Chapter 2: Free Linear Systems one-degree of freedom.

2.1. Free oscillations

- A system oscillating in the absence of any **excitation** force is called a free oscillator.
- The number of independent quantities involved in the movement is called the degree of freedom.
- We call a harmonic oscillator as soon as it is separated from its equilibrium position by a distance x (or angle θ), is subjected to a restoring force opposite and proportional to the separation x (or θ):

$$\boldsymbol{F} = -\boldsymbol{Cx} \qquad (2.1)$$

2.2. Linear oscillator

Vibratory motion is said to be linear if it is governed by a differential equation of the form:

$$\ddot{q} + \omega_0^2 \, q = 0 \qquad (2.2)$$

2.3. Equilibrium conditions

- The equilibrium condition is F=0. If the equilibrium is at $q=q_0$ we write

$$F\big]_{q=q_0} = 0 \qquad \textbf{(2.3)}$$

For a force derived from a potential $(-\frac{\partial U}{\partial q})$, the equilibrium condition is written:

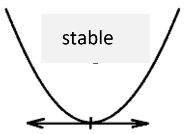
$$\frac{\partial \boldsymbol{U}}{\partial \boldsymbol{q}}\bigg]_{\boldsymbol{q}=\boldsymbol{q_0}} > 0 \qquad \textbf{(2.4)}$$

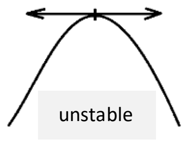
The stability condition (the equilibrium is stable if, once the system moves away from its equilibrium position, it returns to its equilibrium) is given by

$$\frac{\partial^2 U}{\partial q^2} > 0 \qquad \textbf{(2.5)}$$

The equilibrium of a system is unstable if the system does not regain its equilibrium during a deviation, i.e. if C <0. The unstable equilibrium condition is written as

$$\frac{\partial^2 U}{\partial q^2} < 0 \quad \textbf{(2.6)}$$





2.4 The energy of a harmonic oscillator

The energy of an harmonic oscillator is the sum of its kinetic and potential energies: $E=T+U$

- The translational kinetic energy of a body of mass m and velocity v is:

  $$T_{translation} = 1/2\, mv^2 \quad \textbf{(2.7)}$$

- The kinetic energy of rotation of a Body of moment of inertia I_Δ about an axis Δ and angular velocity $\dot{\theta}$ **is:**

  $$T_{rotation} = \frac{1}{2} I_\Delta \dot{\theta}^2 \quad \textbf{(2.8)}$$

- The potential energy of a mass m in a constant gravitational field g is **:**

  $$U = mgh \quad \textbf{(2.9)}$$

  (or $U=-mgh$ in the case of a descent of height h**)**

- The potential energy of a spring of stiffness k at a deformation x is :

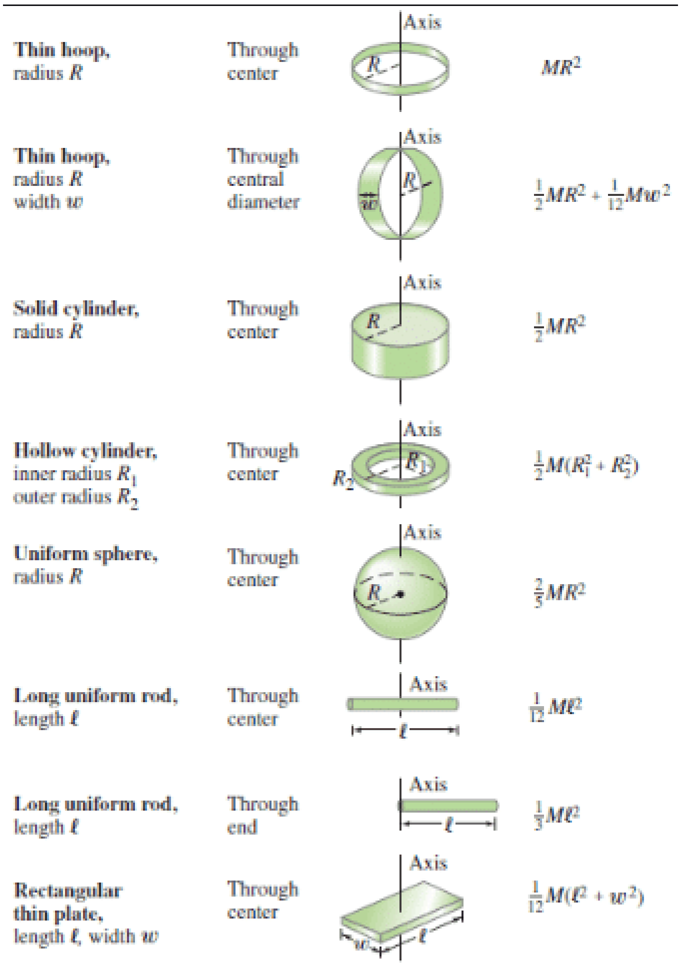
  $$U_{ressort} = \frac{1}{2} Kx^2 \quad \textbf{(2.10)}$$

- The potential energy of a torsion spring of stiffness k when deformation Θ

  $$U = \frac{1}{2} K\Theta^2 \quad \textbf{(2.11)}$$

Note:

The inertia of a body depends on its dimensions, mass and axis of rotation. The figure shows the inertia of different shapes.



Thin hoop, radius R	Through center	MR^2
Thin hoop, radius R width w	Through central diameter	$\frac{1}{2}MR^2 + \frac{1}{12}Mw^2$
Solid cylinder, radius R	Through center	$\frac{1}{2}MR^2$
Hollow cylinder, inner radius R_1 outer radius R_2	Through center	$\frac{1}{2}M(R_1^2 + R_2^2)$
Uniform sphere, radius R	Through center	$\frac{2}{5}MR^2$
Long uniform rod, length ℓ	Through center	$\frac{1}{12}M\ell^2$
Long uniform rod, length ℓ	Through end	$\frac{1}{3}M\ell^2$
Rectangular thin plate, length ℓ, width w	Through center	$\frac{1}{12}M(\ell^2 + w^2)$

Fig.10. inertia of different shapes

2.5. Huygens' theorem :

The moment of inertia varies with the axis of rotation, so if $\mathbf{I_0}$ is the inertia of a body of mass **m** when the axis of rotation passes through the center of mass of the body ,and $\mathbf{I_\Delta}$ is the moment of inertia if the axis of rotation is **Δ** with a distance **d** between the center of mass and the axis of rotation, Huygens' theorem gives the inertia by the following formula:

$$\mathbf{I_\Delta = I_0 + md^2} \qquad (2.12)$$

2.6. Equation of motion of a harmonic oscillator
2.6.1. Example of a mass-spring system

Applying the Newton's laws, we obtain :

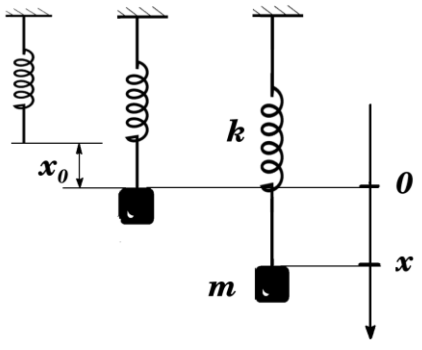


Fig.11.Harmonic oscillator
(mass-spring)

$-kx_0+mg=0$ **(2.13)**

$-k(x_0+x)+mg=m\ddot{x}$ **(2.14)**

So

$$m\ddot{x}+kx=0$$

$$\ddot{x}+\frac{k}{m}x=0 \quad \textbf{(2.15)}$$

On the other hand, we can find the equation of motion using Lagrange's formula

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)-\frac{\partial L}{\partial q}=0 \quad \textbf{(2.16)}$$

Where

q is the generalized coordinate describing the motion of the system (x,y,z,Θ.......) and the number of degrees of freedom of the motion is the number of independent generalized coordinates.

L=T-U **(2.17)**

L is the system's Lagrangian equal to the difference between kinetic energy and potential energy.
T is the kinetic energy of the system
U is the potential energy of the system
For the example of a mass-spring system, we have:
U=$\frac{1}{2}kx^2$ and $T = \frac{1}{2}m\dot{x}^2$

L= $\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$ **(2.18)**

Then the equation of motion of undamped free system by Lagrange's formula is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \quad \textbf{(2.19)}$$

$m\ddot{x}+kx=0$

$$\ddot{x} + \frac{k}{m}x = 0 \quad \textbf{(2.20)}$$

Which is the equation obtained using the fundamental principle of dynamics or the conservation equation.

2.7. Solution of the equation of motion:
The differential equation of the harmonic oscillator has the following sinusoidal solution:

$$x = A\sin(\omega_0 t + \varphi) \quad \textbf{(2.21)}$$

The amplitude A and phase φ depend on the initial conditions, and to find their values we need two initial conditions (usually $q(t_0)$ and $\dot{q}$ (t_0). We can therefore vary these constants by varying the initial conditions.

$$\begin{cases} q(t=0) = q_0 \\ \dot{q}(t=0) = \dot{q}_0 \end{cases}$$

In other words

$$\begin{cases} A\sin\varphi = q_0 \\ -A\omega_0\cos\varphi = \dot{q}_0 \end{cases}$$

To calculate the constant A, we squareq_0 and $\dot{q}_0$then add them term by term, we get:

$$(\frac{q_0}{A})^2 + (\frac{\dot{q}_0}{A\omega_0})^2 = 1 \Rightarrow (A\omega_0)^2 = (q_0\omega_0)^2 + \dot{q}_0{}^2$$

So : $A = \sqrt{\frac{(q_0\omega_0)^2+\dot{q}_0{}^2}{\omega_0{}^2}}$ **(2.22)**

And :tg $\varphi = -\frac{q_0}{\dot{q}_0\omega_0} \Rightarrow \varphi = arc\ tg(-\frac{q_0}{\dot{q}_0\omega_0})$ **(2.23)**

2.8. The natural frequency

The natural pulsation is called ω_0 because it depends only on the oscillator's own quantities ($\omega_0=\sqrt{\frac{k}{m}}$) for the mass-spring system).

2.9. The total energy of a harmonic oscillator :

We saw earlier that the solution to the differential equation of motion has the form

$x(t)=A\sin(\omega_0 t+\varphi)$ **(2.24)**

So:

$\dot{x}(t)=A\ \omega_0\cos(\omega_0 t+\varphi)$ **(2.25)**

The total energy of the system is:

E=T+U **(2.26)**

With : $U=\frac{1}{2}kx^2$ **(2.27)**

And $T = \frac{1}{2}m\dot{x}^2$ **(2.28)**

Then :

$E=\frac{1}{2}mA^2\omega_0^2\cos^2(\omega_0 t+\varphi)+\frac{1}{2}kA^2\sin^2(\omega_0 t + \varphi)$ **(2.29)**

On the other hand, we have:

$\omega_0=\sqrt{\frac{k}{m}} \Rightarrow k = m\omega_0^2$ **(2.30)**

We remplace $k = m\omega_0^2$,

The result is:

$$E=\frac{1}{2}kA^2\cos^2(\omega_0 t+\varphi)+\frac{1}{2}kA^2\sin^2(\omega_0 t+\varphi) \quad \textbf{(2.31)}$$

$$E=\frac{1}{2}kA^2 \quad \textbf{(2.32)}$$

2.10. Variation de l'énergie d'un système vibratoire

In vibratory motion, total energy is constant. Energy is transformed from kinetic to potential energy. When kinetic energy decreases, potential energy increases, and vice versa. This property is known as the conservation of the system's total energy. The variation of kinetic, potential and total energies as a function of displacement x is

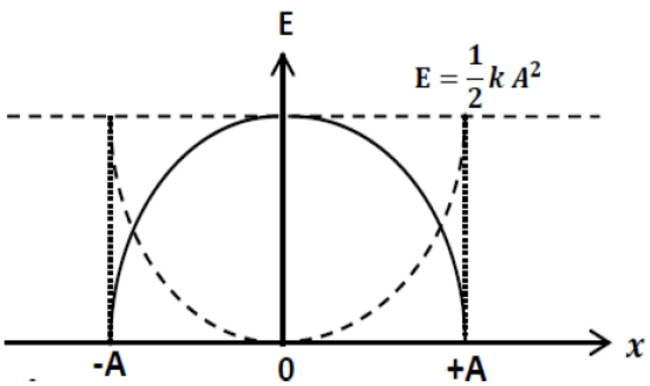


Fig.12.Variation of kinetic, potential and total energies as a function of x

Notes

- The total energy of a system E=T+U is constant (dE/dt=0 means that the system is conservative).
- The restoring force of a spring F=-kx is related by the potential energy as follows:

$$F=-kx=-\frac{\partial}{\partial x}\left(\frac{1}{2}kx^2\right)=-\frac{\partial U}{\partial x}$$

- The harmonic oscillator, whatever its nature, is a conservative system.

Example

Consider the following system

1-What is the kinetic energy and potential energy of the system.

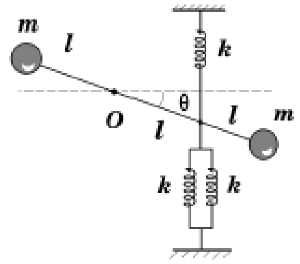
2-What is the Lagrangian of the system?

3-Find the differential equation of motion.

4-At initial conditions $\Theta(0)=0$ and $\dot{\theta}(0)=1$,

find the amplitude of motion and the phase shift

with K=10-4N/met m=1Kg



Response

The equivalent system is: $k_{eq} = 3k$

$$u = \frac{3}{2}K(x_0 + x_1)^2 - mgx_2 + mgx_3$$

$$x_1 = -l\,sin\theta = -l\theta$$
$$x_2 = -2l\,sin\theta = -2l\theta$$
$$x_3 = -l\,sin\theta = -l\theta$$

By replacing the expression of these coordinates in U:

$$u = \frac{3}{2}K(x_0 - l\theta)^2 + 2mgl\theta - mgl\theta$$
$$u = \frac{3}{2}K(l^2\theta^2) + (mg - 3Kx_0)l\theta + \frac{3}{2}Kx_0^2$$

At equilibrium $\frac{\partial u}{\partial \theta} = 0$

$$\frac{\partial u}{\partial \theta} = 3Kl^2\theta + (mg - 3Kx_0)l = 0$$

But also, at equilibrium $\theta=0$, so the equilibrium condition becomes:

$$(mg - 3Kx_0) = 0 \Rightarrow x_0 = \frac{mg}{3K}$$
$$u = \frac{3}{2}K(l^2\theta^2) + \frac{3}{2}Kx_0^2$$

The kinetic energy of the rotating system is:

$$T = \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}(ml^2 + m(2l)^2)\dot{\theta}^2$$
$$\Rightarrow T = \frac{5}{2}ml^2\dot{\theta}^2$$

So the Lagrangian of the system is:

$L = \frac{5}{2}ml^2\dot{\theta}^2 - \frac{3}{2}K(l^2\theta^2) - \frac{3}{2}Kx_0^2$

And the motion equation is :

$$\frac{\boldsymbol{d}}{\boldsymbol{dt}}\left(\frac{\boldsymbol{\partial L}}{\boldsymbol{\partial}\dot{\theta}}\right)-\frac{\boldsymbol{\partial L}}{\boldsymbol{\partial}\theta}=5ml^2\ddot{\theta}+3Kl^2\theta=0$$

$\Rightarrow\ddot{\theta}+\frac{3K}{5m}\,\theta=0$

Where the natural pulsation is: $\omega_0=\sqrt{\frac{3K}{5m}}$