Chapter 2 : Sets theory, relations and functions

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24 octobre 2024

1 Introduction to sets

1.1 Definition

— In mathematics, a set is a collection of objects such as numbers, points or functions. The object in a set is called **element**of the set.

Example.

Let $A = \{3, 7, 9, 5\}$. This formula show that A is a set with 4 elements, and we can write that $5 \in A$, to denote that 5 is an element of A and $10 \notin A$ to means that 10 is not an element of A. Also the number of elements in A called cardinal of A in which $card(A) = 4$ in this case.

— If a set has infinite number of elements, it is said to be infinite, otherways is called a finite set. In the case of infinite set we describe the set by using some property which defines the set

$$\{x \mid \ A(x)\}$$

— For example, the set of natural numbers $\mathbb{N}$ it can be defined as $\mathbb{N} = \{k \in \mathbb{Z} \mid \ k > 0\}$.

— The two sets A and B are equal ($A = B$) if they have exactly the same elements, or in other wordes every element of A is an element of B and every element of B is an element of A. Thus $\{2, 3, a\} \neq \{2, 4, a\}$ and $\{2, 3, b\} = \{b, 2, 3\}$.

The empty set : The set that has no elements is called the empty set denoted by $\emptyset$

Properly, there are several cases of the empty set such as

$$\emptyset = \{x \in \mathbb{R} \mid x^2 < 0\}$$
$$\emptyset = \{x \mid x \neq x\}$$

We should notice that $\emptyset \not\equiv \{\emptyset\}$.

Famous sets

$\mathbb{R}$: *Set of real numbers*
$\mathbb{Q}$: *Set of rational numbers*
$\mathbb{N}$: *Set of natural numbers*
$\mathbb{C}$: *Set of complex numbers*
$\mathbb{Z}$: *Set of integer numbers*

1.2 Operations on sets

Inclusion

A set A is said to be a subset of a set B if all elements of A are also elements of B. We write $A \subseteq B$ to indicate that A is included in B.

$$\forall x \mid (x \in A) \Rightarrow (x \in B)$$

If there is a possibility to show that $A = B$, then we write $A \subseteq B$ else $A \subset B \Rightarrow A\ is\ proper\ subset\ of\ B$.

Example

$B = \{a, b, c, d\}, \quad A = \{a, b\}$

$b, a \in B \Rightarrow A \subset B$

Intersection

We write $A \cap B$ or A intersection B to refer to all elements in A and B

$$(x \in A \cap B) \Leftrightarrow ((x \in A) \wedge (x \in B))$$

Union

The union of A and B denoted by $A \cup B$ refers to a set of every element in A and every element in B

$$(x \in A \cup B) \Leftrightarrow ((x \in A) \vee (x \in B))$$

The difference of two sets

The difference between A and B is a collection of the elements which are belong to A and does not belong to B denoted $A \backslash B$ or $A - B$ or C_A^B such that

$$A - B = \{x \in A \ \wedge \ x \notin B\}$$

Complement of a set

Let $A \subset U$ where U is called universal set which is a large set that is implicitly defined. The complement of A is the set of elements not in A.

$$\overline{A} = C_U^A = \{x \in U \ \wedge \ x \notin A\}$$

Power set

$P(A)$ is the power set of A which represents all of the subsets of A

Example $A = \{0, 1, 2\}$

$$P(A) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$$
$$card(P(A)) = 2^{card(A)} = 2^3 = 8$$

Cartesian product

The cartesian product of A by B is the set denoted by $A \times B$ and defined by

$$A \times B = \{(x, y) \mid x \in A \wedge x \in B\}$$

$$card(A \times B) = card(A) \times card(B)$$

Example

A=$\{x, y, z\}\,, \quad B = \{1, 2, 3\}$

The cartesian product $A \times B$ it can be shown as follows

$\times$	1	2	3	B
x	$(x, 1)$	$(x, 2)$	$(x, 3)$	
y	$(y, 1)$	$(y, 2)$	$(y, 3)$	
z	$(z, 1)$	$(z, 2)$	$(z, 3)$	
A				

$$A \times B = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3), (z, 1), (z, 2), (z, 3)\}$$
$$card(A \times B) = card(A) \times card(B) = 3 \times 3 = 9$$

1.3 Properties

Let $A,\ B$,and C be a subset in E, we can show that

$$A \subset C \text{ and } B \subset C \Rightarrow (A \cap B \subset C) \text{ and } (A \cup B \subset C)$$
$$A \cap B = B \cap A$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$
$$A \cap \emptyset = \emptyset,\ A \cap A = A,\ (A \subset B \Leftrightarrow A \cap B = A)$$
$$A \cup B = B \cup A$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$C_E^{AC_E^A} = A$$
$$C_E^{A \cap B} = C_E^A \cup C_E^B$$
$$C_E^{A \cup B} = C_E^A \cap C_E^B$$

Exercise 1.

Prove that $\quad C_E^{A \cap B} = C_E^A \cup C_E^B$

Solution

$$x \in C_E^A \cup C_E^B \Rightarrow \quad \{ \quad \begin{matrix} x \in C_E^A & x \notin A \Rightarrow x \notin A \cap B \Rightarrow x \in C_E^{A \cap B} \\ x \in C_E^B & x \notin B \Rightarrow x \notin A \cap B \Rightarrow x \in C_E^{A \cap B} \end{matrix}$$

Exercise 2.

Let A, and B be two sets defined in an interval on $\mathbb{R}$ as follows
$A = [-2, 2] \quad B = [-5, 1]$
— Find : $A \cup B, \quad A \cap B, \quad A - B, \quad C_{\mathbb{R}}^{A}$

Solution

$A \cup B = [-5, 2)$
$A \cap B = [-2, 1)$
$A - B = [1, 2)$
$C_{\mathbb{R}}^{A} = -\infty, -2) \cup (2, +\infty$

2 Binary relations

When we deal with numbers take for example a and b, we can easly supose some relations between them
$a < b, \quad a \geq b, \quad a = b, \quad a \neq b, \quad a^2 + 1 = b$
This is not much different when we talk about sets where some relationships can be expected between the elements of two sets or between the elements of the set itself.

2.1 What does it mean a relation in sets :

Let A and B be two sets. Giving a relation between elements of the two sets it can be shown as a look for a link or some common properties between them.
Let $A = \{7, 5, 6\}$,and $B = \{5, 9, 8\}$ and let $\Re$ be relation from A to B defined by

$$\forall a \in A, \ b \in B \mid a > b$$

We can show that the only pair (a, b) which satisfies this relation is $(6, 5)$ and we can express that by $6\Re5$ and thus $\Re = \{(6, 5)\}$

Definition

A binary relation $\Re$ on a set A is a subset of $A \times A$ and the elements of $\Re$ are defined by the paires $(a, b) \in \Re$, or $a\Re b$ to means that a is related to b by $\Re$.

Examples of some relations

1. The subset $\Re = \{\forall (a, b) \in \mathbb{N}^2 \mid a \ divides \ b\}$. It can be shown that the element $(8, 2)$ satisfy the dividing relation but $(2, 8)$ is not. This imply $(8, 2) \in \Re, \quad (2, 8) \notin \Re$.
2. A relation S on $\mathbb{N}$ defined by $m \, S \, n$ if and only if $m > 10 \, n$.
3. The relation G on $\mathbb{R}^+$defined by xGy if and only if $y = e^x$.

2.2 Properties of relations :

Reflexive relation

If $\Re$ is reflexive relation on a set A, then

$$\forall x \in A \mid x\Re x$$

Reflexivity asserts that each element is related to itself.

Example 1.

The relation $\leq$ on $\mathbb{N}$ is reflexive : Fo every $k \in \mathbb{N}$, $k \leq k$. But the relation $<$ is not reflexive.

Example 2.

The relation G on $\mathbb{R}$ defined by : $a\,G\,b$ if and only if $ab > 0$ is not reflexive because $0^2 \not> 0$. The relation G is reflexive if G is a relation on $\mathbb{R}^*$

Symmetric relation

The relation $\Re$ on a set A is said to be symmetric if and only if

$$\forall a, b \in A \mid (a\Re b) \Rightarrow (b\Re a)$$

Example

The relation $\Re$ on $\mathbb{N}$ defined by :

$$k\Re n \Leftrightarrow k + n \quad is\ even$$

is a symmetric relation.

Anti-symmetric relation

Anti-symmetric property of a relation $\Re$ on a set A is defined by

$$\forall a, b \in A \mid (a\Re b) \wedge (b\Re a) \Rightarrow (x = y)$$

You can check that the relation $\leq$ is an anti-symmetric relation.

Transitive relation

We say that a relation $\Re$ on a set A is transitive if and only if

$$\forall a, b, and\ c \in A \mid ((a\Re b) \wedge (b\Re c)) \Rightarrow (a\Re c)$$

Example 1 : The relation $<$ on $\mathbb{R}$ is transitive : $(a < b) \wedge (b < c) \Rightarrow (a < c)$

Example 2 : The parallelism property of straight-lines is a transitive relation.

2.3 Equivalence relation

A relation $\Re$ is an equivalence relation if it is reflexive, symmetric and transitive.

Example

The relation $\Re$ on $\mathbb{N} \times \mathbb{N}$ defined by $(m, n)\Re(p, q) \Leftrightarrow mq = np$ is an equivalence relation.

2.4 Equivalence class

Let $\Re$ be an equivalence class on a non-empty set A. The equivalence class of an element $x \in A$ (w.r.t $\Re$) is the set denoted by $[x]$ and defined by.

Example

The relation $\mathcal{S}$ on $\mathbb{R}_+^*$ is an equivalence relation defined by

$$\forall x, y \in \mathbb{R}_+^* \mid \frac{1}{x^2} + x = \frac{1}{y^2} + y$$

you can check that if it is an equivalence relation.
In this example the class of equivalence of 1 it can be given by

$$[1] = \{y \in \mathbb{R}_+^* \mid y\mathcal{S}1$$
$$\Rightarrow \frac{1}{y^2} + y = \frac{1}{1^2} + 1$$
$$(y^2 - y - 1)(y - 1) = 0$$
$$\Rightarrow [1] = \{1, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\}$$

The general form of the equivalence class w.r.t $\mathcal{S}$ is given by

$$[x] = \{y \in \mathbb{R}_+^* \mid y\mathcal{S}x$$
$$\Rightarrow \frac{1}{y^2} + y = \frac{1}{x^2} + x$$
$$(x + y - x^2y^2)(x - y) = 0$$
$$[x] = \{x, \frac{1+\sqrt{1+4x^3}}{2x^2}, \frac{1-\sqrt{1+4x^3}}{2x^2}\}$$

2.5 Ordering relation

The relation $\mathcal{R}$ on a set A is said to be an ordering relation if it is reflexive, anti-symmetric and transitive.

Example : You can check that the following relation is an ordering relation.
Let $\mathcal{R}$ be a relation on $\mathbb{N}^*$ defined by :

$$x\mathcal{R}y \Leftrightarrow (\kappa \in \mathbb{N}^* \mid y = \kappa x)$$

Total order relation A relation $\mathcal{R}$ is said to be a total order if :

$\forall x, y \in A : (x\mathcal{R}y) \vee (y\mathcal{R}x)$

If $\mathcal{R}$ is not a total order, then is said to be a partial order.
In the previous example $\mathcal{R}$ is a partial order relation becaus $(2, 3)$ is not satisfies the relation.

3 Functions

3.1 Definition

Let E and F be any non empty sets. A $function$ from E to F is a rule or generally a relation that associates with each element of E a unique element of F. In this case the set E is the domain and F is called the codomain.

A function from f from E to F is defined by :

$$\forall x \in E, \forall y \in F \mid f : F \longrightarrow E$$
$$x \longmapsto f(x) = y$$

— $f(E)$ is the image of E under f

— E is the pre-image of $f(E)$ under f

Example

Let E and F be two sets where $E = \{1, 2, 3\}$, and let f a function defined by

$$f : F \longrightarrow E$$
$$x \longmapsto f(x) = 2x + 1$$

This can imply that $F = \{3, 5, 7\}$

3.2 Injective and surjective function

Let f be a function from E to F :

— The function f is injective if $\forall x_1, x_2 \in E \mid (f(x_1) = f(x_2)) \Rightarrow (x_1 = x_2)$

— The function f is surjective if $\forall y \in F; \exists x \in E \mid y = f(x)$

— The function f is bijective if it is both injective and surjective.

Example

Let h be a function defined by :

$$h : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto h(x) = x^2$$

— Is h injective ? $(h(x_1) = h(x_2)) \overset{?}{\Longrightarrow} (x_1 = x_2)$

— $x_1^2 = x_1^2 \not\Rightarrow x_1 = x_2 \Rightarrow h$ is not injective

— I h surjective ?$\forall y \in \mathbb{R}; \exists x \in \mathbb{R} \mid y = f(x)$?

— $h(x) = y = x^2 \Rightarrow x = \sqrt{y}$ and it is obvous that since $y \in \mathbb{R}^-$, x is not defined. This means that for all pre-images $y \in \mathbb{R}^-$ in the codomain $\mathbb{R}$ is not have an image x in the domain $\mathbb{R}$. leading to h to be not sujective.

3.3 Composition of two functions

Let $f : E \longrightarrow F$ and $g : F \longrightarrow H$ be two functions. The composition of the two functions denoted by $g \circ f$ is another function defined by

$$\begin{aligned} g \circ f : \mathbb{E} &\longrightarrow \mathbb{H} \\ x &\longmapsto g(f(x)) \end{aligned}$$

Example 1.

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto x+1 \end{aligned} \qquad \begin{aligned} g : \mathbb{R} &\longrightarrow \mathbb{R}^+ \\ x &\longmapsto x^2+2 \end{aligned}$$

The composition of the two functions gives $g \circ f = g(f(x)) = (x+1)^2 + 2$ which is defined by

$$\begin{aligned} g \circ f : \mathbb{R} &\longrightarrow \mathbb{R}^+ \\ x &\longmapsto g(f(x)) = x^2 + 2x + 3 \end{aligned}$$

Example 2.

Let A, B and C be three sets, and f and g be two functions such that

$$\begin{array}{ccc} A = \{a, b, c\} & B = \{0, 1\} & C = \{1, 2, 3\} \\ f : A \longrightarrow B & & g : B \longrightarrow C \\ f = \{(a, 0), (b, 1), (c, 0)\} & & g = \{(0, 3), (1, 1)\} \end{array}$$

The composition of the two functions gives $g \circ f = \{(a, 3), (b, 1), (c, 3)\}$

3.4 Inverse function

Definition (identity function)

The Identity function on a set E is the function denoted $\imath_E : E \longrightarrow E$ defined as

$$\forall x \in E \mid \imath_E(x) = x$$

you can show that $\imath_E(x) = \imath_E(y) \Rightarrow x = y$

Definition (inverse function)

Let $f : E \longrightarrow F$ be a bijective function. The inverse function of f is denoted by f^{-1} and defined by : $f : F \longrightarrow E$ such that $f^{-1} \circ f = \imath_E$ and $f \circ f^{-1} = \imath_F$. Also we have $(y = f(x)) \Leftrightarrow (x = f^{-1}(y))$. The function f admits an inverse f^{-1} if and only if it is bijective.

Example

The bijective function f is defined by

$$\begin{aligned} f : \mathbb{R}^{-\{-2\}} &\longrightarrow \mathbb{R}^{-\{1\}} \\ x &\longmapsto f(x) = \frac{x+1}{x+2} \end{aligned}$$

We can show that $y = \frac{x+1}{x+2} \Rightarrow x = \frac{2y-1}{1-y}$. Thus, the in verse function of f is given with

$$\begin{aligned} f : \mathbb{R}^{-\{1\}} &\longrightarrow \mathbb{R}^{-\{-2\}} \\ y &\longmapsto f^{-1}(y) = \frac{2y-1}{1-y} \end{aligned}$$