Chapter 4: Forced linear systems with one degree of freedom.

4.1 Excitation force

We define a forced oscillation as any system in motion under the action of an external force, and call this force the excitation force.

4.2 Lagrange equation of forced systems

If, in addition to friction f= -$\alpha\dot{q}$, there is an external excitation force $(\boldsymbol{t})$,

Lagrange's equation is written :$\boldsymbol{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + \frac{\partial D}{\partial \dot{q}} = F(t)}$ **(4.1)**

So :

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} + \frac{\partial D}{\partial \dot{x}} = F(t)$ (translational movement) **(4.2)**

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} + \frac{\partial D}{\partial \dot{\theta}} = M(t)$ (rotation motion) **(4.3)**

Where: $M(t)$ is the moment of force F(t)

4.3. Equation of motion of forced systems

We define the equation of forced motion in the presence of the friction force as follows:

$\ddot{q} + 2\delta\dot{q} + \omega_0^2 q = F(t)$ **(4.4)**

Where : $F(t)$ is called the external excitation function. This equation is non-homogeneous second order linear with constant coefficients.

We can distinguish two types of excitation, sinusoidal excitation called harmonic excitation and periodic excitation

4.4. Solving the differential equation of motion

The differential equation obtained is therefore a second order linear equation with constant coefficients with second member having the solution q(t).

The solution q(t) of the differential equation which presents the response of the system to the external force is the sum of two terms:

$$\mathbf{q(t)=q_g(t)+q_p(t)} \quad \textbf{(4.5)}$$

Where: $q_g(t)$ and $q_p(t)$ represent respectively the general solution of the homogeneous equation and the particular solution.

• qg(t) is the (transient) solution of the homogeneous equation (without F). It is called transient because it goes out over time.

• (t) is the (permanent) solution of the non-homogeneous equation (with). It is called permanent because it lasts throughout the movement.

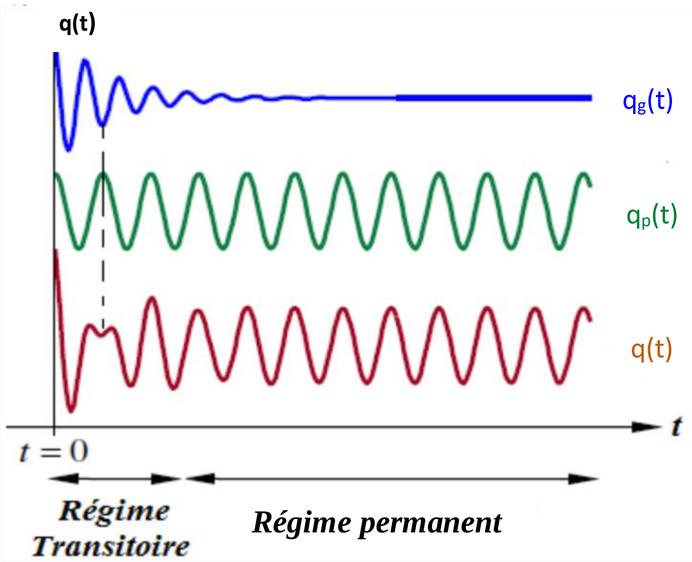


Fig.18.Representation of the general solution and the particular solution.

For sinusoidal excitation of type

$$F(t)=f_0\cos\omega t=f_0e^{j\omega t}$$

The particular solution to the following form

$$q_p(t)=A\cos(\omega t+\varphi)$$

Where **A** is the amplitude, **φ** the phase shift of the total solution**.**

or q(t) is written in the following complex form

$$q_p(t)= Ae^{j(\omega t+\varphi)}$$

We replace in the differential equation of motion

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 q = F(t) \quad \textbf{(4.6)}$$

We have: $\dot{q} = Aj\omega e^{j(\omega t+\varphi)}$

And $\ddot{q} = -A\omega^2 e^{j(\omega t+\varphi)}$

So : $-A\omega^2 e^{j(\omega t+\varphi)} + 2\delta\ Aj\omega e^{j(\omega t+\varphi)} + \omega_0^2\ Ae^{j(\omega t+\varphi)} = f_0 e^{j\omega t}$

$$A[(\omega^2 - \omega_0^2) + 2\delta\ Aj\omega]e^{j\varphi} = f_0$$

By comparing the two sides of the equation we obtain

$$A = \frac{f_0}{\sqrt{(\omega^2-\omega_0^2)^2+4\delta^2\omega^2}} \qquad \textbf{(4.7)}$$

And : $\varphi = Arctg\,\frac{2\delta\omega}{\omega^2-\omega_0^2}$ **(4.8)**

Example

Consider an excitation force of the form F(t)= $f_0\cos\omega t$ applied to the spring mass system.

The equation of motion following the Lagrange formula is given by

$$\boldsymbol{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} + \frac{\partial D}{\partial \dot{x}} = F(t)}$$

Where :L = T-U

The kinetic energy is $T = \frac{1}{2}m\dot{x}^2$

The potentiel energy is U=$\frac{1}{2}kx^2$

So $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$

And $\frac{\partial L}{\partial x} = kx$

Equation of motion is

$$m\ddot{x} + kx = f_0\cos\omega t$$

So

$$\ddot{x} + \frac{k}{m}x = \frac{f_0}{m}\cos\omega t \Leftrightarrow \ddot{x} + \omega_0^2 x = \frac{f_0}{m}\cos\omega t$$

Where the homogeneous equation $\ddot{x} + \omega_0^2 x = 0$

Has the solution $x_g(t) = A_1 cosw_0 t + A_2 cosw_0 t$

On the other hand $\omega \neq \omega_0$

Then the particular solution of the non-homogeneous equation has the following form:

$$x_p(t) = C_1 cosw\ \ t + C_2 sinw\ \ t$$

$$\dot{x}_p(t) = -c_1\omega sin\omega t + C_2\omega cos\omega t$$

$$\ddot{x}_p(t) = -c_1\omega^2 cos\omega t - C_2\omega^2 sin\omega t$$

We replace in the differential equation we obtain:

$$C_1 = \frac{\frac{f_0}{m}}{\omega_0^2 - \omega^2}$$

$$C_2 = 0$$

So : $x_p(t) = \frac{\frac{f_0}{m}}{\omega_0^2 - \omega^2} cos\omega t$

We define the static deflection of the mass under an excitation force by the following factor:

$\boldsymbol{\delta_{st} = \frac{f_0}{k}}$ **(4.9)**

So we can write the solution to the differential equation of motion in the following form:

$x_p(t) = \frac{\delta_{st}}{1-(\frac{\omega}{\omega_0})^2} cos\omega t$

4.5. Amplification factor:

The amplification factor is given by the ratio $\frac{A}{\delta_{st}}$ **(4.10)**

Where: $A = \frac{\delta_{st}}{1-(\frac{\omega}{\omega_0})^2}$

So : $\frac{A}{\delta_{st}} = \frac{1}{1-(\frac{\omega}{\omega_0})^2}$

We distinguish three cases depending on the value of $\frac{A}{\delta_{st}}$

- $\frac{A}{\delta_{st}} > 0 \Rightarrow \frac{\omega}{\omega_0} < 1 \Rightarrow x(t) et\ f(t)$ are in phase
- $\frac{A}{\delta_{st}} < 0 \Rightarrow \frac{\omega}{\omega_0} > 1 \Rightarrow x(t) et\ f(t)$ in phase opposition
- $\frac{A}{\delta_{st}} \rightarrow \Rightarrow \frac{\omega}{\omega_0} = 1 \Rightarrow$ in resonance

4.6. Resonance pulse:

The excitation pulse ω for which the amplitude A reaches its maximum is called the resonance pulse ω_r. A is maximum when

$\frac{\partial A}{\partial \omega}=0$

$$\frac{\partial A}{\partial \omega} = \frac{\partial}{\partial \omega}\frac{f_0}{\sqrt{(\omega^2-\omega_0^2)^2+4\delta^2\omega^2}}$$

$$=\frac{f_0(4\omega(\omega^2-\omega_0^2)+8\delta^2\omega)}{2\sqrt{(\omega^2-\omega_0^2)^2+4\delta^2\omega^2}}$$

$\frac{\partial A}{\partial \omega}=0 \Rightarrow f_0(4\omega(\omega^2-\omega_0^2)+8\delta^2\omega)=0$

$$\Rightarrow \begin{cases} \omega \equiv \omega_r = 0 \\ ou \\ \omega \equiv \omega_r = \sqrt{\omega_0^2 - 2\delta^2} \end{cases} \quad \textbf{(4.11)}$$

So the maximal amplitude is :

$$A_{max} = \frac{f_0}{\sqrt{4\delta^2\omega^2 - 4\delta^4}} \quad \textbf{(4.12)}$$

For there to be resonance, it is necessary that:

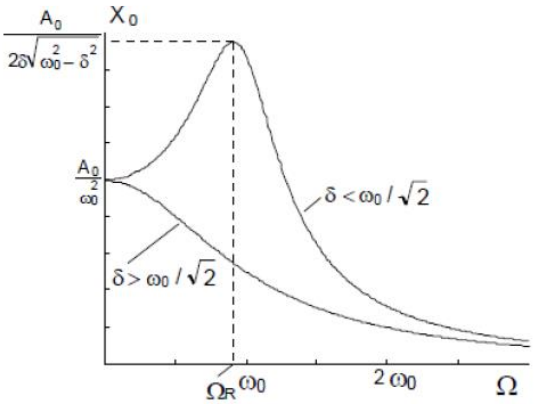
$$\omega_0^2 - 2\delta^2 > 0 \Rightarrow 1 - \frac{1}{2Q^2} > 0$$

$$\Rightarrow Q > \frac{1}{\sqrt{2}}$$

That means that the quality factor is $Q > \frac{1}{\sqrt{2}} \Rightarrow$ Low damping

The variation of the amplitude A and the phase shift φ are presented in the following figures

ω



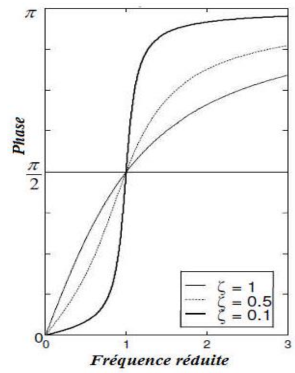


Fig.19. Amplitude as a function of ω **Fig. 20.**Shiftphase as fonction of ω

4.7.Bandwidth:

In the case of a sinusoidal excitation of variable pulsation ω and in the case where $\delta<\frac{\omega_0}{\sqrt{2}}$, we define the pulsation bandwidth of the oscillator by the interval:$\Delta\omega = \omega_2 - \omega_1$ (avec $\omega_2 > \omega_1$)

Where the pulsations ω_1 and $\omega_2$correspond to the amplitudes A(ω_1) and A(ω_2) such that

$$A(\omega_1)=A(\omega_2)=\frac{A_{(\omega_r)}}{\sqrt{2}} \qquad \textbf{(4.13)}$$

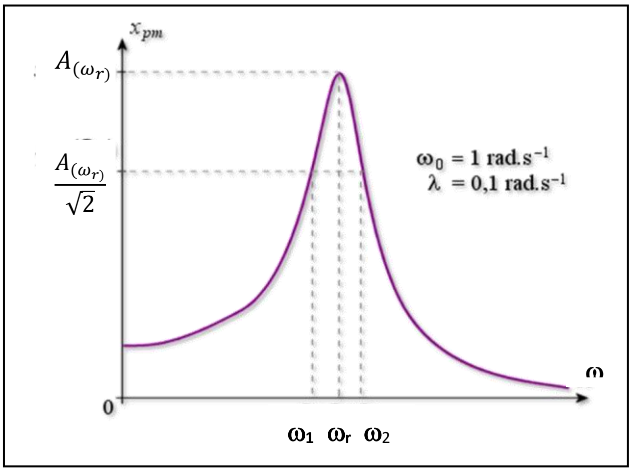


Fig.21. Bandwidth

Then the bandwidth is given by:

$$B = \Delta\omega = \omega_2 - \omega_1 = 2\delta \qquad \textbf{(4.14)}$$

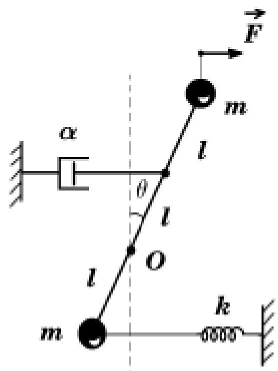
4.8. The quality factor

The quality factor is defined by the ratio of the specific pulsation to the bandwidth

$$Q = \frac{\omega_0}{B} = \frac{\omega_0}{2\delta} \qquad \textbf{(4.15)}$$

Exemple :

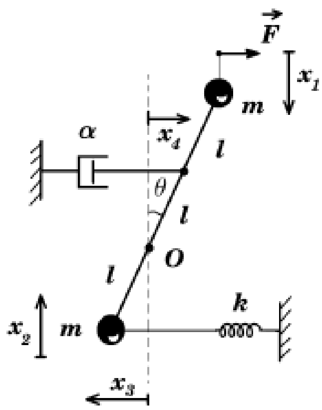
The system opposite is forced to oscillate around the vertical, which is the position of equilibrium, by a sinusoidal force F which remains horizontal during movement. It is given by $F=F_0 \cos\omega t$ and the positive direction is chosen towards the right. The coefficient of friction is α.



It is assumed that the system oscillates at small angles.

1. Express and simplify the expression for the potential energy U.
2. Give the expression for the kinetic energy T of the system.
3. Give the expression for the Lagrangian of the system and deduce the equation of motion.
4. Give the permanent solution. Specify its amplitude and phase.

Answer :



The potential energy of the system is given by:

$$U = \frac{1}{2}K(x_3 + x_0)^2 + mgx_2 - mgx_1$$

Où

$$x_1 \approx l\theta^2$$

$$x_2 \approx \frac{1}{2}l\theta^2$$

$x_3 \approx -l\theta$

So :

$U = \frac{1}{2}K(x_0 - l\theta)^2 + \frac{1}{2}mgl\theta^2 - mgl\theta^2$

$\Rightarrow$

$U = \frac{1}{2}(Kl - mg)l\theta^2 - Klx_0\theta + \frac{1}{2}Kx_0{}^2$

At equilibrium $\frac{\partial U}{\partial \theta} = 0 \Rightarrow (Kl - mg)l\theta - Klx_0 = 0$

For $\theta = 0 \Rightarrow x_0 = 0$

So $U = \frac{1}{2}(Kl - mg)l\theta^2$

The kinetic enrgy is :

$T = \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}(ml^2 + m(2l)^2)\dot{\theta}^2 = \frac{5}{2}ml^2\dot{\theta}^2$

The lagrangien is :

L=T-U=$\frac{5}{2}ml^2\dot{\theta}^2 - \frac{1}{2}(Kl - mg)l\theta^2$

In case of existance of friction force , the dissipation energy is given by

$$D = \frac{1}{2}\alpha\dot{\theta}^2$$

On other hand existance of external force , we add the following terme to teh equation of motion :

$$F = F_0 cos\omega t$$

So the equation of motion is :

$$\ddot{\theta} + 2\delta\dot{\theta} + \omega_0^2\theta = bcos\omega t$$

Where :$\delta = \frac{\alpha}{10m}\,\omega_0^2 = \frac{Kl - mg}{5l}$ et $b = \frac{-2F_0}{5ml}$

The particular solution of this equation has the following form:

$$x_p(t) = a\cos(\omega t + \varphi)$$

Or we use the complex notation

$$x_p(t) = a\cos(\omega t + \varphi) \leftrightarrow X = ae^{i\omega t}$$

$$bcos\omega t \leftrightarrow be^{i\omega t}$$

By replacement in the differential equation we obtain
$(\omega_0^2 - \omega^2 + 2\delta\omega)$A=b⇒A=$\frac{b}{(\omega_0^2-\omega^2+2\delta\omega)}$

We can whrite

$$(\omega_0^2 - \omega^2 + 2\delta\omega = ((\omega_0^2 - \omega^2)^2 + (2\delta\omega)^2)^{\frac{1}{2}}e^{i\varphi}$$

where :$\varphi = arc\ tg\,\frac{2\delta\omega}{\omega_0^2-\omega^2}$donc la solution X s'écrit :

$$X = \frac{b}{((\omega_0^2 - \omega^2)^2 + (2\delta\omega)^2)}e^{i(\omega t-\varphi)}$$

$$=\frac{\frac{-2F_0}{5ml}}{((\omega_0^2-\omega^2)^2+(2\delta\omega)^2)}e^{i(\omega t-\varphi)} = \frac{\frac{2F_0}{5ml}}{((\omega_0^2-\omega^2)^2+(2\delta\omega)^2)}e^{i(\omega t+\pi-\varphi)}$$

The particular solution is given by the real part of

$$x_p(t) = a\cos(\omega t + \varphi)\begin{cases} a = \dfrac{\frac{2F_0}{5ml}}{((\omega_0^2 - \omega^2)^2 + (2\delta\omega)^2)^{\frac{1}{2}}} \\ \varphi = \pi - arc\ tg\,\dfrac{2\delta\omega}{\omega_0^2 - \omega^2} \end{cases}$$