{x}^2 \\ U = \frac{1}{2}kx^2 \end{cases}$$

$$x(t) = A\cos(\omega_0 t + \varphi)//\dot{x(t)} = -A\omega_0\sin(\omega_0 t + \varphi)$$

$$\begin{cases} T = \frac{1}{2}m(-A\omega_0\sin(\omega_0 t + \varphi))^2 \\ U = \frac{1}{2}k(A\cos(\omega_0 t + \varphi))^2 \end{cases}$$

From an energetic point of view, this oscillator transforms elastic energy into kinetic energy and vice versa.

$$E_m = \frac{1}{2}kA^2\sin^2(\omega_0 t + \varphi) + \frac{1}{2}kA^2\cos^2(\omega_0 t + \varphi) \Leftrightarrow E_m = \frac{1}{2}kA^2$$

$$\omega_0^2 = \frac{k}{m}$$

- The mechanical energy of an oscillator is proportional to the square of the amplitude (see Chapter 1: Presentation of Mechanical Energy).

2.1.4 Electro-mechanical Analogy

By comparing the differential equations of the elastic pendulum and the LC electrical circuit, the following electromechanical analogy can be established:

Potential Energy: corresponds to the energy in the capacitor.

Electrical System	Mechanical System
$\ddot{x} + \frac{k}{m}x = 0$	$\ddot{q} + \frac{1}{LC}q = 0$
Elongation: x	Charge : q
Mass : m	Inductance : L
Ressort : k	Inverse de la capacité $\frac{1}{C}$
Kinetic energy: $\frac{1}{2}m\dot{x}^2$	Energy of the coil $\frac{1}{2}L\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2$
Potential energy: $\frac{1}{2}kx^2$	Capacitor energy $\frac{1}{2}\left[\frac{q^2}{c}\right]$

TABLE 2.1: Electromechanical Analogy (Free Motion)

Chapter 3

Single degree of freedom damped free linear systems

3.1 Introduction to Damped Free Oscillation Types of Friction

"A part of the oscillator's energy is transferred to the external environment (dissipated through friction or radiation). The amplitude of the oscillations decreases over time, and the oscillator eventually comes to a stop."

3.2 Types of Friction

The equations of motion depend on the nature of friction. The solution to the equation of motion is only possible with certain types of friction.

3.2.1 Solid Friction

The force of friction is proportional $\sim$ to the normal reaction of the support.

$$F_f = -sgn(\nu)\mu R \tag{3.1}$$

With:
F : Restoring force $F = -kx$
μ : Coefficient of dynamic friction $\neq$ static friction.

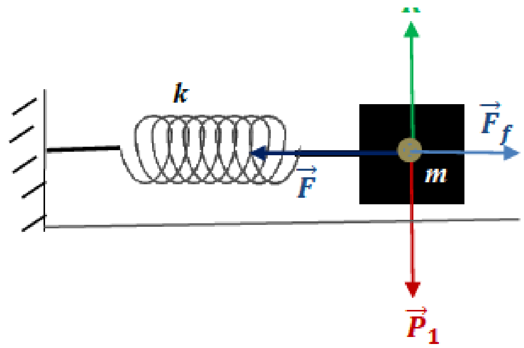

FIGURE 3.1: Solid Friction

3.2.2 Fluid or Viscous Friction

The fluid friction force is proportional $\neq$ to and opposite to the velocity.

$$F_f = -\alpha \upsilon \tag{3.2}$$

With $\alpha \succ 0$

3.2.3 Friction in Highly Viscous Media

The friction in highly viscous media is proportional $\neq$ to the square of the velocity. The equation of motion is non-linear and generally does not have an analytical solution.

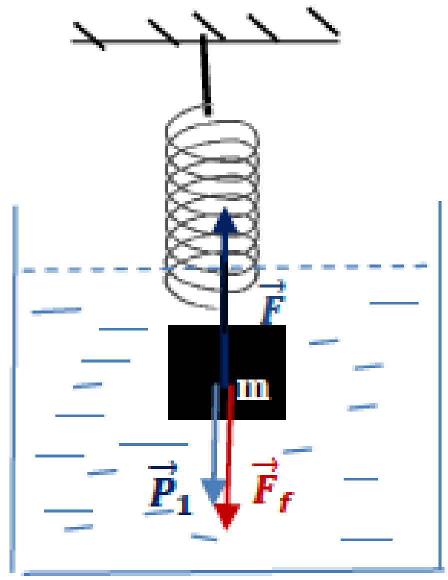

FIGURE 3.2: Viscous Friction

3.2.4 Other Complex Types of Friction

In this course, we will limit ourselves to viscous friction forces that are proportional to velocity. The expression for the viscous friction force is as follows:

$$F_q = -\alpha\dot{q} \tag{3.3}$$

With: α The coefficient of viscous friction $\alpha : [N.\frac{s}{m}$ q : The generalized coordinate of the system $\dot{q}$: The generalized velocity of the system. In a one-dimensional x motion, the force is expressed as:

$$\overrightarrow{f} = -\alpha\overrightarrow{v} = -\alpha\dot{x}\overrightarrow{u}$$

In mechanics, the damper is represented by figure 3.3.

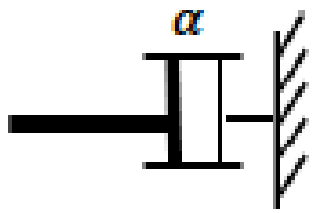

FIGURE 3.3: Represent of damper

3.3 Lagrange's equation in a damped system

"If friction $f = -\alpha\dot{q}$ exists, the Lagrange equation becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = F_q$$

Under the action of friction forces, the system dissipates (loses) mechanical energy in the form of heat, so there is a relationship between the forceF_q and the dissipation function D on one side, and the viscous friction coefficient on the other side α.

$$F_q = -\frac{\partial D}{\partial \dot{q}} \tag{3.4}$$

with :

$$D = \frac{1}{2}\alpha\dot{q}^2$$

The Lagrange equation for a damped system becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = -\frac{\partial D}{\partial \dot{q}}$$

3.4 Mass-Spring-Damper System Differential Equation

3.4.1 The equation of motion

To study the motion of such a system, we can use the fundamental relation of dynamics (FRD) and the Lagrange relations.

A- Fundamental Relation of Dynamics FRD :

$$\sum \overrightarrow{F}_{ext} = m\overrightarrow{\gamma}$$

$$\overrightarrow{P} + \overrightarrow{F}_R + \overrightarrow{F}_\alpha = m\overrightarrow{\gamma}$$

Projection on ox:

$$mg - k(x + x_0) - \alpha\dot{x} = m\ddot{x}$$

$$\underbrace{mg - kx_0}_{=0} + kx - \alpha\dot{x} = m\ddot{x}$$

$$\Rightarrow m\ddot{x} + \alpha\dot{x} + kx = 0$$

$$\ddot{x} + \frac{\alpha}{m}\dot{x} + \frac{k}{m}x = 0$$

This is homogeneous second degree linear differential equation.

B-Lagrange method

Kinetic energy:

$$T = \frac{1}{2}m\dot{x}^2$$

Potential energy:

$$U = \frac{1}{2}kx^2$$

Dissipation function:

$$D = \frac{1}{2}\alpha\dot{x}^2$$

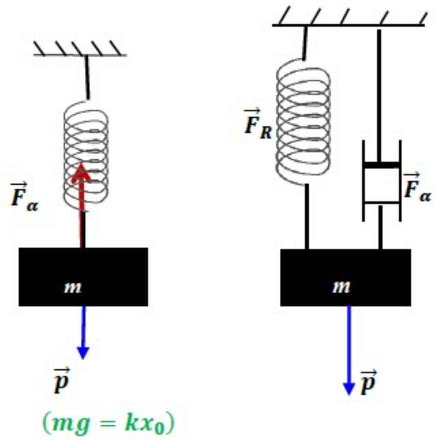

FIGURE 3.4: Mass-spring-damper system In balance and in motion

Lagrange function:

$$L = T - U$$

$$\Rightarrow L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

Lagrange formalism:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = -\frac{\partial D}{\partial \dot{x}}$$

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) & = m\ddot{x} \\ \frac{\partial L}{\partial x} & = -kx \\ \frac{\partial D}{\partial \dot{x}} & = \alpha\ddot{x} \end{cases}$$

By replacing in equation (III.1) we will have:

$$m\ddot{x} + kx = -\alpha\dot{x}$$

$$\Rightarrow \ddot{x} + \frac{\alpha}{m}\dot{x} + \frac{k}{m}x = 0$$

This is the differential equation of motion in the case of a free damped system.

As in the case of the undamped harmonic oscillator, the natural frequency of the system is $\omega_0 = \frac{k}{m}$. However, a new term associated with the damping parameter appears $\left(\frac{\alpha}{m}\right)$. This coefficient is equal to $\lambda = \frac{\alpha}{2m}$, with λ being the damping factor.

Therefore, the equation of motion is written as:

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = 0 \tag{3.5}$$

3.4.2 Solution of the Equation of Motion

Equation (3.5) is a second-order differential equation without an external term. The set of solutions to this equation forms a 2-dimensional vector space. The general solution of this equation can be expressed as a linear combination of two solutions that form a basis. This basis can be found by focusing on exponential time solutions.

$$\begin{cases} x(t) = Ae^{rt} \\ \dot{x(t)} = Are^{rt} \\ \ddot{(x(t))} = Ar^2e^{rt} \end{cases}$$

We substitute the three terms into equation (3.5), which gives:

$$Ar^2e^{-i\omega t} + 2\lambda Are^{-i\omega t} + \omega_0^2 Ae^{-i\omega t} = 0$$

By factoring $Ae^{-i\omega t}$, we obtain: (3.6)

$$Ae^{-i\omega t}(r^2 + 2\lambda r + \omega_0^2) = 0 \tag{3.6}$$

Equation (3.6) is known as the characteristic equation. It is a second-degree equation and can yield either two distinct real roots, a double root (within the real numbers), or two complex roots. The reduced discriminant is calculated as follows:

$$\Delta = \lambda^2 - \omega_0^2$$

Three regimes are to be studied:

Heavily Damped (Aperiodic) Regime

In this regime, the discriminant of the characteristic equation is positive, leading to two distinct real roots. This indicates that the system returns to equilibrium without oscillating, meaning it is aperiodic. The solution to the equation of motion is therefore a sum of two exponentially decaying terms, each associated with one of the real roots.

$$\Delta > 0 \Rightarrow \lambda > \omega_0$$

$$\begin{cases} r_1 = & \frac{-2\lambda+\sqrt{\Delta}}{2} = & -\lambda + \sqrt{\lambda^2 - \omega_0^2} \\ r_2 = & \frac{-2\lambda-\sqrt{\Delta}}{2} = & -\lambda - \sqrt{\lambda^2 - \omega_0^2} \end{cases}$$

$$x(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t} \Rightarrow x(t) = A_1 e^{-\lambda+\sqrt{\lambda^2-\omega_0^2}t} + A_2 e^{-\lambda-\sqrt{\lambda^2-\omega_0^2}t}$$

$$x(t) = e^{-i\omega t}\left[A_1 e^{-\lambda+\sqrt{\lambda^2-\omega_0^2}t} + A_2 e^{-\lambda-\sqrt{\lambda^2-\omega_0^2}t}\right]$$

The coefficients A_1 and A_2 are determined by the initial replacement and speed conditions.

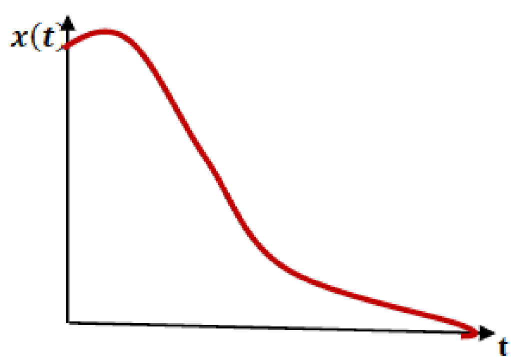

Figure 3.5: The aperiodic regime

By displacing the system from its equilibrium position, it no longer oscillates and comes to a complete stop after a certain amount of time, which depends on the damping coefficient. The larger the damping coefficient, the shorter the stopping time. This regime is called aperiodic, and the damping is heavy.

The term $\sqrt{\lambda^2 - \omega_0^2}$ is not considered an angular frequency because, in the case of a heavily damped regime, there is no oscillation around the equilibrium position.

Critical Aperiodic Regime

This corresponds to the case where the reduced discriminant is zero.

$$\Delta = 0 \Leftrightarrow \lambda = \omega_0$$

The characteristic equation has a real double root.

$$r_1 = r_2 = -\lambda$$

Thus, the function $e^{-\lambda t}$ is a solution of the differential equation.
The second solution can be obtained by noting that $te^{-\lambda t}$ it is also a solution. The general solution is then written as:

$$x(t) = Ate^{-\lambda t} + Be^{-\lambda t} \Rightarrow x(t) = (At + B)e^{-\lambda t}$$

In the case of the "critical aperiodic regime", where the discriminant of the characteristic equation is zero, we have a double real root. For a second-order differential equation of the form:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

its characteristic equation is:

$$r^2 + 2\zeta\omega_0 r + \omega_0^2 = 0$$

where:

- $\omega_0 = \sqrt{\frac{k}{m}}$ is the undamped natural frequency, - $\zeta = \frac{b}{2\sqrt{km}}$ is the damping ratio.

In the critical damping case, $\zeta = 1$, so the discriminant is zero, and the characteristic equation has a double root $r = -\omega_0$. The general solution is then:

$$x(t) = (A + Bt)e^{-\omega_0 t}$$

where A and B are constants determined by initial conditions.

$$\begin{cases} \lambda = & \frac{\alpha}{m} \\ \omega_0 = & \sqrt{\frac{k}{m}} \end{cases} \Rightarrow \lambda = \omega_0 \Rightarrow \frac{\alpha^2}{4m^2} = \frac{k}{m}$$

Where C: critical $\alpha_C = 2\sqrt{km}$. The critical aperiodic regime is the one in which the system returns to its equilibrium position more quickly than in any other aperiodic regime.

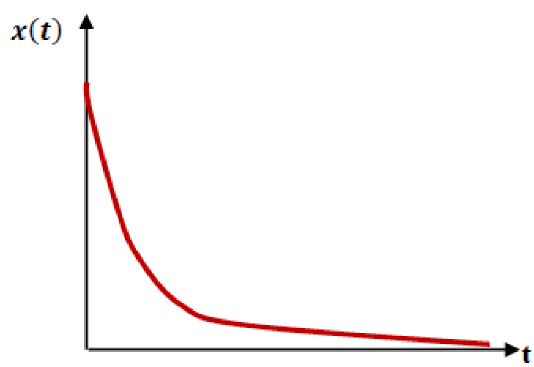

FIGURE 3.6: Critical aperiodic regime

The damping is known as critical damping. The critical regime plays an important role in certain practical applications and the design of measuring instruments, as after a disturbance, the system returns to its rest position as quickly as possible without overshooting it.

Note

For strong damping ($\lambda \geq \omega_0$), the system returns to its equilibrium position without oscillating; thus, a damped oscillator does not always oscillate.

Pseudo-periodic regime

A pseudo-periodic regime is a term often used in the context of dynamic systems, particularly in physics, engineering, and mathematics. It describes a behavior that appears to have a regular, periodic structure but doesn't precisely repeat in the same way as true periodic behavior. In other words, while the system may seem to exhibit a repetitive pattern, the intervals or amplitudes might vary slightly or drift over time, preventing exact repetition.
Corresponds to the case where the reduced discriminant is negative:

$$\Delta < 0 \Leftrightarrow \lambda < \omega_0$$

$$\Delta \acute{=} (-1)(\omega_0^2 - \lambda^2) = i^2(\omega_0^2 - \lambda^2)$$

With:ω_α the damping pulsation or the pseudo-pulsation : $\omega_\alpha = \sqrt{\omega_0^2 - \lambda^2}$ Thus, the solutions to the differential equation are:

$$x(t) = A_1 e^{(-\lambda - i\sqrt{\Delta})t} + A_2 e^{(-\lambda + i\sqrt{\Delta})t}$$

$$x(t) = e^{-\lambda t}(A_1 e^{-i\sqrt{\Delta}t} + A_2 e^{+i\sqrt{\Delta}t})$$

The solution to the equation:

$$x(t) = Ae^{-\lambda t}(A\cos\omega_\alpha t + B\sin\omega_\alpha t)$$

$$x(t) = A\cos(\omega_\alpha t - \varphi)$$

The constants A and φ are determined by the initial conditions. The system performs oscillations of decreasing amplitudes and of "pseudo-periodic regime" given by:

$$T_\alpha = \frac{2\pi}{\omega_\alpha}$$

$$T_\alpha = \frac{2\pi}{\omega_\alpha} \Rightarrow T_\alpha = \frac{2\pi}{\sqrt{\omega_0^2 - \lambda^2}}$$

$$\omega_\alpha^2 = \omega_0^2\sqrt{1-\zeta^2}$$

$$T_\alpha = \frac{2\pi}{\sqrt{\omega_0^2 - \lambda^2}} = \frac{T_0}{\sqrt{1-\zeta^2}}$$

$$T_\alpha = \frac{T_0}{\sqrt{1-\zeta^2}} \Rightarrow T_0 < T_\alpha$$

if $\lambda \ll \omega_0 \Rightarrow \zeta^2 = 1$ so $T - \alpha \simeq T_0$ The curve $x(t)$ is enveloped by the two exponentials $Ae^{-\lambda t}$ and $-Ae^{-\lambda t}$ since, in modulus, $\cos(\omega_\alpha t - \varphi)$ cannot exceed one. We see that x becomes zero as t tends towards its equilibrium position (see figure 3.7). There is a pseudo-frequency, and the motion is described as pseudo-periodic. The damping is weak. It should be noted that the pseudo-frequency ω_α is less than the

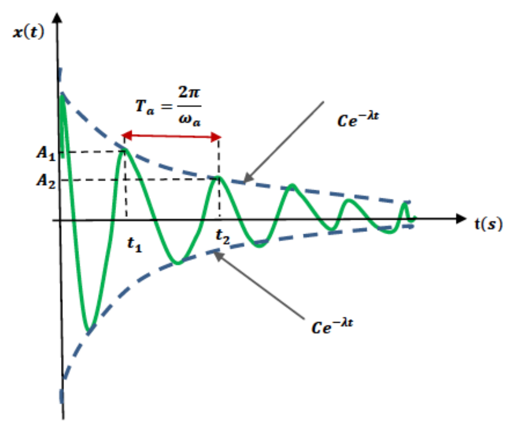

Figure 3.7: Oscillations (pseudo-period)

natural frequency ω_0, and the pseudo-period T_α is greater than the period T_0 of the corresponding undamped oscillator.

In mechanical vibration, a *pseudo-periodic regime* is generally represented by mathematical models that combine periodic and quasi-periodic terms or include non-linear effects. Such models often use coupled differential equations or introduce slowly varying terms to capture deviations from true periodicity. Here are some commonly used equations and formulations for describing pseudo-periodic behavior in mechanical systems:

Harmonic Oscillator with a Slowly Varying Frequency or Amplitude

A basic model for pseudo-periodic vibration can start with a harmonic oscillator with slowly varying parameters:

$$x(t) = A(t)\cos(\omega(t)t + \phi)$$

where: - $A(t)$ is a slowly varying amplitude function, - $\omega(t)$ is a time-dependent angular frequency, - ϕ is a phase constant.

In the pseudo-periodic regime, $A(t)$ and $\omega(t)$ change slowly over time, introducing small variations in the otherwise periodic motion.

Damped Forced Oscillator with Two Competing Frequencies

A damped oscillator driven by two or more incommensurate frequencies can also exhibit pseudo-periodic behavior:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F_1\cos(\omega_1 t) + F_2\cos(\omega_2 t)$$

where: - m is the mass, - c is the damping coefficient, - k is the stiffness, - F_1 and F_2 are forces applied at two different frequencies ω_1 and ω_2.

When ω_1 and ω_2 are not integer multiples of each other, the system will exhibit a pseudo-periodic regime due to the quasi-periodic forcing.

Nonlinear Oscillator with Weak Nonlinearity

A weakly nonlinear oscillator can also exhibit pseudo-periodic behavior, particularly if there's a resonance or near-resonance between different vibrational modes:

$$\frac{d^2x}{dt^2} + \omega_0^2 x + \alpha x^3 = 0$$

where α is a small nonlinear term. This equation is known as the *Duffing equation*.

In the pseudo-periodic regime, nonlinear effects (here through αx^3) introduce slight irregularities in the oscillation patterns, causing a deviation from simple periodicity.

3.4.3 The logarithmic decrement

(or *logarithmic decrease*) of a pseudo-periodic regime is a measure of the rate at which the amplitude of oscillations decays in a damped vibrational system. It quantifies the rate of energy dissipation over each cycle, helping to characterize damping in systems that do not oscillate with a true periodicity.

In the pseudo-periodic regime, where the oscillations are damped but not strictly periodic, the logarithmic decrement can still be applied to approximate the decay behavior. It's defined by the natural logarithm of the ratio of successive peak amplitudes in the damped oscillation.

Definition of Logarithmic Decrement

The logarithmic decrement δ is defined as:

$$\delta = \ln\left(\frac{x(t)}{x(t+T_\alpha)}\right)$$

where: - $x(t)$ is the amplitude at a given time t, - $x(t+T_\alpha)$ is the amplitude after one pseudo-period T_α.

-Equation for Logarithmic Decrement in Damped Systems

For a lightly damped system in a pseudo-periodic regime, the logarithmic decrement can be approximated as:

$$\delta = \frac{2\pi\lambda}{\omega_\alpha}$$

where: - λ is the damping coefficient (associated with the decay rate of the envelope $Ae^{-\lambda t}$), - ω_α is the pseudo-frequency (the damped angular frequency).

- Decay of Amplitude in Pseudo-Periodic Motion

The amplitude $A(t)$ of the pseudo-periodic motion decreases over time according to the exponential envelope:

$$A(t) = A_0 e^{-\lambda t}$$

The motion $x(t)$ in the pseudo-periodic regime is given by:

$$x(t) = A_0 e^{-\lambda t} \cos(\omega_\alpha t + \phi)$$

In this equation: - A_0 is the initial amplitude, - ϕ is the phase shift, - $\cos(\omega_\alpha t + \phi)$ represents the oscillatory component with the pseudo-frequency ω_α.

The logarithmic decrement, δ, provides a way to quantify how quickly the oscillations decay over time and is particularly useful for identifying damping characteristics in lightly damped systems. For a damped system:

$$x(t) = Ce^{-\lambda t} sin(\omega_\alpha t + \varphi)$$

$$\Rightarrow \delta = \ln \frac{Ce^{-\lambda t} sin(\omega_\alpha t + \varphi)}{Ce^{-\lambda(t_1+T\alpha)} sin(\omega_\alpha(t_1 + T_\alpha) + \varphi)}$$

$$\Rightarrow \delta = \ln e^{\lambda T_\alpha} = \lambda T_\alpha$$

$$\lambda T_\alpha = \lambda \frac{T_\alpha}{\sqrt{1-\zeta^2}} = \zeta\omega_0 \frac{T_\alpha}{1-\zeta^2} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

with :

$$\delta = \ln \frac{x(t_1)}{x(t_2)} = 2\pi \frac{\zeta}{\sqrt{1-\zeta^2}} = \lambda T_\alpha$$

Note:

For multiple periods:

$T = T_\alpha(t_1 = t_2 + nT_\alpha) \Rightarrow \delta \ln(\frac{x(t_1)}{x(t_1+nT_\alpha)}) = 2\pi \frac{n\zeta}{\sqrt{1-\zeta^2}}$

The pseudo-period and the logarithmic decrement are only meaningful if the regime is pseudo-periodic

3.4.4 Total Energy of a Damped Harmonic Oscillator

We consider that:

$$x(t) = Ae^{-\lambda t} \sin(\omega t + \varphi)$$

$$\dot{x(t)} = Ae^{-\lambda t}\omega \cos(\omega t + \varphi) - A\lambda e^{-\lambda t} \sin(\omega t + \varphi)$$

In the case of very weak damping $\lambda \to 0$, the pseudo-frequency is approximately equal to the natural frequency of the system, meaning:

$$\omega = \omega_0 \Rightarrow \omega = \sqrt{\frac{k}{m}}$$

The total energy is written as:

$$E_T(T) = U + T \Rightarrow$$

$$E_T(r) = \frac{1}{2}kA^2 e^{-i\omega t} \sin^2(\omega t + \varphi) \frac{1}{2} mA^2 \left[e^{-2\lambda t}\omega^2 \cos^2(\omega t + \varphi) + e^{-2\lambda t} \sin^2(\omega t + \varphi)\right]$$

$$-2\lambda\omega \sin(\omega t + \varphi) \cos(\omega t + \varphi)$$

If we make the second and third terms of the kinetic energy tend towards 0, we obtain for the total energy the following three expressions:

$$E_T(t) = \frac{1}{2}kA^2e^{-2\lambda t}$$

Where:

$$E_T(t) = \frac{1}{2}m\omega_0^2e^{-2\lambda t}$$

3.4.5 The quality factor

In a damped harmonic oscillator, there is a dissipation of mechanical energy. This dissipation is characterized by the coefficient or quality factor,Q, which accounts for the oscillator's efficiency or quality. The quality factor, denoted as Q, is a measure of an oscillator's efficiency in retaining its energy. A high Q-factor indicates that the oscillator effectively conserves its energy, experiencing minimal energy loss per cycle. In a damped harmonic oscillator (such as a pendulum or an oscillating electrical circuit), damping causes a gradual loss of mechanical energy due to friction or resistance. The quality factor Q is inversely related to this energy dissipation: the higher the Q-factor, the better the oscillator can sustain oscillations over a longer period without significant energy loss.

$$Q = 2\pi\frac{E_m}{\Delta E}$$

$$Q = 2\pi\frac{E_T(t)}{\left[\frac{1}{2}kA^2e^{-i\omega t} - \frac{1}{2}kA^2e^{-2\lambda(t+T)}\right]}$$

$$\Rightarrow Q = 2\pi\frac{\frac{1}{2}kA^2e^{-2\lambda t}}{\left[\frac{1}{2}kA^2e^{-i\omega t} - \frac{1}{2}kA^2e^{-2\lambda(t+T)}\right]}$$

$$\Leftrightarrow Q = 2\pi\frac{1}{1 - e^{-2\lambda t}}$$

It is assumed that the damping is very low.
$\lambda \to 0$ and $e^{-2\lambda t} = 1 - 2\lambda T$

$$Q = \frac{\pi}{\lambda T}$$

Either :

$$Q = \frac{\omega}{2\lambda}$$

$$Q = \frac{\omega_0}{2\lambda}$$

$$\Delta \geq \Rightarrow \lambda \geq \omega_0$$

$$Q \leq \frac{1}{2}$$

There are no oscillations, the regime is aperiodic, the pseudo periodic is written:

$$T = \frac{2\lambda}{\omega_0 \sqrt{1 - \frac{1}{4Q^2}}}$$

$$T_0 = \frac{T_a}{\sqrt{1 - \frac{1}{4Q^2}}}$$

3.4.6 Electric Harmonic Oscillator

The oscillating circuit, in addition to having an inductance L and a capacitance C, also includes an ohmic resistance R. In this type of circuit, the inductance L and capacitance C allow oscillations of electric charge or current, while the resistance R introduces damping. This damping causes the oscillations to gradually decrease in amplitude over time, resulting in energy dissipation through the resistor.

$$U_R + U_c + U_L = 0$$

$$Ri(t) + \frac{1}{c}q + L\frac{\mathrm{d}i}{\mathrm{d}t} = 0$$

$$R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{c}q + L\frac{\mathrm{d}^2q}{\mathrm{d}t^2} = 0$$

$$R\dot{q} + \frac{1}{C}q + L\ddot{q} = 0$$

$$\ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC}q = 0$$

$$\Rightarrow \ddot{q} + 2\lambda\dot{q} + \omega_0^2 q = 0$$

With:

$$\begin{cases} \lambda = & \frac{R}{2L} \\ \omega_0^2 = & \frac{1}{LC} \end{cases}$$

SO :

$$\ddot{q} + 2\lambda\dot{q} + \omega_0^2 q = 0 \Rightarrow \begin{cases} \lambda = & \frac{R}{2L} \\ \omega_0 = & \sqrt{\frac{1}{LC}} \end{cases}$$

Note

For critical damping $\lambda = \omega_0 \Rightarrow \frac{R}{2L} = \sqrt{\frac{1}{LC}}$ So : $R = R_c = 2\sqrt{\frac{L}{C}}$

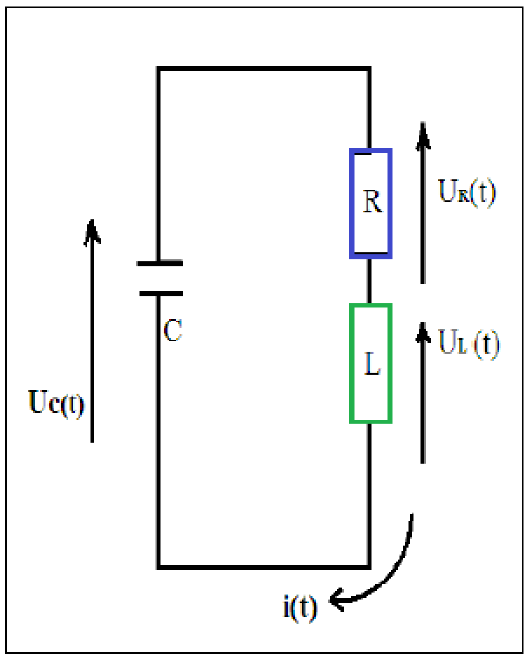

FIGURE 3.8: Electric harmonic oscillator

Feature	Mechanical oscillations	Electrical oscillations
Equation of motion	$m\ddot{x} + \lambda\dot{x} + kx = 0$	$L\ddot{q} + R\dot{q} + \frac{1}{C}q = 0$
Own pulse	$\omega_0 = \sqrt{\frac{k}{m}(rds^{-1})}$	$\omega_0 = \sqrt{\frac{1}{LC}}$
Coefficient of viscous friction	$\lambda(N.s.m^{-1})$	$R(\Omega)$
Damping coefficient	$\alpha = \frac{\lambda}{2m}(s^{-1})$	$\alpha = \frac{R}{2L}$
Damped pulsation	$\omega_\alpha = \sqrt{\frac{k}{m} - \frac{2\lambda}{4m^2}}(rd.s^{-1})$	$\omega_\alpha = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$
Quality factor	$Q = \sqrt{\frac{mk}{\lambda}}$	$Q = \frac{l}{R}\sqrt{\frac{L}{R}}$
Kinetic energy	$T = E_K = \frac{1}{2}m\dot{x}^2(J)$	$E_{BO} = \frac{1}{2}Li^2$
Potential energy	$E_p = \frac{1}{2}k\dot{x}^2(J)$	$E_{co} = \frac{q^2}{C}$
Dissipation function	$D = \frac{1}{2}\alpha\dot{x}^2$	$E_D = \frac{1}{2}Ri^2$

TABLE 3.1: Analogy between mechanical and eclectic oscillations