Objectives of the vibration and waves course

- Describe the harmonic oscillator model and apply it to the study to the study of oscillating physical systems
- Study the responses of these systems, taking into account their characteristic parameters and initial conditions,
- Know how to study the energy of such systems.

Chapter 1: General information on vibrations.

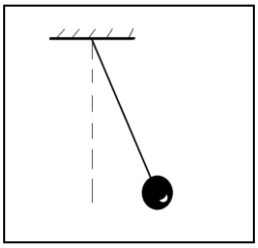
1.1. Definitions :

1.1.1 Definition of vibratory motion (oscillation)

Oscillatory motion is a repetitive back-and-forth movement. Examples of this type of motion are the simple pendulum, the oscillating electric circuit, the mass-spring system.....

- The simple pendulum

Composed of a mass attached to a wire, moved away from its equilibrium position its position of



equilibrium and then released, performs a back-and-forth movement which repeats it self over time.

Fig.1.Simple pendulum

- Oscillating electrical circuit

Linear circuit containing an electrical resistor and a capacitor (capacity) and a coil (inductance) and capable of electrical oscillation.

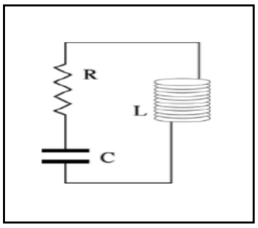


Fig.2.Electrical oscillating circuit

- Mass-spring system

Consisting of a mass attached to a spring, moved away from its equilibrium position and then released performs a motion that repeats itself over time.As soon as the body is moved away from its equilibrium position, a force appears to try to bring it back to equilibrium, This force is known as a restoring force.

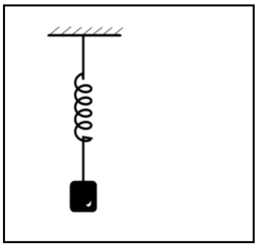


Fig.3.mass-spring system

1.1.2.Periodic movement

Periodic movement is an oscillatory movement which occurs in an identical manner. We say that a movement is periodic if after a time T necessary to carry out a complete oscillation around the equilibrium position and we call the time T the period measured in seconds s.

The number of repetitions per second is called frequency (denoted f, measured in Hertz or s^{-1}.) It is linked to the period by

$$\boldsymbol{f=\frac{1}{T}} \qquad \textbf{(1.1)}$$

The pulsation is defined by the number of revolutions per second (noted ω, measured in rad/s.)

$$\boldsymbol{\omega = 2\pi f = \frac{2\pi}{T}} \qquad \textbf{(1.2)}$$

Mathematically, periodicity is expressed as $x(t+T) = x(t)$.

Example :

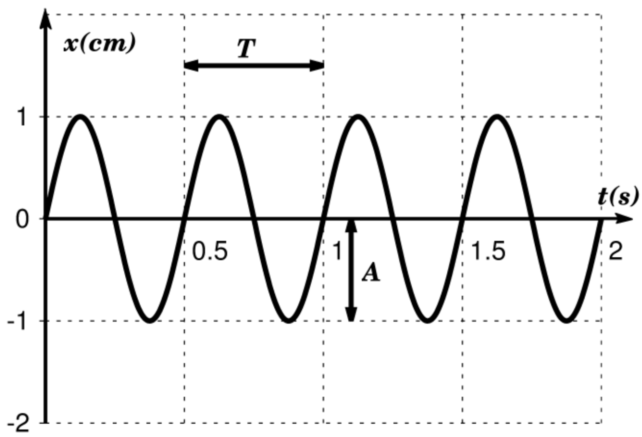


Fig.4 periodic motion.

Example :

Let us consider the periodic functions f(t) whose graphs are represented in Figures (5), (6) and (7).

1. Derive the period, frequency and pulsation and amplitude of each function.

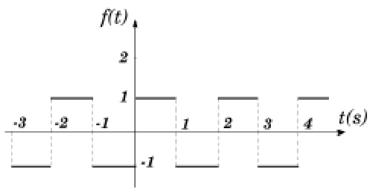


Fig.06.

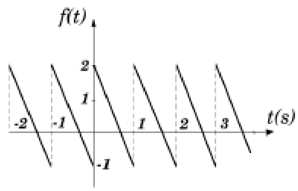


Fig.07.

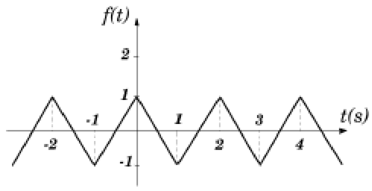


Fig.08.

1.1.3. Sinusoidal Motion and Complex Notation

A periodic quantity is said to be sinusoidal when it is of the form

$$x(t) = A\sin(\omega t+\varphi) \quad (1.3)$$

Or

$$x(t) = A\cos(\omega t+\varphi) \quad (1.4)$$

A is called amplitude,

ω: the pulsation,

: the initial phase.

To facilitate calculations, we transform the sinusoidal quantities into exponentials which are simpler to handle. We can consider the oxy plane as a complex plane.

The point with coordinates (x, y) corresponds to a complex number z

$$z = x + iy \quad (1.5)$$

with x= r cosθ

and y= r sinθ

cosθ+i sinθ= $e^{i\theta}$ (1.6)

with $i^2 = -1$

So :

z = r (cos θ+ isin θ) = r $e^{i\theta}$ (1.7)

with $i^2 = -1$

and from this relationship we can even deduce that:

sin θ = $\frac{e^{i\theta}-e^{-i\theta}}{2}$ **(1.8)**

and cos θ =$\frac{e^{i\theta}+e^{-i\theta}}{2}$ **(1.9)**

1.1.4. Superposition of sinusoidal quantities of the same pulsation

The superposition of two sinusoidal quantities with the same pulsation ω is a sinusoidal quantity with pulsation ω.

Example: Let the two sinusoidal quantities be

$x_1(t)= a\cos(\omega t + \alpha)$ **(1.10)**

and $x_2(t)= b\cos(\omega t+\beta)$. **(1.11)**

The superposition of $x_1(t)$ et $x_2(t)$ donne $x_3(t)$

$x_3(t)= x_1(t) + x_2(t)= a\cos(\omega t + \alpha)+ b\cos(\omega t+\beta)$. **(1.12)**

Suppose the function

$y_3(t)= y_2(t)+ y_1(t)$

With :$y_1(t) = a\,sin(\omega t + \alpha)$ **(1.13)**

et $y_2(t) = b\,sin(\omega t + \beta)$ **(1.14)**

Then :$x_3+jy_3= (x_1+iy_1)+(x_2+iy_2)$

$= (a\,cos(\omega t + \alpha)+i\,a\,sin(\omega t + \alpha))+(b\sin(\omega t+\beta)+i\,b\,sin(\omega t + \beta))$

$=ae^{ij(\omega t + \alpha)} + be^{i(\omega t + \beta)}$

$= \left(ae^{i\alpha)} + be^{i\beta}\right)e^{i\omega t}$

$=Ae^{i\omega t}$ **(1.15)**

The nomber A = $ae^{i\alpha} + be^{i\beta}$ is a constant complex number that has a norm

|A| =$\sqrt{AA^*} = \sqrt{(ae^{i\alpha} + be^{i\beta})(ae^{-i\alpha} + be^{-i\beta})} = \sqrt{a^2 + b^2 + abcos(\alpha - \beta)}$

and a phase ϕ defined by

tg ϕ= $\frac{Im(A)}{Re(A)} = \frac{asin\alpha+bsin\beta}{acos\alpha+bcos\beta}$

So A=|A| $e^{j\phi}$

Finally we arrive at

$x_3 = |A| \cos(\omega t + \phi)$ **(1.17)**

1.1.5. Velocity and Acceleration in Simple Harmonic Motion

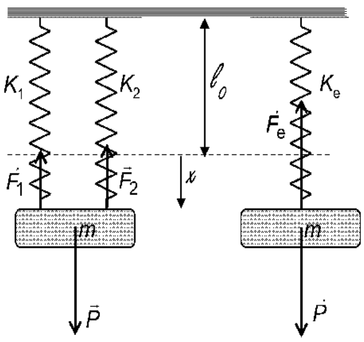
The values of the velocity and acceleration in simple harmonic motion for

$x(t) = A\sin(\omega t + \varphi)$

are given by $\frac{dx}{dt} = \dot{x} = A\omega\cos(wt + \varphi)$

and $\frac{d^2X}{dt^2} = \ddot{x} = -A\omega^2\sin(wt + \varphi)$

The maximum value of the velocity $A\omega$ is called the velocity amplitude and the acceleration amplitude is given by $A\omega^2$.



1.1.6. The connection of the springs

1.1.6.1. Parallel springs

Let two springs k_1 and k_2 have the same length empty l_0 and undergo the same elongation x. When we hang a mass m at the end of the two springs. The equivalent spring of stiffness k_{eq} has the same elongation.

At equilibrium we have:

So $\begin{cases} mg = k_1x + k_2x \\ mg = k_{eq}x \end{cases}$ **(1.18)**

$k_{eq} = k_1 + k_2$

1.1.6.2. Springs in series

Consider two springs k_1 and k_2, their elongation x_1 and x_2 respectively, the equivalent spring of stiffness k_{eq} at the elongation $x = x_1 + x_2$, such that:

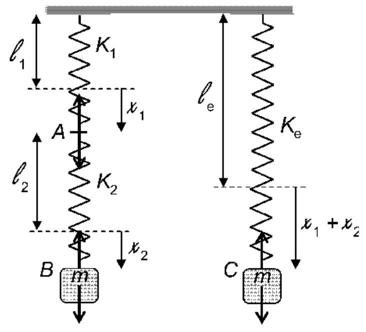


Fig.9.*Springs in series*

$$\begin{cases} \boldsymbol{k_1 x_1 = k_2 x_2} \\ \boldsymbol{mg = k_2 x_2} \\ \boldsymbol{mg = k_{eq}(x_1 + x_2)} \end{cases}$$

$$\Rightarrow \begin{cases} \boldsymbol{x_1 = \frac{k_2}{k_1} x_2} \\ \boldsymbol{k_2 x_2 = k_{eq}(x_1 + x_2)} \end{cases}$$

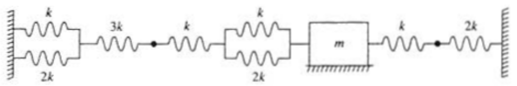
$$\Rightarrow \boldsymbol{k_2 x_2} = k_{eq}\left(\boldsymbol{\frac{k_2}{k_1} x_2 + x_2}\right)$$

$$\Rightarrow k_{eq} = \frac{k_1 k_2}{k_1 + k_2} \qquad \textbf{(1.19)}$$

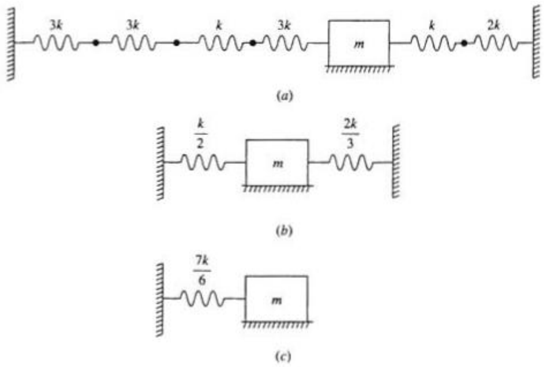
$$\text{Or } \frac{1}{k_{eq}} = \frac{1}{\boldsymbol{k_1}} + \frac{1}{\boldsymbol{k_2}} \qquad \textbf{(1.20)}$$

Example :

Find the equivalent spring in the following system



Réponse :



1.1.6. Equivalent mass and equivalent moment:

From the total kinetic energy of the mechanical system, we can find the equivalent mass and equivalent moment of the system as follows:

$$\begin{cases} T_{totale} = \frac{1}{2}(masse\ équivalente)V^2 \\ T_{totale} = \frac{1}{2}(moment\ équivalent)\dot{\theta}^2 \end{cases}$$

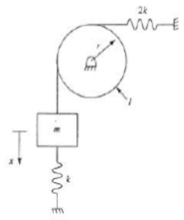
where

V is the velocity of the body's mass center

$\dot{\theta}$ is the angular velocity

Example :

Given the following system, find the equivalent mass and the equivalent moment.



Answer:

$$U = \frac{1}{2}Kx^2 + \frac{1}{2}(2K)x^2 = \frac{1}{2}(3K)x^2 \rightarrow K_{eq} = 3K$$

$$T = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\ I(\frac{\dot{x}}{r})^2 = \frac{1}{2}\left(m + \frac{I}{r^2}\right)\dot{x}^2 \rightarrow m_{eq} = m + \frac{I}{r^2}$$