Chapter 3: Damped free systems with one degree of freedom.

3.1 Introduction :

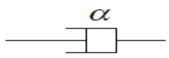
In practice some energy is always dissipated by a resistive or viscous process; for example, the amplitude of a freely swinging pendulum will always decay with time as energy is lost. The presence of resistance to motion means that another force is active, which is taken as being proportional to the velocity. The frictional force acts in the direction opposite to that of the velocity and friction forces have the following form

***f*=-cv** **(3.1)**

where :

c is the coefficient of friction

v is the velocity

The damped system is characterized by 

3.2 Lagrange equation for the damped system:

In the case of the damped system, there is a friction force of the form *f*=-c$\dot{q}$ and the energy loss is defined by the dissipation function $D = \frac{1}{2}c\dot{q}^2$ and the equation of motion in the case of a damped free system is :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + \frac{\partial D}{\partial \dot{q}} = 0 \qquad (3.2)$$

Example:

In the case of the mass-spring system, we have :

The kinetic energy of the mass is $T = \frac{1}{2}m\dot{x}^2$

The potential energy of the spring is U=$\frac{1}{2}kx^2$

The dissipation energy is $D = \frac{1}{2}\alpha\dot{x}^2$

So $D = \frac{1}{2}\alpha\dot{x}^2$

And $\frac{\partial L}{\partial x} = kx$

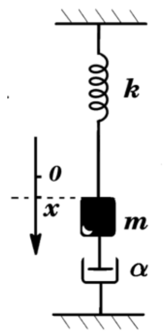


Fig.13.*Damped free system*

and $\frac{\partial D}{\partial \dot{x}} = \alpha\dot{x}$

The equation of motion is

$$m\ddot{x} + kx + \alpha\dot{x} = 0$$

$$\Rightarrow \ddot{x} + \frac{\alpha}{m}\dot{x} + \frac{k}{m}x = 0$$

Which is a second-order linear differential equation
More generally, for a generalized coordinate q it is written

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 q = 0$$

3.3. Solving the differential equation of motion

The second-order linear differential equation

$$\boldsymbol{\ddot{q} + 2\delta\dot{q} + \omega_0^2 q = 0} \quad \textbf{(3.3)}$$

has the following characteristic equation:

$$\lambda^2 + 2\delta\lambda + \omega_0^2 = 0$$

Depending on the nature of the roots of the characteristic equation, there are three types of damping.

$\grave{\Delta}$<0⇒ $\delta^2 - \omega_0^2 < 0$ weakly damped system.

$\grave{\Delta}$=0⇒ $\delta^2 - \omega_0^2 = 0$ criticaly damped system.

$\grave{\Delta}$>0⇒ $\delta^2 - \omega_0^2 > 0$ strongly damped system.

3.3.1. Weakly damped regime:

If δ<ω_0the solution of the differential equation of motion takes the form:

$$\boldsymbol{q(t) = Ae^{-\delta t}cos(\omega_a t - \varphi)} \quad (3.4)$$

Such that: $\omega_a = \sqrt{\omega_0^2 - \delta^2}$ (3.5)

ω_a *is* the frequency damped system.

We define the period of the system T called the pseudo-period as follows:

$$\mathbf{T}=\frac{\boldsymbol{2\pi}}{\omega_a} \quad \textbf{(3.6)}$$

3.3.2. Critical regime :

If $\delta = \omega_0$the system no longer performs oscillatory motion and the system returns to equilibrium without any oscillation the solution of the differential equation of motion takes the form :

$$\boldsymbol{q(t) = e^{-\delta t}(A + Bt)} \quad \textbf{(3.7)}$$

We use the initials conditions to find the two constants A and B.

3.3.3. Highly damped regime :

If $\delta > \omega_0$ in this case too the system no longer performs oscillatory motion and the system returns directly to equilibrium without any oscillation and the solution of the differential equation of motion takes the following form:

$$\boldsymbol{q(t) = e^{-\delta t}(Ae^{-\sqrt{-\omega_0^2+\delta^2}\,t} + Be^{\sqrt{-\omega_0^2+\delta^2}\,t})}$$

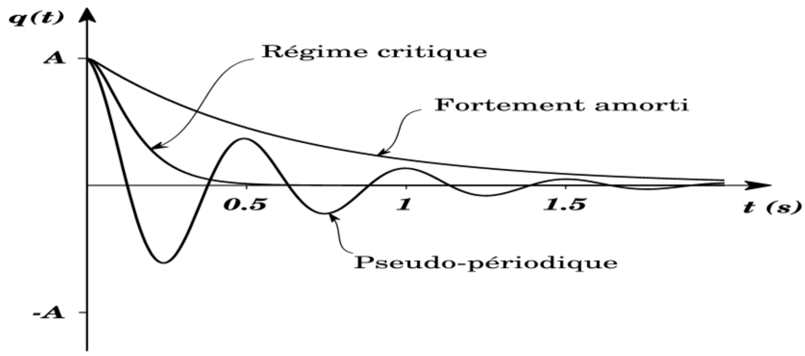


Fig.14.*les types d'amortissement.*

Example:

We found the differential equation of the mass-spring system in the case of the damped regime is $\ddot{\boldsymbol{x}} + \frac{\boldsymbol{\alpha}}{\boldsymbol{m}}\dot{\boldsymbol{x}} + \frac{\boldsymbol{k}}{\boldsymbol{m}}\boldsymbol{x} = \boldsymbol{0}$

The characteristic equation is :

$$\boldsymbol{\lambda}^{\boldsymbol{2}} + \frac{\alpha}{m}\boldsymbol{\lambda} + \frac{k}{m} = 0$$

We have : $\Delta = (\frac{\alpha}{m})^2 - 4\frac{k}{m}$

Depending on the sign of Δ, there are three types of friction

$\Delta = 0 \Rightarrow (\frac{\alpha}{m})^2 = 4\frac{k}{m} \Rightarrow \lambda \quad = -\frac{\alpha}{2m}$ (critical regime case) and the solution to the equation is :

$$x(t) = e^{\lambda t}(A + Bt)$$

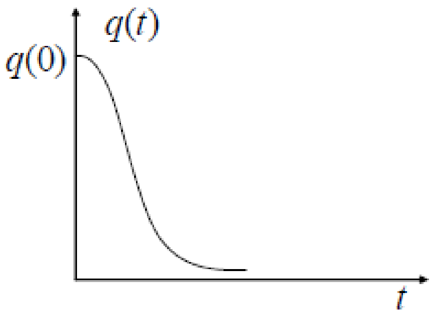


Fig.15. Variation of q(t) as a function of time for the critical regime

- Δ>0$\Rightarrow (\frac{\alpha}{m})^2 > \frac{k}{m} \Rightarrow \lambda_1 = -\frac{\alpha}{2m} - \sqrt{(\frac{\alpha}{2m})^2 - \frac{k}{m}}$ et

$\lambda_2 = -\frac{\alpha}{2m} - \sqrt{(\frac{\alpha}{2m})^2 - \frac{k}{m}}$ (case of the highly damped regime) and the solution of the equation is: $x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$

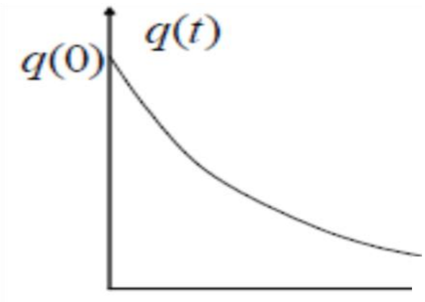


Fig.16.Variation of q(t) as a function of time for the highly damped regime

- Δ<0$\Rightarrow (\frac{\alpha}{m})^2 > \frac{k}{m} \Rightarrow \lambda_1 = -\frac{\alpha}{2m} - i\sqrt{\frac{k}{m} - (\frac{\alpha}{2m})^2}$

$\lambda_2 = -\frac{\alpha}{2m} + i\sqrt{\frac{k}{m} - (\frac{\alpha}{2m})^2}$ (Case of the weakly damped regime) and the solution of the equation is:

$$x(t) = e^{-\frac{\alpha}{2m}}(A\cos\sqrt{\frac{k}{m} - \left(\frac{\alpha}{2m}\right)^2}\, t + B\sin\sqrt{\frac{k}{m} - \left(\frac{\alpha}{2m}\right)^2})$$

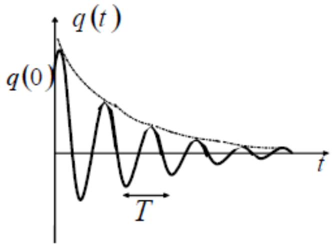


Fig.17. Variation of q(t) as a function of time

for the weakly damped regime

3.4. Critical damping coefficient

C_c is the value of C corresponding to Δ=0, that is to say

$$\left(\frac{C_c}{m}\right)^2 = 4\frac{k}{m}$$

$$\Rightarrow C_c = 2m\sqrt{\frac{k}{m}} = 2m\omega_0$$

$$C_c = 2m\omega_0 \qquad \textbf{(3.9)}$$

3.5. Damping ratio

The damping ratio is defined by

$$\varepsilon = \frac{c}{C_c} \qquad \textbf{(3.10)}$$

$$\Rightarrow \frac{C}{2m} = \varepsilon\omega_0 \qquad \textbf{(3.11)}$$

3.6. The quality factor

The quality factor is defined by

$$\boldsymbol{Q = 2\pi\frac{E}{\Delta E} = \frac{\omega_0}{2\delta}} \qquad \textbf{(3.12)}$$

Where the energy of the harmonic oscillator ΔE is the energy dissipated during one cycle.
Whene the damping is low, the quality of the system is good. where Q is greater, hence the name quality factor.

3.7. The frequency of the pseudo-period system

We define the pulsation of the weakly damped system as follows:

$$\boldsymbol{\omega_a = \sqrt{{\omega_0}^2 - \delta^2}} \tag{3.13}$$

So the period of the system is

$$\boldsymbol{T_a = \frac{2\pi}{\omega_a}} \tag{3.14}$$

3.8. The logarithmic decrement

We define the logarithmic decrement which represents the decrease of the single-period amplitude of the system as follows:

$$D = \ln \frac{q(t)}{q(t+T)} \tag{3.15}$$

$$D = ln \frac{\boldsymbol{Ae^{-\delta t} cos(\omega_a t - \varphi)}}{\boldsymbol{Ae^{-\delta(t+T_a)} cos(\omega_a (t+T_a) - \varphi)}}$$

where :$\boldsymbol{\omega_a T_a = 2\pi}$

And: $\boldsymbol{cos(\omega_a(t+T_a) - \varphi) = cos(\omega_a t - \varphi)}$

So : D= $ln \frac{\boldsymbol{Ae^{-\delta t}}}{\boldsymbol{Ae^{-\delta(t+T_a)}}} = ln\boldsymbol{e^{\delta T_a}}$

$$\boldsymbol{D = \delta T_a} \tag{3.16}$$

3.9. The energy dissipated

Because of the friction force, the system suffers a total energy loss due to the work of the friction forces.

$$\boldsymbol{dE_T(t) = -dw_{fr}} \tag{3.17}$$

Exemple1 :

We define a damped oscillator governed by the following differential equation:

$$\ddot{x} + \frac{\alpha}{m}\dot{x} + \frac{k}{m}x = 0$$

Where m is the mass of the body, k is the spring coefficient and x is the displacement of the body. We launch the system with an initial speed v_0=25cm/s.

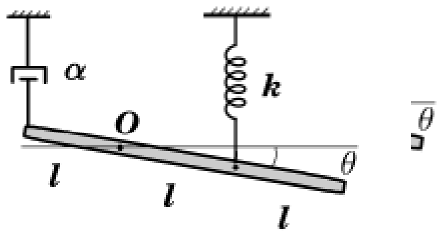
We have at : t=0, x=0 et $\dot{x} = v_0$

- Calculate the natural period of the system, Knowing that m=150g et k=3.8N/m.
- Show that if α=0.6kg/s, the body has a damped oscillatory motion. In this case, solve the differential equation.
- Calculate the pseudo-period of the motion.
- Calculate the time t_m after which the first amplitude x_m is reached. and

deduce x_m.

- Calculate the speed of a pseudo-period.

Exemple 2 :



In the previous system, the bar of mass m and length 3l can rotate around the axis passing through O. We symbolize all the friction by a damper of coefficient α.

At equilibrium the rod is horizontal.

The rod is moved away from the horizontal by a small angle to admit that sin Ɵ=Ɵ

1. Find the differential equation of motion.
2. What is the value that the coefficient of friction α must not exceed to have an oscillatory motion.

Calculate this value if m = 1 kg and k = 1 N/m.

3. For the value calculated in the previous question takes,What is the nature of the motion?

4-Give the time equation Θ (t) knowing that initially conditions are

$\Theta(0) = 5°$et $\dot{\Theta}$ (0) = 0°/s.

Réponse :

The equation of motion of this system is given by:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + \frac{\partial D}{\partial \dot{q}} = 0$$

Où L=T-U

Avec T= et U=

So the equation of motion is

$$\ddot{\theta} + \frac{\alpha}{m}\dot{\theta} + \frac{K}{m}\theta = 0$$

For there to be an oscillatory motion is necessary to be in the pseudo-periodic regime $\delta^2 - \omega_0^2 < 0$

$(\frac{\alpha}{2m})^2 - \frac{K}{m} < 0 \Rightarrow \alpha < 2\sqrt{Km}$

So :α<2Ns /m

Whene :α=2Ns /m we are in the critical regime.

the time equation of motion knowing that $\delta = \frac{\alpha}{2m} = 1s^{-1}$ is

$$\theta(t) = e^{-t}(A_1 + A_2 t)$$

$A_1$and A_2 are determined from the initial conditions $\theta(0) = 5° \Rightarrow A_1 = 5°$

$$\dot{\theta}(t) = -e^{-t}(A_1 + A_2 t) + A_2 e^{-t}$$

$$\dot{\theta}(0) = 0°/s \Rightarrow A_2 = 5°/s$$

So the equation of motion is:

$$\theta(t) = e^{-t}(5 + 5t)$$