Chapter 5: multi-degree-of-freedom systems.

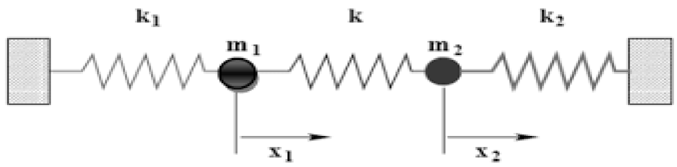
5.1. Degree of freedom

The number of degrees of freedom is defined by the number of independent variables required to describe the motion of a system.Let's consider a system with n degrees of freedom, subject to forces deriving from a potential, frictional forces due to viscosity and external forces. If the generalized coordinates are q1,q2........qn, Lagrange's equations can be written as follows:

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_1}}\right) - \left(\frac{\partial L}{\partial q_1}\right) + \left(\frac{\partial D}{\partial \dot{q_1}}\right) = F_1 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_2}}\right) - \left(\frac{\partial L}{\partial q_2}\right) + \left(\frac{\partial D}{\partial \dot{q_2}}\right) = F_2 \\ \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_n}}\right) - \left(\frac{\partial L}{\partial q_n}\right) + \left(\frac{\partial D}{\partial \dot{q_n}}\right) = F_n \end{cases} \quad (5.1)$$

5.2 Equations of motion for a system with two degrees of freedom

The two independent variables are *x1* and *x2*, the coupling element is the spring K as shown in the following figure



The kinetic energy of this system is given by

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2$$

And the potential energy is given by

U=$\frac{1}{2}kx_1^2+\frac{1}{2}kx_2^2+\frac{1}{2}k(x_1-x_2)^2$

The Lagrangian is

L=T-U=$\frac{1}{2}m\dot{x}_1^2+\frac{1}{2}m\dot{x}_2^2-\frac{1}{2}kx_1^2-\frac{1}{2}kx_2^2-\frac{1}{2}k(x_1-x_2)^2$

The system of differential equations is written as follows:

$$\begin{cases}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x_1}}\right)-\left(\frac{\partial L}{\partial x_1}\right)=0\\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x_2}}\right)-\left(\frac{\partial L}{\partial x_2}\right)=0\end{cases}\Rightarrow\begin{cases}m_1\ddot{x}_1+(K_1+K)x_1-Kx_2=0\\ m_2\ddot{x}_2+(K_2+K)x_2-Kx_1=0\end{cases} \quad \textbf{(5.2)}$$

Sinusoidal solutions are proposed to solve this system of linear differential equations, where the masses oscillate at the same pulsation w_p with different amplitudes and phases.

5.3. Propres modes

The solution of the previous system is of the following form

$$x_1(t)=Acos(\omega_p t+\varphi)$$

Et **(5.3)**

$$x_2(t)=Bcos(\omega_p t+\varphi)$$

So :

$$\dot{x}_1(t)=-A\omega sin(\omega_p t+\varphi)$$

$$\dot{x}_2(t)=-B\omega sin(\omega_p t+\varphi)$$

And :

$$\ddot{x}_1(t)=A\omega^2cos(\omega_p t+\varphi)$$

$$\ddot{x}_2(t)=B\omega^2cos(\omega_p t+\varphi)$$

By replacing with $x_1(t)$, $\ddot{x}_1(t)$, $x_2(t)$ et $\ddot{x}_2(t)$,

The system becomes as follows:

$$\begin{cases} \left(-\omega_p{}^2 + \frac{K_1+K}{m_1}\right)A - \frac{K}{m_1}B = 0 \\ -\frac{K}{m_2}A + \left(-\omega_p{}^2 + \frac{K_2+K}{m_2}\right)B = 0 \end{cases} \quad \textbf{(5.4)}$$

$$\Leftrightarrow \begin{bmatrix} -\omega_p{}^2 + \frac{K_1+K}{m_1} & -\frac{K}{m_1} \\ -\frac{K}{m_2} & -\omega_p{}^2 + \frac{K_2+K}{m_2} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0 \quad \textbf{(5.5)}$$

System (5.3) has a solution if and only if A=$B = 0$ or

$$\begin{vmatrix} -\omega_p{}^2 + \dfrac{K_1 + K}{m_1} & -\dfrac{K}{m_1} \\ -\dfrac{K}{m_2} & -\omega_p{}^2 + \dfrac{K_2 + K}{m_2} \end{vmatrix} = 0$$

$$\omega_p{}^4 - (\omega_1^2 + \omega_2^2)\omega_p{}^2 + \omega_1^2\omega_2^2(1 - k^2) = 0$$

where

$$\omega_1^2 = \frac{K_1}{m_1} \quad \omega_2^2 = \frac{K_2}{m_2} \qquad \text{et} \quad k = \frac{K^2}{(K_1+K)(K_2+K)}$$

Where k is the coupling coefficient. The two natural pulsations are

$$\omega_{p1=}^2 \frac{\omega_1^2 + \omega_2^2}{2} - \frac{1\sqrt{(\omega_1^2 + \omega_2^2) + 4k\omega_1^2\omega_2^2}}{2}$$

$$\omega_{p2=}^2 \frac{\omega_1^2 + \omega_2^2}{2} + \frac{1\sqrt{(\omega_1^2 + \omega_2^2) + 4k\omega_1^2\omega_2^2}}{2}$$

Let's assume $K = K_1 = K_2$ et m =$m_1 = m_2$ to simplify the calculations; so the system of equations becomes :

$$\begin{cases} m\ddot{x}_1 + 2Kx_1 - Kx_2 = 0 \\ m\ddot{x}_2 + 2Kx_2 - Kx_1 = 0 \end{cases} \quad \textbf{(5.6)}$$

So the proper pulsations are

$$x_1(t) = A\cos(\omega_p t + \varphi)$$

And

$$x_2(t) = Bcos\left(\omega_p t + \varphi\right)$$

In system (5.6), we obtain

$$\begin{cases}(-m\omega_p^2+2K)A-KB=0\\(-m\omega_p^2+2K)B-KA=0\end{cases}\Leftrightarrow\begin{vmatrix}-m\omega_p^2+2K & -K\\ -K & -m\omega_p^2+2K\end{vmatrix}=0$$

$$(-m\omega_p^2+2K)^2-K^2=0\Rightarrow\begin{cases}3K-m\omega_p^2=0\\K-m\omega_p^2=0\end{cases}$$

$$\begin{cases}\omega_{p1}=\sqrt{\dfrac{K}{m}}\\[2ex]\omega_{p2}=\sqrt{\dfrac{3K}{m}}\end{cases}$$

So the general solutions are :

$$\begin{cases}x_1(t)=A_1\cos(\omega_{p1}t+\varphi_1)+A_2\cos(\omega_{p2}t+\varphi_2)\\x_2(t)=B_1\cos(\omega_{p1}t+\varphi_1)+B_2\cos(\omega_{p2}t+\varphi_2)\end{cases}\quad\textbf{(5.7)}$$

$$\begin{cases}x_1(t)=A_1\cos\left(\sqrt{\frac{K}{m}}t+\varphi_1\right)+A_2\cos\left(\sqrt{\frac{3K}{m}}t+\varphi_2\right)\\[2ex]x_2(t)=B_1\cos\left(\sqrt{\frac{K}{m}}t+\varphi_1\right)+B_2\cos\left(\sqrt{\frac{3K}{m}}t+\varphi_2\right)\end{cases}\quad\textbf{(5.8)}$$

Supposons que les deux masses oscillent avec le même battement ω_{p1} puis ω_{p2}

1er cas ω=$\boldsymbol{\omega_{p1}}$

$$\begin{cases}x_1(t)=A_1\cos\left(\sqrt{\frac{K}{m}}t+\varphi_1\right)\\[2ex]x_2(t)=B_1\cos\left(\sqrt{\frac{K}{m}}t+\varphi_1\right)\end{cases}\quad\textbf{(5.9)}$$

Where $x_1(t)$ et $x_2(t)$ are the solutions of differential equation

$$\begin{bmatrix}-m\omega_{p1}^2+2K & -K\\ -K & -m\omega_{p1}^2+2K\end{bmatrix}\begin{bmatrix}A_1\\B_1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

And

$$\begin{cases} (-m\omega_{p1}^2 + 2K)\, A_1 - KB_1 = 0 \\ -KA_1 + \left(-m\omega_{p1}^2 + 2K\right)B_1 = 0 \end{cases} \tag{5.10}$$

So : $\frac{A_1}{B_1} = \frac{-K}{-m\omega_{p1}^2+2K}$ et $\frac{A_1}{B_1} = \frac{-m\omega_{p1}^2+2K}{-K}$

And : $\frac{A_1}{B_1} = 1 \Rightarrow A_1 = B_1$ (x_1 et x_2 are in phase)

1st case ω=ω_{p2}

$$\begin{cases} x_1(t) = A_2 \cos\left(\sqrt{\frac{3K}{m}}t + \varphi_2\right) \\ x_2(t) = B_2 \cos\left(\sqrt{\frac{3K}{m}}t + \varphi_2\right) \end{cases}$$

$\Rightarrow$

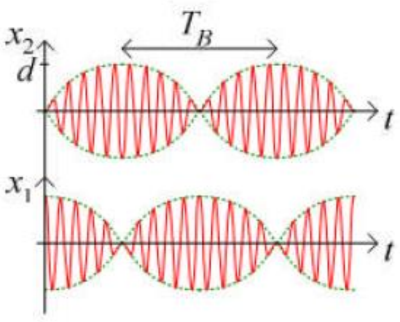
$$\begin{bmatrix} -m\omega_{p2}^2 + 2K & -K \\ -K & -m\omega_{p2}^2 + 2K \end{bmatrix} \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow A_2 = -B_2$(x_1 et x_2 Are in phase opposition

So:
$$\begin{aligned} x_1(t) &= A_1 \cos\left(\sqrt{\frac{K}{m}}t + \varphi_1\right) + A_2 \cos\left(\sqrt{\frac{3K}{m}}t + \varphi_2\right) \\ x_2(t) &= A_1 \cos\left(\sqrt{\frac{K}{m}}t + \varphi_1\right) - A_2 \cos\left(\sqrt{\frac{3K}{m}}t + \varphi_2\right) \end{aligned}$$

We find A_1,A_2,$\varphi_1$et φ_2 from initial conditions.

The phenomenon studied is the beat:



Systems with two degrees of freedom

Let an external force be applied to the first subsystem; this force to the next force:

The new equations of motion are :

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x_1}}\right) - \left(\frac{\partial L}{\partial x_1}\right) = f(t) \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x_2}}\right) - \left(\frac{\partial L}{\partial x_2}\right) = 0 \end{cases} \Rightarrow \begin{cases} m_1\ddot{x}_1 + (K_1 + K)x_1 - Kx_2 = f(t) \\ m_2\ddot{x}_2 + (K_2 + K)x_2 - Kx_1 = 0 \end{cases}$$

Particular solutions have the form

$$\begin{cases} x_1(t) = A_1 e^{i(\omega_p t+\varphi)} \\ x_2(t) = A_2 e^{i(\omega_p t+\varphi)} \end{cases} \Rightarrow \begin{cases} \ddot{x}_1(t) = -\omega_p^2 e^{i(\omega_p t+\varphi)} \\ \ddot{x}_2(t) = -\omega_p^2 e^{i(\omega_p t+\varphi)} \end{cases}$$

By replacing the solutions in the differential system, we obtain

$$\begin{cases} (-m\omega_p^2+2K)A_1 e^{i\varphi} - KA_2 e^{i\varphi} = f_0 \\ (-m\omega_p^2+2K)A_2 e^{i\varphi} - KA_1 e^{i\varphi} = O \end{cases}$$

The amplitude modules are
$$\begin{cases} A_1 = \dfrac{\begin{vmatrix} f_0 & -K \\ 0 & -m\omega_p^2+2K \end{vmatrix}}{\begin{vmatrix} -m\omega_p^2+2K & -K \\ -K & -m\omega_p^2+2K \end{vmatrix}} = \dfrac{\frac{f_0}{m}(-\omega^2+\frac{K}{m})}{(\omega^2-\omega_{1p}^2)(\omega^2-\omega_{2p}^2)} \\ A_1 = \dfrac{\begin{vmatrix} -m\omega_p^2+2K & f_0 \\ -K & 0 \end{vmatrix}}{\begin{vmatrix} -m\omega_p^2+2K & -K \\ -K & -m\omega_p^2+2K \end{vmatrix}} = \dfrac{\frac{f_0 K}{m^2}}{(\omega^2-\omega_{1p}^2)(\omega^2-\omega_{2p}^2)} \end{cases}$$

The phenomena studied are

- Resonance $\begin{cases} A_1 \to \infty \\ A_2 \to \infty \end{cases}$ where $\begin{cases} \omega \to \omega_{1p} \\ \omega \to \omega_{2p} \end{cases}$

- Anti-resonance $\begin{cases} A_1 \to 0 \\ A_2 \to constante \end{cases}$ where $\omega \to \frac{K}{m}$